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Representations of the symmetric group and the Schur-Weyl Duality

Introduction : The aim of these lectures is to study representation theory of symmetric groups and use it to study some representations of the general linear group. This can be treated as an introduction to the material covered in chapters 4 and 6 of the book by Fulton and Harris. Let V be a (finite dimensional) complex vector space. For a fixed positive integer n, we have natural action of $\mathrm{GL}(V)$ (from the left) and the symmetric group S_n on $V^{\otimes^n}$ (from the right), given by

$$g.(v_1 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n,\ g \in \mathrm{GL}(V)$$

and

$$(v_1 \otimes \cdots \otimes v_n)\sigma = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)},\ \sigma \in S_n.$$

It is an easy exercise to see that these two actions commute with each other. This fact plays an important role in the theory. The representaion theory of S_n and that of the general linear group $\mathrm{GL}(V)$ tie up in the study of representation of $\mathrm{GL}(V) \times S_n$ on $V^{\otimes^n}$. There is a pairing between irreducible representations of S_n and irreducible representations of $\mathrm{GL}(V)$, which will be made precise among other results. We shall first breifly review some results in representations of finite groups. The notes are mostly self-contained, have been (shamelessly!) copied from the book by Fulton and Harris. **Warning :** Watch out for mistakes!!!

A quick review of representation theory

In this section, we will briefly recall some main features of representation theory of finite groups, for proofs of results we refer the reader to Fulton and Harris. **All representations in these lectures are finite dimensional and are defined over the complex field**. A **representation** of a group G on a (complex) vector space V is a homomorphism $\rho : G \longrightarrow \mathrm{GL}(V)$. We will sometimes denote this by saying (V, ρ) is a representation of G.

Examples : The constant homomorphism $G \longrightarrow \mathrm{GL}(V)$ is called the **trivial representation** of G on V. Let X be any set with a (left) G-action (denoted by gx) and let V be a vector space with basis $\{e_x | x \in X\}$. The G has a representation on V by permuting the basis, i.e. for $g \in G$ we define $ge_x = e_{gx}$. This is called the **permutation representation** of G on X. By a map between two representations (V, ρ) and (W, π) we mean a linear map $\phi : V \longrightarrow W$ such that $\phi(\rho(g)v) = \pi(g)\phi(v)$ for all $g \in G,\ v \in V$. We write gv for $\rho(g)v$ when the context is clear. Isomorphic representations are called **equivalent**. A **sub-representation** of a representation (V, ρ) is a subspace $W \subset V$ such that $\rho(g)W = W$ for all $g \in G$. We call a representation **irreducible** if it has no sub-representations other than zero and itself.

Group algebra and the Regular representation : Let G be a (finite) group. The **group algebra** $\mathbb{C}[G]$ of G is the vector space of all (complex valued) functions on G. Let $\phi, \psi \in \mathbb{C}$. We define the product on $\mathbb{C}[G]$ by

$$\phi\psi(x) = \sum_{yz=x} \phi(y)\psi(z).$$

This makes $\mathbb{C}[G]$ a $\mathbb{C}$-algebra. The characteristic functions of elements of G form a basis of $\mathbb{C}[G]$ (prove this!). Namely, let e_g for $g \in G$ denote the function $e_g(h) = 1$ if $h = g$ and $e_g(h) = 0$ if $h \neq g$. Then any $\phi \in \mathbb{C}[G]$ has the expression $\phi = \sum \phi(g)e_g$. We identify the functions e_g with $g \in G$ and treat elements of $\mathbb{C}[G]$ as linear combinations of elements of G. We have $\dim\mathbb{C}[G] = |G|$. The representation of G on $\mathbb{C}[G]$ via left translations (i.e. by permuting the basis G) is called the **left regular representation** of G. Clearly the regular representation is the permutation representation of G for the left translation action on G. The regular reprewsentation of contains all irreducible representations of G (upto equivalence). The representations of G are precisely (left) modules over $\mathbb{C}[G]$. **Irreducible representations** are **simple** modules over $\mathbb{C}[G]$, which in turn, upto a $\mathbb{C}[G]$-isomorphism, are the **simple left ideals** of $\mathbb{C}[G]$. The group algebra $\mathbb{C}[G]$ is **semisimple**, i.e., is the direct sum of its simple (left) ideals. As a consequence, any $\mathbb{C}[G]$-module is semisimple. We have therefore

Theorem 0.0.1. *Every irreducible representation of G is contained in the regular representation.*

Theorem 0.0.2. *Any representation of G is a direct sum of irreducible representations, i.e. is* **completely reducible**.

Remark : We maybe given a **right** action of a group G on a vector space V, making V into a **right** $\mathbb{C}[G]$-module. We have an **involution** (i.e. anti-automorphism of order 2) on G, namely $g \mapsto g^{-1}$. This induces an isomorphism of $\mathbb{C}[G]$ with the **opposite** group algebra $\mathbb{C}[G]^{\mathrm{op}}$, where the product is the flipped product of the one in $\mathbb{C}[G]$. This converts the right representations of G into left representations of G. We will apply this to the situation $G = S_n$ and the representation on $V^{\otimes^n}$ as described in the introduction and treat $V^{\otimes^n}$ as a left $\mathbb{C}[S_n]$-module via this gadget (write the left action explicitly!).

A representation which is direct sum of irreducible representations, all isomorphic to a representation S, is called **isotypical** of type S. Clearly any representation can be broken into its isotypical components. Since any simple $\mathbb{C}[G]$-module (i.e. an irreducible representation of G) is isomorphic, as a $\mathbb{C}[G]$-module, to a simple left ideal of $\mathbb{C}[G]$, to find the irreducible representations of G, we must locate the simple ideals making up $\mathbb{C}[G]$. This is done by constructing projections of $\mathbb{C}[G]$ onto the simple ideals, which is same as constructing certain idempotents in $\mathbb{C}[G]$ (why?). We shall do this in the case of symmetric groups. The number of irreducible representations of a finite group is finite, in fact,

Theorem 0.0.3. *The number of inequivalent irreducible representations of a finite group G is equal to h, the number of distinct conjugacy classes in G. We have $\mathbb{C}[G] \simeq \prod_{i=1}^{h} M_{m_i}(\mathbb{C})$, $M_{m_i}(\mathbb{C})$ is isotypical of type S_i, where S_i can be taken to be the set of all matrices in $M_{m_i}(\mathbb{C})$ whose all coulmns but the first are zero. We have $|G| = \sum_{i=1}^{h} m_i^2$, $m_i = \dim(S_i)$, $1 \le i \le h$.*

Given a representation V of G, there are non-negative integers d_i, uniquely determined by V, such that $V \simeq V_1^{\oplus d_1} \oplus \cdots \oplus V_h^{\oplus d_h}$, where V_i are the inequivalent irreducible representations of G. We call d_i the **multiplicity** of V_i. Hence

Theorem 0.0.4. *An irreducible representation of G occurs in the regular representation with multiplicity equal to its dimension.*

Character theory : Let (V, ρ) be a representation of G. The function $\chi_\rho : G \longrightarrow \mathbb{C}$, $g \mapsto \mathrm{trace}(\rho(g))$ is called the **character** of (V, ρ).

Examples : For the trivial representation (V, ρ) of dimension d, $\chi_\rho(g) = d$ for all $g \in G$. For the regular representation we have $\chi_{\mathrm{reg}}(g) = |G|$ if $g = 1$ and $\chi_{\mathrm{reg}}(g) = 0$ if $g \neq 1$. This is a special case of the following situation. Let H be a subgroup of G and consider the permutation representation on $X = G/H$. We describe the character of this representation below :

Proposition 0.0.5. *Let $g \in G$ and let C_g be the conjugacy class of g. Then the character of the permutation representation of G on G/H is given by*

$$\chi(x) = [G : H]\frac{|C_x \cap H|}{|C_x|}.$$

Functions which are constant along conjugacy classes of G are called **class functions**. The vector space $\mathcal{C}(G)$ of all class functions on G has a natural basis, namely the set of characteristic functions of conjugacy classes in G (prove this). We introduce an **inner product** on the space of all functions on G by

$$< \phi, \psi >= \frac{1}{|G|}\sum_{x\in G}\phi(x^{-1})\psi(x).$$

We have,

Proposition 0.0.6. *The set of mutually inequivalent ireducible characters of G is an* **orthonormal basis** *for $\mathcal{C}(G)$.*

Corollary 0.0.7. *The multiplicity of an irreducible character χ_i in a character χ is $< \chi, \chi_i >$.*

The high point of character theory is

Theorem 0.0.8. *Two representations of G are equivalent if and only if their characters are equal, i.e. the character determines the representation.*

Representations of the Symmetric group

Exercise : Let $\theta = (a_1, \cdots, a_r)$ be an r-cycle in S_n and $\sigma \in S_n$. Show that $\sigma\theta\sigma^{-1} = (\sigma(a_1), \cdots, \sigma(a_r))$. Show that any two permutations have the same cycle structure (i.e. the lengths of cycles in an expresession as a product of disjoint cycles) if and only if they are conjugate.

Theorem 0.0.9. *The set of conjugacy classes of S_n is in* **natural bijection** *with the set of all partitions of n. This simply says that a conjugacy class is uniquely determined by a partition of n.*

In what follows, we will associate, to a given partition λ of n, a **simple left ideal** of the group algebra $\mathbb{C}[S_n]$, thereby giving an irreducible representation V_λ of S_n. These will be shown to be inequivalent for distinct partitions and hence we would have all the irreducibles for S_n. These modules are called the **Frobenius-Young modules** for S_n. We now exploit the above bijection between partitions of n and the conjugacy classes in S_n.

Definition 0.0.10. *By a* **partition** λ *of* n *we mean a decreasing sequence of* **positive** *integers* $\lambda = (\lambda_1, \cdots, \lambda_r)$ *with* $\lambda_1 \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 1$ *such that* $\sum \lambda_i = n$.

We also write $n = \sum_{i=1}^{n} \gamma_i i$, where γ_i is the number of times i occurs in the partition λ of n. We book-keep this by the notation $\lambda = (1^{\lambda_1}, \cdots, i^{\lambda_i}, \cdots, n^{\lambda_n})$. In the above definition, r is called the **depth** of λ. Two partitions λ and μ of n are **equal** if they have equal depth and $\lambda_i = \mu_i$ for all i.

Lexicographic order : On the set of partitions of n, we have a natural total order $\geq$, called the lexicographic order, given by $\lambda \geq \mu$ if either $\lambda = \mu$ or, at the first place i where λ and μ differ, $\lambda_i > \mu_i$. Here $\lambda = (\lambda_1, \cdots, \lambda_k)$ and $\mu = (\mu_1, \cdots, \mu_l)$. For example, $(2,1) > (1,2) > (1,1,1)$

The number of partitions of n is denoted by $P(n)$. For example, $P(1) = 1,\ P(2) = 2,\ P(3) = 3,\ P(4) = 5,\ P(5) = 7$ and so on.

Proposition 0.0.11. *Let* $\lambda = (1^{\gamma_1}, \cdots, n^{\gamma_n})$ *be a partition of* n. *Let* $\mathcal{C}_\lambda$ *be the conjugacy class in* S_n *of cycle type* λ. *Then*

$$|\mathcal{C}_\lambda| = \frac{n!}{1^{\gamma_1}\gamma_1!.2^{\gamma_2}\gamma_2! \cdots n^{\gamma_n}\gamma_n!}.$$

Proof. We can enumerate the elements in $\mathcal{C}_\lambda$ by laying out γ_i blank cycles of lengths $i,\ 1 \leq i \leq n$ and filling numbers from 1 to n in them. We have $n!$ ways of doing this without repetitions. Now, each cycle of length i has i expressions giving the same cycle and disjoint cycles commute with each other. Hence the product of all i-cycles in a reprsentative of $\mathcal{C}_\lambda$ in the above count repeats $i^{\gamma_i}\gamma_i!$ times. The result now follows. □

Characters of S_n : Let $\lambda = (\lambda_1, \cdots, \lambda_r)$ be a partition of n. We call the subgroup $Y_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_r}$ of S_n a **Young subgroup** of shape λ. Clearly the index of this subgroup is $n!/(\lambda_1!, \cdots, \lambda_r!)$. The **permutation representations** of Young subgroups are of special importance.

Theorem 0.0.12. *Let* λ *be a partition of* n *as above and let* ζ_λ *be the character of the permutation representation* U_λ *defined by the Young subgroup* Y_λ. *For a conjugacy class* $\mathcal{C}_\mu$, *the character value is given by*

$$\zeta_\lambda(\mathcal{C}_\mu) = \frac{n!|\mathcal{C}_\mu \cap Y_\lambda|}{(\lambda_1! \cdots \lambda_r!)|\mathcal{C}_\mu|}.$$

Proof. This is immediate from Proposition 0.0.5. □

Hence we need to compute $|\mathcal{C}_\mu \cap Y_\lambda|$ in the above, since we already have computed $|\mathcal{C}_\mu|$. Let $\lambda = (\lambda_1, \cdots, \lambda_r) = (1^{\gamma_1}, \cdots, n^{\gamma_n})$ and $\mu = (\mu_1, \cdots, \mu_s) = (1^{\epsilon_1}, \cdots, n^{\epsilon_n})$. Now, any element $\theta \in Y_\lambda$ is of the form $\theta = \theta_1 \cdots \theta_r$ for uniquely determined elements $\theta_i \in S_{\lambda_i},\ 1 \leq i \leq r$. Let l_{ij} be the number of j-cycles in the cycle decomposition of $\theta_i,\ 1 \leq j \leq n$. Hence we have,

$$\lambda_i = \sum_{j=1}^{n} j l_{ij}, \quad 1 \leq i \leq r \qquad (1)$$

If $\theta \in \mathcal{C}_\mu$ then the total number of j-cycles in θ must be ϵ_j, for all j. So that we also have

$$\epsilon_j = \sum_{i=1}^{r} l_{ij}, \quad 1 \leq j \leq n \qquad (2)$$

Hence to each $\theta \in \mathcal{C}_\mu \cap Y_\lambda$, we have associated an $r \times n$ matrix (l_{ij}) of non-negtive entries satisfying the above equations and conversely, given any such matrix, we can get an element $\theta \in \mathcal{C}_\mu \cap Y_\lambda$ associated to it. Let us now count the number of elements in $\mathcal{C}_\mu \cap Y_\lambda$ associated to the **same** matrix (l_{ij}) satisfying (1) and (2). For a fixed i, equation (1) determines a conjugacy class in S_{λ_i} of cycle type $(1^{l_{i1}}, \cdots, j^{l_{ij}}, n^{l_{in}})$, which is a partition of λ_i. By [**?**], this conjugacy class contains $\delta_i = \lambda_i!/(1^{l_{i1}} l_{i1}! \cdots n^{l_{in}} l_{in}!)$ elements. Hence $\mathcal{C}_\mu \cap Y_\lambda$ contains $\prod_{i=1}^{r} \delta_i$ elements corresponding to the same matrix satisfying (1) and (2). Hence, we have,

$$|\mathcal{C}_\mu \cap Y_\lambda| = \sum \prod_{i=1}^{r} \frac{\lambda_i!}{1^{l_{i1}} l_{i1}! \cdots n^{l_{in}} l_{in}!},$$

the sum being over all sets $\{l_{ij}\}$ of non-negative integers satisfying (1) and (2). Combining this with Theorem 0.0.12 and simplifying a bit, we get

Theorem 0.0.13. *The value of the permutation character ζ_λ corresponding to a Young subgroup of shape λ on a conjugacy class $\mathcal{C}_\mu$ is given by*

$$\zeta(\mathcal{C}_\mu) = \sum \prod_{j=1}^{n} \frac{\epsilon_j!}{l_{j1}! \cdots l_{jn}!},$$

the sum is over sets $\{l_{st}\}$ of all non-negative solutions of (1) *and* (2).

The set of all irreducible characters of S_n is a basis of the group of all characters of S_n. We have in fact,

Theorem 0.0.14. *The set $\{\zeta_\lambda\}$ of permutation characters corresponding to all Young subgroups of S_n is a basis of the group of characters of S_n. In particular, all characters of S_n take integer values.*

We will supply a proof of this later (after we introduce Schur polynomials in the next section).

Young diagrams and tableau : We represent a partition $\lambda = (\lambda_1 \cdots, \lambda_r)$ of n combinatorially by arranging boxes in rows, with the ith row containing λ_i many boxes and such that the left most boxes in all the rows are in the same column and then the next and so on. We get then an array of (empty) boxes, called a **Young frame** or a **Young diagram** (of shape λ) . Given a partition $\lambda = (\lambda_1, \cdots, \lambda_r)$ of n, the partition $\lambda' = (\lambda'_1, \cdots, \lambda'_s)$ is called the **conjugate partition** of λ where λ'_i is the number of boxes in the ith column of T_λ. The Young diagram below depicts the partition $\lambda = (5, 5, 3)$ of $n = 13$.

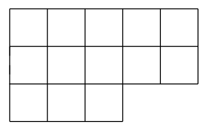

Figure 1:

Young diagrams provide us with a tool to construct certain projection operators for the **regular representation** of S_n, which in turn provide us with the irreducible representations of S_n. To acheive this, we fill the empty boxes with **distinct** numbers from 1 to n, resulting in a **Young tableau** as pictured above.

1	7	4	12	10
5	3	9	11	13
6	8	2		

Figure 2:

Exercise : Prove that for a Young diagram with n boxes, there are $n!$ Young tableau.

Row and Column groups: For a given Young diagram, the Young tableau are in a natural bijection with elements of S_n. We associate to a Young tableau T_λ the permutation $\sigma \in S_n$ such that $\sigma(1), \sigma(2), \cdots, \sigma(\lambda'_1)$ are the entries in the first column from top to bottom, $\sigma(\lambda'_1 + 1), \cdots, \sigma(\lambda'_2)$ are the entries in the second column and so on. Hence we may write $T_\lambda(\sigma)$ for a Young tableau corresponding to the shape λ. Note that S_n acts on the set of tableau of a fixed shape λ by permuting the entries, $\sigma(T_\lambda(\theta)) = T_\lambda(\sigma\theta)$ for $\sigma, \theta \in S_n$. We can associate two natural subgroups of S_n to a fixed tableau $T_\lambda(\sigma)$. Let $P(T_\lambda(\sigma)) = P_\lambda$ be the set of all permutations in S_n leaving the rows of $T_\lambda(\sigma)$ invariant. We call this the **row**

group of the tableau. The **column group** $Q_\lambda = Q(T_\lambda(\sigma))$ of a tableau is defined similarly. The proposition below lists some basic properties of these groups.

Proposition 0.0.15. *Let $\lambda = (\lambda_1, \cdots, \lambda_r)$ be a partition of n and T_λ be a Young tableau of shape λ. We then have*
1. $Q(T_\lambda) = P(T_{\lambda'})$, *where λ' is the conjugate partition of λ.*
2. $P(T_\lambda) \simeq S_{\lambda_1} \times \cdots \times S_{\lambda_r}$, *where S_{λ_i} is the symmetric group on the entries of the ith row of T_λ. Hence $P(T_\lambda)$ is a Young subgroup of shape λ.*
3. $P(T_\lambda) \cap Q(T_\lambda) = \{1\}$, $\;P(\sigma(T_\lambda)) = \sigma P(T_\lambda)\sigma^{-1}$ *and* $Q(\sigma(T_\lambda)) = \sigma Q(T_\lambda)\sigma^{-1}$ *for* $\sigma \in S_n$.

Proof. The proof is an easy exercise. □

Young symmetrizers : For a fixed tableau of shape λ, let P and Q denote the row and coulmn groups respectively (with respect to a partition λ of n). We single out two special elements of the group algebra $\mathbb{C}[S_n]$ as follows. Let

$$a_\lambda = \sum_{g \in P} e_g, \quad b_\lambda = \sum_{g \in Q} \mathrm{sign}(g) e_g.$$

Since the coefficients of both a_λ and b_λ are nonzero, as elements of $\mathbb{C}[S_n]$ both $a_\lambda, b_\lambda \neq 0$. Before we go on, let us look at an example. Consider the partition $\lambda = (2, 1)$ of 3 and the tableau as below

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Figure 3:

Here the row group $P_\lambda = S_{\{1,2\}}$. Hence $a_\lambda = e_1 + e_{(1,2)}$. While, the column group $Q_\lambda = S_{\{1,3\}}$, hence $b_\lambda = e_1 - e_{(1,3)}$. Let us now see what a_λ and b_λ do. Let V be any complex vector space. Then we have seen that S_n acts on $V^{\otimes^n}$ by permuting the factors. Thinking of elements of S_n as endomorphisms of $V^{\otimes^n}$ in this fashion, we can identify $\mathbb{C}[G]$ with a subalgebra of $\mathrm{End}(V^{\otimes^n})$. Hence both a_λ and b_λ can be thought of as endomorphisms of $V^{\otimes^n}$. If $\lambda = (\lambda_1, \cdots, \lambda_r)$, then the row group $P_\lambda \simeq S_{\lambda_1} \times \cdots \times S_{\lambda_r}$ is a Young subgroup and a_λ is the sum of all elements of this group. Taking into account the action of S_n on $V^{\otimes^n}$, it follows that

$$\mathrm{Im}(a_\lambda) = \mathrm{Sym}^{\lambda_1}(V) \otimes \mathrm{Sym}^{\lambda_2}(V) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_r}(V) \subset V^{\otimes^n},$$

where we group the factors of $V^{\otimes^n}$ according to the rows of the Young tableau. Similarly we have

$$\mathrm{Im}(b_\lambda) = \bigwedge^{\mu_1}(V) \otimes \bigwedge^{\mu_2}(V) \otimes \cdots \otimes \bigwedge^{\mu_l}(V) \subset V^{\otimes^n},$$

where $\mu = (\mu_1, \cdots, \mu_l)$ is the conjugate partition of λ. The Young diagram of the conjugate partition is simply the **tranpose** of the Young diagram of λ.

Example: In the figure 3 above, these calculations yield, for $x_1 \otimes x_2 \otimes x_3 \in V \otimes V \otimes V$,

$$a_\lambda(x_1 \otimes x_2 \otimes x_3) = (e_1 + e_{(12)})(x_1 \otimes x_2 \otimes x_3) = x_1 \otimes x_2 \otimes x_3 + x_2 \otimes x_1 \otimes x_3 \in \mathrm{Sym}^2(V) \otimes V,$$

and

$$b_\lambda(x_1 \otimes x_2 \otimes x_3) = (e_1 - e_{(13)})(x_1 \otimes x_2 \otimes x_3) = x_1 \otimes x_2 \otimes x_3 - x_3 \otimes x_2 \otimes x_1 \in \bigwedge^2(V) \otimes V.$$

Thus we see that $a_\lambda(V^{\otimes^3}) = \mathrm{Sym}^2(V) \otimes V$ and $b_\lambda(V^{\otimes^3}) = \bigwedge^2(V) \otimes V$. The element $c_\lambda = a_\lambda . b_\lambda \in \mathbb{C}[S_n]$ is called a **Young symmetrizer**.

Example : For the partition of 3 in figure 3, the symmetrizer

$$c_\lambda = (e_1 + e_{(12)})(e_1 - e_{(13)}) = e_1 + e_{(12)} - e_{(13)} - e_{(132)}.$$

We have

$$c_\lambda(x_1 \otimes x_2 \otimes x_3) = x_1 \otimes x_2 \otimes x_3 + x_2 \otimes x_1 \otimes x_3 - x_3 \otimes x_2 \otimes x_1 - x_3 \otimes x_1 \otimes x_2 \in \bigwedge^2(V) \otimes V.$$

When $\lambda = (n)$, we have $P_\lambda = S_n, \ \ Q_\lambda = \{1\}$. Hence $a_\lambda = \sum_{g \in S_n} e_g, \ \ b_\lambda = 1$ and $c_\lambda = a_\lambda . b_\lambda = a_\lambda = \sum_{g \in S_n} e_g$. Further $c_\lambda(V^{\otimes^n}) = \mathrm{Sym}^n(V)$. When $\lambda = (1, \cdots, 1)$, we have $P_\lambda = \{1\}, \ \ Q_\lambda = S_n$. Hence $a_\lambda = 1, \ \ b_\lambda = \sum_{g \in S_n} \mathrm{sign}(g) e_g$ and $c_\lambda = b_\lambda = \sum_{g \in S_n} \mathrm{sign}(g) e_g$. We have, in this case, $c_\lambda(V^{\otimes^n}) = \bigwedge^n(V)$. We will see later that the images of various symmetrizers give almost all **irreducible** representations of $GL(V)$. First let us record some basic properties of the elements constructed above:

Proposition 0.0.16. *With notation as above we have*
1. $c_\lambda \neq 0$
2. $\sigma a_\lambda = a_\lambda \sigma = a_\lambda$ *for all* $\sigma \in P_\lambda$.
3. $\tau b_\lambda = b_\lambda \tau = \mathrm{sign}(\tau) b_\lambda$ *for all* $\tau \in Q_\lambda$.
4. $\sigma c_\lambda = c_\lambda$ *for all* $\sigma \in P_\lambda$.
5. $c_\lambda \tau = \mathrm{sign}(\tau) c_\lambda$ *for all* $\tau \in Q_\lambda$.

Proof. **1.** We have

$$c_\lambda = (\sum_{\sigma\in P}\sigma)(\sum_{\tau\in Q}\operatorname{sign}(\tau)\tau) = \sum_{\sigma\in P}\sum_{\tau\in Q}\operatorname{sign}(\tau)\sigma\tau.$$

Now $P \cap Q = \{1\}$. Hence if $\sigma\tau = \sigma'\tau'$ with $\sigma, \sigma' \in P$ and $\tau, \tau' \in Q$ then $\sigma = \sigma'$ and $\tau = \tau'$. Hence c_λ is a linear combination of **distinct** elements in S_n (as an element of $\mathbb{C}[S_n]$) with nonzero coefficients. Hence $c_\lambda \neq 0$.
2 and **3** are immediate from the definitions.
4. For $\sigma \in P_\lambda$ we have, using **2**,

$$\sigma c_\lambda = \sigma a_\lambda b_\lambda = (\sigma a_\lambda) b_\lambda = a_\lambda b_\lambda = c_\lambda,$$

5. For $\tau \in Q_\lambda$ we have, using **3**,

$$c_\lambda \tau = a_\lambda b_\lambda \tau = \operatorname{sign}(\tau) a_\lambda b_\lambda = \operatorname{sign}(\tau) c_\lambda.$$

□

We now come to the much awaited result, that tells us how to get the irreducible representations of S_n from the regular representation.

Theorem 0.0.17. *$c_\lambda^2 = n_\lambda c_\lambda$ for some complex (in fact rational) number n_λ. The ideal $\mathbb{C}[S_n].c_\lambda$ is a simple left ideal, i.e. $V_\lambda = \mathbb{C}[S_n].c_\lambda$ is an irreducible representation of S_n. All irreducible representations of S_n arise this way, for a unique partition λ of n.*

We will break the proof of this theorem into easier steps in the form of lemmas.

Lemma 0.0.18. *For all $\sigma \in P$ and $\tau \in Q$, $\quad \sigma c_\lambda(\operatorname{sign}(\tau)\tau) = c_\lambda$ and, up to a scalar factor, c_λ is the only such element in $\mathbb{C}[S_n]$.*

Proof. For $\sigma \in P$ and $\tau \in Q$ we have

$$\sigma c_\lambda(\operatorname{sign}(\tau)\tau) = \operatorname{sign}(\tau) c_\lambda \tau = \operatorname{sign}(\tau)^2 c_\lambda = c_\lambda.$$

Conversely suppose $\sum_{g\in S_n} n_g g \in \mathbb{C}[S_n]$ satisfies the above condition. Then, for all $p \in P$ and $q \in Q$ we have $\operatorname{sign}(q) p (\sum n_g g) q = \sum n_g g$. Hence $n_{pgq} = \operatorname{sign}(q) n_g$ for all g, p, q. Putting $g = 1$ we get $n_{pq} = \operatorname{sign}(q) n_1$. Since $c_\lambda = \sum_{p\in P}\sum_{q\in Q}\operatorname{sign}(q) pq$, we would be done if we prove $n_g = 0$ if $g \notin PQ$. Let g be such an element. Suppose we manage to find a **transposition** t such that $t = p \in P$ and $q = g^{-1} t g \in Q$. Then $g = pgq$ and by the above calculation,

$$n_g = \operatorname{sign}(g^{-1} t g) n_g = \operatorname{sign}(t) n_g = -n_g,$$

and hence $n_g = 0$. We will prove next that such a transposition indeed exists. □

Lemma 0.0.19. *Let λ and μ be partitions of n and suppose $\lambda \geq \mu$ in the lexicographic order. Let S_λ and T_μ be fixed Young tableau of shapes λ and μ respectively. Suppose no two numbers appearing in the same row in S_λ are in the same column of T_μ. Then $\lambda = \mu$ and $T_\lambda = \sigma\tau S_\lambda$ for some $\sigma \in P(S_\lambda)$ and $\tau \in Q(S_\lambda)$.*

Proof. The number of entries in the first row of S_λ is equal or bigger than that in the first row of T_μ, since $\lambda \geq \mu$. Suppose it is bigger. Then since the number of columns in T_μ is the number of entries in the first row of T, by pigeon hole principle, some two entries of the first row of S_λ must occur in the same column of T_μ, contradicting the hypothesis. Hence the number of entries in the first row of both the diagrams must be same. There is a column permutation (i.e. an element of the column group) τ_1 of T_μ so that the first row of $\tau_1 T_\mu$ has the same entries as the first row of S_λ. Now note that the entries in the second row of S_λ occur in distinct columns of $\tau_1 T_\mu$ and in rows different from the first. Arguing as before, it follows that S_λ and $\tau_1 T_\mu$ and thus S_λ and T_μ have the same number of entries in the second row. There is a column permutation τ_2 of $\tau_1 T_\mu$ (and hence of T_μ) so that S_λ and $\tau_1\tau_2 T_\mu$ have the same entries in the second row, and so on. Hence we see that $\lambda = \mu$ and there is $\tau' \in Q(T_\lambda)$ such that the entries of each row of $\tau' T_\lambda$ and of S_λ are the same. Hence there is $\sigma \in P(S_\lambda)$ such that $\sigma S_\lambda = \tau' T_\lambda$. We have

$$\tau' \in Q(T_\lambda) = Q(\tau' T_\lambda) = Q(\sigma S_\lambda) = \sigma Q(S_\lambda)\sigma^{-1}.$$

Therefore, $\tau' = \sigma\tau^{-1}\sigma^{-1}$ for some $\tau \in Q(S_\lambda)$. Hence $\sigma\tau^{-1}\sigma^{-1}T_\lambda = \sigma S_\lambda$ and $T_\lambda = \sigma\tau S_\lambda$. □

Corollary 0.0.20. *Suppose $g \notin PQ$. Then there is a transposition $t = p \in P$ such that $g^{-1}tg \in Q$.*

Proof. Let $T_\mu = g(S_\lambda)$. We claim that there are two distinct integers appearing in the same row of S_λ and in the same column of T_μ. If otherwise, by the lemma above, $\lambda = \mu$ and $T_\lambda = \sigma\tau S_\lambda$ for some $\sigma \in P(S_\lambda)$ and $\tau \in Q(S_\lambda)$. Hence $g(S_\lambda) = \sigma\tau(S_\lambda)$. In other words $g^{-1}\sigma\tau \in P(S_\lambda) \cap Q(S_\lambda)$. Hence $g = \sigma\tau \in PQ$, a contradiction. Hence there are two distinct integers appearing in the same row of S_λ and in the same column of T_μ. Let t be the transposition flipping these two integers. Then $t \in P(S_\lambda) \cap Q(T_\mu)$. But $Q(T_\mu) = gQ(S_\lambda)g^{-1}$. Hence $t = gsg^{-1}$ for some $s \in Q(S_\lambda)$, i.e. $g^{-1}tg \in Q$. □

Lemma 0.0.21. **1.** *If $\lambda > \mu$, then for all $x \in \mathbb{C}[S_n]$, $a_\lambda x b_\mu = 0$. In particular, if $\lambda > \mu$ then $c_\lambda c_\mu = 0$.*
2. *For all $x \in \mathbb{C}[S_n]$, $c_\lambda x c_\lambda$ is a scalar multiple of c_λ. In particular, $c_\lambda^2 = n_\lambda c_\lambda$ for some $n_\lambda \in \mathbb{C}$.*

Proof. **1.** It suffices to prove the assertion when $x = g \in S_n$. Let T_μ be the Young tableau corresponding to b_μ. Then $gb_\mu g^{-1}$ corresponds to the tableau gT_μ. Since these are precisely all the Young tableau of shape μ for $g \in S_n$, it suffices to prove $a_\lambda b_\mu = 0$ when $\lambda > \mu$. By Lemma 0.0.19, there are two integers in the same row of T_λ and the same column of T_μ. Let t be the transposition flipping these two integers. Then t is a row permutation of T_λ and a column permutation of T_μ. Hence $a_\lambda t = a_\lambda, \quad tb_\mu = -b_\mu$. Hence

$$a_\lambda b_\mu = a_\lambda t^2 b_\mu = -a_\lambda b_\mu,$$

and so $a_\lambda b_\mu = 0$. That $c_\lambda c_\mu = 0$ follows immediately.
3. Again, we may assume $x = g \in S_n$. By Lemma 0.0.18, it is enough to prove, for $\sigma \in P$ and $\tau \in Q$,

$$\sigma(c_\lambda g c_\lambda)\text{sign}(\tau)\tau = c_\lambda g c_\lambda.$$

We have,

$$\sigma(c_\lambda g c_\lambda)\text{sign}(\tau)\tau = (\sigma c_\lambda \text{sign}(\tau)\tau)\tau^{-1} g c_\lambda \tau$$
$$= c_\lambda \tau^{-1} g \text{sign}(\tau) c_\lambda = \text{sign}(\tau^{-1}) c_\lambda g \text{sign}(\tau) c_\lambda = c_\lambda g c_\lambda.$$

□

Exercise : Show that if $\lambda \neq \mu$ then $c_\lambda \mathbb{C}[S_n] c_\mu = 0$. In particular $c_\lambda c_\mu = 0$ if $\lambda \neq \mu$.

Lemma 0.0.22. **1.** *For every patition λ of n, $V_\lambda = \mathbb{C}[S_n].c_\lambda$ is an irreducible representation of S_n.* **2.** *If $\lambda \neq \mu$ then V_λ and V_μ are not isomorphic.*

Proof. **1.** We have, by **2** of the above lemma,

$$c_\lambda V_\lambda = c_\lambda \mathbb{C}[S_n].c_\lambda \subset \mathbb{C}c_\lambda.$$

Let $W \subset V_\lambda$ be a subrepresentation, then $c_\lambda W \subset \mathbb{C}c_\lambda$. Hence $W = 0$ or $W = \mathbb{C}c_\lambda$. If $W = \mathbb{C}c_\lambda$ then

$$V_\lambda = \mathbb{C}[S_n]c_\lambda = \mathbb{C}[S_n].\mathbb{C}c_\lambda \subset \mathbb{C}[S_n]W = W.$$

If $c_\lambda W = 0$ then $W.W \subset \mathbb{C}[S_n]c_\lambda.W = 0$. Hence W is a **nilpotent ideal** in the **semisimple ring** $\mathbb{C}[S_n]$. Hence $W = 0$. Note that, we have shown in particular that $c_\lambda V_\lambda \neq 0$, hence $n_\lambda \neq 0$ in the previous lemma.
2. We may assume $\lambda > \mu$. Then, by **2** of the previous lemma, $c_\lambda V_\lambda = c_\lambda \mathbb{C}[S_n]c_\lambda = \mathbb{C}c_\lambda \neq 0$. But $c_\lambda V_\mu = c_\lambda \mathbb{C}[S_n]c_\mu = 0$, hence $V_\lambda \not\simeq V_\mu$. □

Lemma 0.0.23. *For any partition λ of n, $c_\lambda^2 = n_\lambda c_\lambda$ with $n_\lambda = n!/\dim(V_\lambda)$.*

Proof. For this, let ϕ be the **right multiplication** by c_λ on V_λ. Then ϕ acts as scalar multiplication by n_λ on V_λ and as zero on $\ker(\phi)$. Since $V_\lambda = \ker(\phi) \oplus n_{_\lambda}$, we see that $\text{trace}(\phi) = n_\lambda \dim(V_\lambda)$. On the other hand, to compute the trace of ϕ on $V_\lambda = \mathbb{C}[S_n]c_\lambda$, we must compute the coefficient of $g \in S_n$ in $g.c_\lambda$. But

$$g.c_\lambda = g. \sum_{\sigma \in P, \tau \in Q} \text{sign}(\tau)\sigma\tau = g + \cdots ,$$

where the other terms are elements of S_n not equal to g. Hence $\text{trace}(\phi) = n!$. Thus $n_\lambda \dim(V_\lambda) = n!$ and we are done. □

Let us compute some examples to get a feel of what is going on. We must remark here that if we take two Young tableau of the **same shape** then the corresponding irreducible representations are equivalent. One must also note that the theorem provides a natural bijection between conjugacy classes in S_n and irreducible representations of S_n. Namely, given a conjugacy class, we have a unique partition λ of n. Take any Young tableau of shape λ and construct the Young symmetrizer c_λ to get hold of the irreducible representation V_λ. That this is a bijection follows from the basic fact that the number of irreducibles coincides with the number of conjugacy classes, which is again the same as the number of Young diagrams.

Examples : First let us look at the two extreme cases. Let $\lambda = (1, \cdots, 1)$. Then $c_\lambda = \sum_{\sigma \in S_n} \text{sign}(\sigma)\sigma$ and

$$V_\lambda = \mathbb{C}[S_n]c_\lambda = \mathbb{C}[G]. \sum_{\sigma \in S_n} \text{sign}(\sigma)\sigma = \mathbb{C}. \sum_{\sigma \in S_n} \text{sign}(\sigma)\sigma.$$

Since on this **one dimensional** vector space, any element of S_n acts via its signature, this is the **alternating representation** U' of S_n, i.e. the representation of S_n on the one dimensional vector space $\mathbb{C}$ given by multiplication by the signature. Now let us consider $\lambda = (n)$. Then $c_\lambda = a_\lambda = \sum_{\sigma \in S_n} \sigma$ and

$$V_\lambda = \mathbb{C}[S_n]. \sum_{\sigma \in S_n} \sigma = \mathbb{C}. \sum_{\sigma \in S_n} \sigma.$$

Since every vector of this representation is **fixed** by elements of S_n, it follows that this is the **trivial representation** U of S_n. Next, let us take the partition $\lambda = (2, 1)$ of 3, the corresponding Young diagram is as shown in figure 3. We have $c_\lambda = 1 + (12) - (13) - (132)$, hence

$$V_\lambda = \mathbb{C}[S_3].(1 + (12) - (13) - (132)).$$

It is easy to see that $c_{(2,1)}$ and $(13)c_{(2,1)}$ span $V_{(2,1)}$. For example,

$$(12)c_{(2,1)} = (12)(1 + (12) - (13) - (132)) = (12) + 1 - (132) - (13) = c_{(2,1)},$$

$$(13)c_{(2,1)} = (13)(1 + (12) - (13) - (132)) = (13) + (123) - 1 - (23),$$

$$(23)c_{(2,1)} = (23)(1+(12)-(13)-(132)) = (23)+(132)-(123)-(12) = -c_{(2,1)}-(13)c_{(2,1)}.$$

Since the **standard representation** V is the only two dimensional representation of S_3, it follows that $V_{(2,1)}$ is the standard representation V of S_3. The computations for S_4 and for higher S_n become more involved, but can be worked out.

Exercise : Show that $\{v_2, \cdots, v_n\}$ is a basis for $V_{(n-1,1)} = \mathbb{C}[S_n].c_{(n-1,1)}$, where

$$v_j = \sum_{\sigma(n)=j} e_\sigma - \sum_{\tau(1)=j} e_\tau, \ 1 \leq j \leq n.$$

(Note that $v_n = c_{(n-1,1)}$, $v_1 + \cdots + v_n = 0$ and $\sigma(v_j) = v_i$ if $\sigma(j) = i$. Observe that the standard representation V has a basis $\{v'_2, \cdots, v'_n\}$ where $v'_j = e_j - e_{j-1}$, e_i now denotes the standard basis of $\mathbb{C}^n$). Show that $V_{(n-1,1)}$ is isomorphic to the standard representation of S_n.

Dimension of V_λ : For counting the dimension of V_λ, we introduce the notion of a **standard Young tableau**. From a given Young diagram, we get a standard Young tableau by filling up the boxes with numbers as before, but such that the numbers in all rows increase from left to right and the entries in all columns increase from top to bottom. For example, the tableau in figure 3 is standard while the tableau above is **not** standard.

1	3	2
4		
5		

Figure 4:

Young's rule for the dimension of V_λ : The dimension of V_λ is the number of standard Young tableau of shape λ. We will prove this after we have proved the Frobenius character formula.

As a simple example that we have already computed, consider the Young diagram corresponding to trhe partition $\lambda = (2, 1)$. Then $\dim(V_{(2,1)}) = 2$, as we have shown above. Clearly we have exactly two standard tableau corresponding to $(2, 1)$, shown below.

1	2
3	

1	3
2	

Figure 5:

There is yet another way to compute the dimension of V_λ, called the **Hook length formula**. The **hook length** of a box B in a Young diagram is the number of boxes directly below and directly to the right of the box, including the box B itself once. In the picture below, we have a Young diagram with each box labeled by its hook length.

8	7	5	3	2
7	6	4	2	1
4	3	1		
2	1			

Figure 6:

Hook Length Formula :

$$\dim(V_\lambda) = \frac{n!}{\prod(\text{Hook lengths})}$$

For example, in the standard tableau of figure 3, by hook length formula, we have $\dim(V_{(2,1)}) = \frac{3!}{1.3.1} = \frac{6}{3} = 2$.

Exercise : (Determinantal Formula) Let $\lambda = (\lambda_1, \cdots, \lambda_r)$ be a partition of n. Prove the following formula

$$\dim(V_\lambda) = n!\det\left(\frac{1}{(\lambda_i - i + j)!}\right),\ 1 \leq i, j \leq r.$$

Frobenius' character and dimension formula : We will now describe a formula for the character and dimension of the irreducible representation V_λ, corresponding to a partition $\lambda = (\lambda_1, \cdots, \lambda_r)$ of n. Let $\mathcal{C}_\mu$ denote the conjugacy class in S_n, corresponding to the partition $\mu = (1^{\epsilon_1}, \cdots, n^{\epsilon_n}) = (\mu_1, \cdots, \mu_s)$, i.e. $\sum i\epsilon_i = n = \sum \mu_i,\ \mu_1 \geq \mu_2 \geq \cdots \geq \mu_s$. Let $x_1, \cdots, x_r$ be algebraically independent variables, here r is the number of rows in the Young diagram associated to λ. Let

$$\Delta(x) = \prod_{i<j}(x_i - x_j) \ \text{ and } \ P_j(x) = x_1^j + x_2^j + \cdots + x_r^j, \quad 1 \leq j \leq n.$$

For a **formal power series** $f(x_1, \cdots, x_r)$ and a tuple $(l_1, \cdots, l_r)$ of non-negative integers, let
$$[f(x)]_{(l_1,\cdots,l_r)} = \text{ coefficient of } x_1^{l_1}\cdots x_r^{l_r} \text{ in } f(x).$$
For the partition λ of n as fixed above, define

$$l_1 = \lambda_1 + r - 1, \quad l_2 = \lambda_2 + r - 2, \cdots, l_r = \lambda_r.$$

Then clearly $l_1 > l_2 > \cdots > l_r$. Let χ_λ denote the character of the irreducible representation V_λ studied above.

Theorem 0.0.24. (Frobenius Character Formula)

$$\chi_\lambda(\mathcal{C}_\mu) = [\Delta(x). \prod_j P_j(x)^{\epsilon_j}]_{(l_1,\cdots,l_r)}.$$

Proof. (sketch) We will use the computation of the character ζ_λ for the permutation representation corresponding to a Young subgroup of shape λ (Theorem 0.0.13). We have
$$\zeta_\lambda(\mathcal{C}_\mu) = \sum \prod_{j=1}^{n} \frac{\epsilon_j!}{r_{j1}!\cdots r_{jn}!},$$
where $\mu = (1^{\epsilon_1}, \cdots, n^{\epsilon_n})$ and the sum is over all collections $\{r_{ij} : \ 1 \le i \le r, \ 1 \le j \le n\}$ of non-negative integers satisfying

$$\lambda_i = \sum_{j=1}^{n} j r_{ij}, \quad 1 \le i \le r; \quad \epsilon_j = \sum_{i=1}^{r} r_{ij}, \quad 1 \le j \le n.$$

The sum on the right hand side of the above formula can be shown to be the coefficient of $x^\lambda = x_1^{\lambda_1}\cdots x_r^{\lambda_r}$ in the symmetric polynomial

$$P^{(\epsilon)} = (x_1 + \cdots + x_r)^{\epsilon_1}(x_1^2 + \cdots + x_r^2)^{\epsilon_2}\cdots(x_1^n + \cdots + x_r^n)^{\epsilon_n}.$$

We have therefore,

$$\zeta_\lambda(\mathcal{C}_\mu) = [P^{(\epsilon)}]_\lambda = \text{ coefficient of } x^\lambda \text{ in } P^{(\epsilon)}.$$

We need to compare this with

$$\omega_\lambda(\epsilon) = [\Delta.P^{(\epsilon)}]_l, \quad l = (\lambda_1 + r - 1, \lambda_2 + r - 2, \cdots, \lambda_r).$$

To prove the Frobenius character formula, we need to show $\chi_\lambda(\mathcal{C}_\mu) = \omega_\lambda(\epsilon)$. Let P be a homogeneous symmetric polynomial of degree n in r variables and $\lambda = (\lambda_1, \cdots, \lambda_r)$ be a partition of n. Let $[P]_\lambda$ be the coefficient of $x^\lambda = x_1^{\lambda_1}\cdots x_r^{\lambda_r}$ in

P. Let $H_\lambda = H_{\lambda_1}.H_{\lambda_2}\cdots H_{\lambda_r}$, where H_j is the jth complete symmetric polynomial (sum of all distinct monomials of degree j). Let $K_{\mu\lambda} = [\Delta(\mu).H_\lambda]_{(\mu_1+s-1,\mu_2+s-2,\cdots,\mu_l)}$ with $\Delta(\mu) = \prod_{i<j}(x_i - x_j)$ is the discriminant corresponding to $\mu = (\mu_1, \cdots, \mu_s)$. We need

Lemma 0.0.25.

$$[P]_\lambda = \sum_\mu K_{\mu\lambda}[\Delta(\mu).P]_{(\mu_1+s-1,\mu_2+s-2,\cdots,\mu_s)}.$$

Applying this to the polynomial $P = P^{(\epsilon)}$, we have

$$\zeta_\lambda(\mathcal{C}_\mu) = \sum_\nu K_{\nu\lambda}\omega_\nu(\epsilon) = \omega_\lambda(\epsilon) + \sum_{\nu>\lambda} K_{\nu\lambda}\omega_\nu(\epsilon) \qquad (*)$$

Claim : Let χ_λ be the character of V_λ. Then, for any conjugacy class $\mathcal{C}_\mu$ of S_n,

$$\chi_\lambda(\mathcal{C}_\mu) = \omega_\lambda(\epsilon),$$

here $\mu = (1^{\epsilon_1}, \cdots, n^{\epsilon_n})$.

Exercise :1. Let $A = \mathbb{C}[S_n]$, so that $V_\lambda = A.c_\lambda = A.a_\lambda b_\lambda$. Show that $V_\lambda \simeq A.b_\lambda a_\lambda$.
2. Prove that V_λ is the image of the map $Aa_\lambda \longrightarrow Ab_\lambda$ given by right multiplication by b_λ and this is isomorphic to the image of the map $Ab_\lambda \longrightarrow Aa_\lambda$ given by right multiplication by a_λ.
3. Let U_λ be the permutation representation for the Young subgroup of shape λ. Show that $U_\lambda \simeq A.a_\lambda$.
4. Show that $U'_\lambda = A.b_\lambda$ is the representation of S_n **induced** from the (restriction of the) sign representation on the subgroup $Q_\lambda \simeq S_{\mu_1} \times \cdots \times S_{\mu_s}$.
***5.** From the above exercises, note that both U_λ and U'_λ contain a copy of V_λ. Show that V_λ has multiplicity 1 in both U_λ and U'_λ and that V_λ is the unique common irreducible subrepresentation of U_λ and U'_λ having multiplicity 1 (Hint: Use Frobenius reciprocity etc).

Using the exercises, we have a surjection of $\mathbb{C}[S_n]$-modules $U_\lambda \longrightarrow V_\lambda, \quad x \mapsto x.b_\lambda$. Semisimplicity (i.e. complete reducibility) of U_λ tells us that V_λ is a summand of U_λ, i.e. appears in U_λ.

Exercise : Show that $U_{(d-1,1)} \simeq V_{(d-1,1)} \oplus V_{(d)}$. More generally, show that $U_{(n-a,a)} = \bigoplus_{i=0}^a V_{(n-i,i)}$.

Writing the permutation character ζ_λ (the character of U_λ), in terms of irreducible characters of S_n, and using the fact that V_λ appears in U_λ we have,

$$\zeta_\lambda = \sum_\mu n_{\lambda\mu}\chi_\mu, \quad n_{\lambda\lambda} \geq 1 \text{ and all } n_{\lambda\mu} \geq 0 \qquad (**).$$

This, together with (∗) above, we see that each ω_λ is a **virtual character**.

$$\omega_\lambda = \sum m_{\lambda\mu}\chi_\mu, \quad m_{\lambda\mu} \in \mathbb{Z}.$$

One can show that the ω_λ are orthonormal. Hence,

$$1 =< \omega_\lambda, \omega_\lambda >= \sum_\mu m_{\lambda\mu}^2.$$

Therefore $\omega_\lambda = \pm\chi$ for some irreducible character χ of S_n. Fix λ and assume inductively that $\chi_\mu = \omega_\mu$ for all $\mu > \lambda$. Hence by (∗),

$$\zeta_\lambda = \omega_\lambda + \sum_{\mu>\lambda} K_{\mu\lambda}\chi_\mu.$$

Hence, by (∗∗), we have

$$\omega_\lambda + \sum_{\mu>\lambda} K_{\mu\lambda}\chi_\mu = \sum_\mu n_{\lambda\mu}\chi_\mu, \; n_{\lambda\lambda} \geq 1, \; n_{\lambda\mu} \geq 0.$$

Therefore, by the linear independence of irreducible characters, it follows that $\omega_\lambda = \chi_\lambda$. □

Corollary 0.0.26. (Young's rule) *The integer $K_{\mu\lambda}$ is the multiplicity of the irreducible representation V_μ in the permutation representation U_λ,*

$$U_\lambda \simeq V_\lambda \oplus \bigoplus_{\mu>\lambda} K_{\mu\lambda}V_\mu, \quad \zeta_\lambda = \chi_\lambda + \sum_{\mu>\lambda} K_{\mu\lambda}\chi_\mu.$$

Proof. We have $\zeta_\lambda = \omega_\lambda + \sum_{\mu>\lambda} K_{\mu\lambda}\chi_\mu$ and by the proof above, $\omega_\lambda = \chi_\lambda$. Hence the assertion follows. □

Proof of Young's rule for $\dim(V_\lambda)$ **:** We note that for the partition $\lambda = (1, 1, \cdots, 1)$ of n, the representation U_λ is simply the regular representation (explain!). Hence $K_{\mu(1,\cdots,1)} = \dim(V_\mu)$. Now, $K_{\mu\lambda}$ can be shown to be equal to the number of ways one can fill the boxes of the Young diagram of shape μ with λ_1 1's, λ_2 2's upto λ_r r's, such that the entries in each row are **nondecreasing** and in each column are **strictly increasing**. Then $K_{\lambda\lambda} = 1$ and $K_{\mu\lambda} = 0$ if $\mu < \lambda$. This shows that $\dim(V_\lambda) = K_{\lambda(1,\cdots,1)}$ is the number of ways we can fill the Young diagram of $\lambda = (\lambda_1, \cdots, \lambda_r)$ with 1 1's, 1 2's and so on upto 1 n's so that the numbers are strictly increasing in all rows and all columns, i.e. the number of standard Young tableau of shape λ.

Frobenius dimension formula : The dimension of V_λ is precisely the character χ_λ evaluated at the conjugacy class of 1. The partition corresponding to this conjugacy class is $\mu = (n)$. Hence, by the above formula,

$$\dim(V_\lambda) = \chi_\lambda(\mathcal{C}_{(n)}) = [\Delta(x).(x_1 + \cdots + x_r)^n]_{(l_1,\cdots,l_r)}.$$

We observe that $\Delta(x) = \sum_{\sigma \in S_r} \text{sign}(\sigma) x_r^{\sigma(1)-1} \cdots x_1^{\sigma(r)-1}$ and

$$(x_1 + \cdots + x_r)^n = \sum \frac{n!}{t_1! \cdots t_r!} x_1^{t_1} x_2^{t_2} \cdots x_r^{t_r},$$

the sum being over r-tuples $(t_1, \cdots, t_r)$ with $\sum t_i = n$. To compute the coefficient of $x_1^{l_1} \cdots x_r^{l_r}$ in the product $\Delta(x).(x_1 + \cdots + x_r)^n$, we pair the corresponding terms in the above sums and get

$$\sum \text{sign}(\sigma) \frac{n!}{(l_1 - \sigma(r) + 1)! \cdots (l_r - \sigma(1) + 1)!},$$

the sum being over $\sigma \in S_r$ such that $l_{r-i+1} - \sigma(i) + 1 \geq 0, \quad 1 \leq i \leq r$. This sum is rewritten as

$$\frac{n!}{l_1! \cdots l_r!} \sum_{\sigma \in S_r} \text{sign}(\sigma) \prod_{j=1}^{r} l_j(l_j - 1) \cdots (l_j - \sigma(k-j+1) + 2) = \frac{n!}{l_1! \cdots l_r!} \prod_{i<j} (l_i - l_j).$$

Hence, we have

Theorem 0.0.27. (Frobenius Dimension formula)

$$\dim(V_\lambda) = \frac{n!}{l_1! \cdots l_r!} \prod_{i<j} (l_i - l_j), \quad \text{where } l_i = \lambda_i + r - i.$$

Examples :1. Consider $n = 3$, $\lambda = (2, 1)$ with $\lambda_1 = 2, \lambda_2 = 1$ (so $r = 2$) and $\mu = (3)$, so that $\mathcal{C}_{(3)}$ is the conjugacy class of (123). In this situation, we write μ as $\mu = (1^0, 2^0, 3^1)$. So that $\epsilon_1 = \epsilon_2 = 0$, $\epsilon_3 = 1$. We have

$$P_1(x) = x_1 + x_2,\ P_2 = x_1^2 + x_2^2,\ P_3(x) = x_1^3 + x_2^3,\ \Delta(x) = (x_1 - x_2).$$

Also

$$l_1 = \lambda_1 + r - 1 = 2 + 2 - 1 = 3,\ l_2 = \lambda_2 + r - 2 = 1 + 2 - 2 = 1.$$

Hence

$$\chi_{(2,1)}(\mathcal{C}_{(3)}) = [(x_1 - x_2)(x_1 + x_2)^0 (x_1^2 + x_2^2)^0 (x_1^3 + x_2^3)]_{(3,1)} = -1.$$

Now, let $\mu = (2,1) = (1^1, 2^1, 3^0)$, so that $\epsilon_1 = 1,\ \epsilon_2 = 1,\ \epsilon_3 = 0$. Hence

$$\chi_{(2,1)}(\mathcal{C}_{(2,1)}) = [(x_1 - x_2)(x_1 + x_2)(x_1^2 + x_2^2)]_{(l_1,l_2)} = [x_1^4 - x_2^4]_{(3,1)} = 0.$$

Finally, let $\mu = (1,1,1) = (1^3, 2^0, 3^0)$, so that $\epsilon_1 = 3,\ \epsilon_2 = \epsilon_3 = 0$. Hence

$$\dim(V_{(2,1)} = \chi_{(2,1)}(\mathcal{C}_{(1,1,1)}) = [(x_1 - x_2)(x_1 + x_2)^3]_{(3,1)} = 2.$$

Which is what we had calculated for the standard representation of S_3.
2. Let $n = 4$ and $\lambda = (2,1,1)$, so $\lambda_1 = 2,\ \lambda_2 = 1,\ \lambda_3 = 1$ and $r = 3$. Let $\mu = (2,2) = (1^0, 2^2, 3^0, 4^0)$, so that $\epsilon_1 = \epsilon_3 = \epsilon_4 = 0$ and $\epsilon_2 = 2$. Also $l_1 = 2 + 3 - 1 = 4,\ l_2 = 1 + 3 - 2 = 2,\ l_3 = 1 = \lambda_3$. Hence

$$\chi_{(2,1,1)}(\mathcal{C}_{(2,2)}) = [(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_1^2 + x_2^2 + x_3^2)^2]_{(4,2,1)} = -1.$$

Now, let $\mu = (4) = (1^0, 2^0, 3^0, 4^1)$ so that $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$ and $\epsilon_4 = 1$. Hence

$$\chi_{(2,1,1)}(\mathcal{C}_{(4)}) = [(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_1^4 + x_2^4 + x_3^4)]_{(4,2,1)} = 1.$$

Next, let $\mu = (3,1) = (1^1, 2^0, 3^1, 4^0)$, so $\epsilon_1 = 1,\ \epsilon_2 = \epsilon_4 = 0,\ \epsilon_3 = 1$. Therefore,

$$\chi_{(2,1,1)}(\mathcal{C}_{(3,1)}) = [(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_1 + x_2 + x_3)(x_1^3 + x_2^3 + x_3^3)]_{(4,2,1)} = 0.$$

Let $\mu = (2,1,1) = (1^2, 2^1, 3^0, 4^0)$, so that $\epsilon_1 = 2,\ \epsilon_2 = 1,\ \epsilon_3 = \epsilon_4 = 0$. Hence,

$$\chi_{(2,1,1)}(\mathcal{C}_{(2,1,1)}) = [(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_1 + x_2 + x_3)^2(x_1^2 + x_2^2 + x_3^2)]_{(4,2,1)} = -1.$$

Finally, let $\mu = (1,1,1,1) = (1^4, 2^0, 3^0, 4^0)$, so that $\epsilon_1 = 4,\ \epsilon_2 = \epsilon_3 = \epsilon_4 = 0$. Hence

$$\chi_{(2,1,1)}(\mathcal{C}_{(1,1,1,1)}) = [(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_1 + x_2 + x_3)^4]_{(4,2,1)} = 3.$$

We see that these match with the character values of the 3-dimensional representation $V \otimes U'$, where V is the standard representation and U' the sign representation of S_4 (verify this).

Proof of Hook length dimension formula : Let $\lambda = (\lambda_1, \cdots, \lambda_r)$ and $l_1 = \lambda_1 + r - 1,\ l_2 = \lambda_2 + r - 2, \cdots, l_r = \lambda_r$. Then the hook lengths of the boxes in the first column of the Young diagram of shape λ are precisely $l_1, \cdots, l_r$. We will prove the hook length dimension formula

$$\dim(V_\lambda) = \frac{n!}{\prod(\text{Hook lengths})}$$

by induction on the number of columns in the Young diagram, assuming the Frobenius dimension formula. By comparision of the two formulae, we have to show

$$\prod(\text{Hook lengths for } \lambda) = \frac{l_1! \cdots l_r!}{\prod_{i<j}(l_i - l_j)}.$$

If there is only one column, we have $\lambda = (1,,1,\cdots,1)$ and $V_{(1,1,\cdots,1)} \simeq U'$, the sign (alternating) representation of S_n. We have, in this case, $r = n$ and $l_1 = 1+n-1 = n,\ l_2 = 1+n-2 = n-1,\ \cdots, l_n = 1$. Hence

$$\frac{n!}{\prod(\text{Hook lengths})} = \frac{n!}{n!} = 1 = \dim(U').$$

Omitting the first column gives a Young diagram for $\lambda' = (\lambda_1 - 1, \cdots, \lambda_r - 1)$. In this case $l_1' = l_1 - 1,\ l_2' = l_2 - 1, \cdots, l_r' = l_r - 1$. By induction, we have

$$\prod(\text{Hook lengths for } \lambda') = \frac{l_1'! \cdots l_r'!}{\prod_{i<j}(l_i' - l_j')} = \frac{(l_1-1)! \cdots (l_r-1)!}{\prod_{i<j}(l_i - l_j)}.$$

Hence

$$\prod(\text{hook lengths for } \lambda) = [\prod(\text{Hook lengths for } \lambda')](l_1.l_2 \cdots l_r)$$

$$= \frac{(l_1-1)! \cdots (l_r-1)!}{\prod_{i<j}(l_i - l_j)}(l_1.l_2 \cdots l_r) = \frac{l_1! \cdots l_r!}{\prod_{i<j}(l_i - l_j)}.$$

Hence we are done.

Tensor product of Frobenius-Young modules : We will now decompose the tensor product $V_\lambda \otimes_{\mathbb{C}} V_\mu$. We have

Proposition 0.0.28. *With the notation fixed in the proof of the Frobenius character formula above, we have*

$$V_\lambda \otimes_{\mathbb{C}} V_\mu \simeq \sum_\nu V_\nu^{C_{\lambda\mu\nu}},$$

where

$$C_{\lambda\mu\nu} = \sum_i \frac{1}{z(\mathbf{i})} \omega_\lambda(\mathbf{i}) \omega_\mu(\mathbf{i}) \omega_\nu(\mathbf{i}),$$

the sum being over all $\mathbf{i} = (i_1, \cdots, i_n)$ *with* $\sum j i_j = n$ *and* $\omega_\lambda(\mathbf{i}) = \chi_\lambda(\mathcal{C}_{\mathbf{i}})$ *and* $z(\mathbf{i}) = i_1! 1^{i_1}.i_2! 2^{i_2} \cdots i_n! n^{i_n}$.

Proof. Let $\xi_{\mathbf{i}}$ denote the characteristic function of the conjugacy class $\mathcal{C}_{\mathbf{i}}$ in S_n corresponding to the partition $\mathbf{i} = (1^{i_1}, \cdots, n^{i_n})$. Then

$$\chi_\lambda = \sum_{\mathbf{i}} \chi(\mathcal{C}_{\mathbf{i}})\xi_{\mathbf{i}} = \sum_{\mathbf{i}} \omega_\lambda(\mathbf{i})\xi_{\mathbf{i}}$$

and $\xi_{\mathbf{i}} = (1/z(\mathbf{i})) \sum_\nu \omega_\nu(\mathbf{i})\chi_\nu$ (prove this). Hence

$$\chi_\lambda\chi_\mu = \sum_{\mathbf{i}} \omega_\lambda(\mathbf{i})\omega_\mu(\mathbf{i})\xi_{\mathbf{i}},$$

the result now follows by substituting the expression for $\xi_{\mathbf{i}}$ and using the fact that character determines a representation.

□

Representations of $\mathrm{GL}(V)$, Schur-Weyl duality

We now begin our study of representations of $\mathrm{GL}(V) \times S_n$ (note that here $\dim(V)$ is independent of n) in light of what we have learnt for the symmetric group. Let us go back to the actions of $\mathrm{GL}(V)$ and S_n on $V^{\otimes^n}$ and use the Young symmetrizers to see what we get. Let us denote the image of c_λ on $V^{\otimes^n}$ by $\mathbb{S}_\lambda(V)$, i.e. $\mathbb{S}_\lambda(V) = c_\lambda(V^{\otimes^n})$. The association $V \mapsto \mathbb{S}_\lambda(V)$ is functorial.

Exercise : Show explicitly that given a linear map $f : V \longrightarrow W$, there is a natural linear map $\mathbb{S}_\lambda(f) : \mathbb{S}_\lambda(V) \longrightarrow \mathbb{S}_\lambda(W)$ such that $\mathbb{S}_\lambda(f \circ g) = \mathbb{S}(f) \circ \mathbb{S}_\lambda(g)$. In particular, we have a representation of $\mathrm{GL}(V)$ into $\mathcal{S}_\lambda(V)$.

The functor $V \mapsto \mathcal{S}_\lambda(V)$ is called the **Schur functor** corresponding to λ and the representation $\mathbb{S}_\lambda(V)$ of $\mathrm{GL}(V)$ is called **Weyl's construction** or the **Weyl module** corresponding to λ. We will see a sort of correspondence between representations of symmetric groups and representations of the general linear groups, and under this correspondence, they occur in pairs in the decomposition of $V^{\otimes^n}$ as a $\mathrm{GL}(V) \times S_n$-representation.

Example : Let $\lambda = (n)$. Then we have seen that $\mathbb{S}_\lambda(V) = \mathrm{Sym}^n(V)$. If $\lambda = (1, \cdots, 1)$ we have $\mathbb{S}_\lambda(V) = \bigwedge^n(V)$ and the Schur functors in these examples are $V \mapsto \mathrm{Sym}^n(V)$ and $V \mapsto \bigwedge^n(V)$ respectively. Let us look at the partition $(2, 1)$ of 3 again. We have computed the symmetrizer $c_\lambda = 1 + (12) - (13) - (132)$ and seen that the image of c_λ in $V^{\otimes^3}$ is the subspace spanned by all vectors of the form

$$x_1 \otimes x_2 \otimes x_3 + x_2 \otimes x_1 \otimes x_3 - x_3 \otimes x_2 \otimes x_1 - x_3 \otimes x_1 \otimes x_2.$$

We have an embedding of $\bigwedge^2(V) \otimes V \longrightarrow V^{\otimes^3}$ given by

$$(x_1 \wedge x_3) \otimes x_2 \mapsto x_1 \otimes x_2 \otimes x_3 - x_3 \otimes x_2 \otimes x_1.$$

We identify $\bigwedge^2(V) \otimes V$ with its image under this embedding in $V^{\otimes^3}$. Then the image of c_λ is the subspace of $\bigwedge^2(V) \otimes V$ spanned by vectors of the form

$$(x_1 \wedge x_3) \otimes x_2 + (x_2 \wedge x_3) \otimes x_1.$$

Clearly these vectors are in the kernel of the **canonical map** $\bigwedge^2(V) \otimes V \longrightarrow \bigwedge^3(V)$.

Exercise : Show that the kernel is spanned by these vectors, i.e.,

$$\mathbb{S}_{(2,1)}(V) = \mathrm{Ker}(\bigwedge^2(V) \otimes V \longrightarrow \bigwedge^3(V)).$$

Exercise : Show that

$$V^{\otimes^3} \simeq \mathrm{Sym}^3(V) \oplus \bigwedge^3(V) \oplus (\mathcal{S}_{(2,1)}(V))^{\oplus^2}.$$

Now, $g \in \mathrm{GL}(V)$ defines an endomorphism $\mathbb{S}_\lambda : \mathbb{S}_\lambda(V) \longrightarrow \mathbb{S}_\lambda(V)$. We denote the trace of this endomorphism by $\chi_{\mathbb{S}_\lambda(V)}(g)$.

Let $x_1, \cdots, x_k$ be the **eigenvalues** of g on V, where $k = \dim(V)$. Let $\lambda = (n)$. Then $\mathbb{S}_{(n)}(V) = \mathrm{Sym}^n(V)$.

Exercise : Prove that $\chi_{\mathbb{S}_{(n)}(V)}(g) = H_n(x_1, \cdots, x_k)$, where the quantity on the right denotes the **complete symmetric polynomial** of degree n, i.e., the sum of all monomials of degree n in the variables x_i. (Hint: Use the Jordan canonical form).

When $\lambda = (1, \cdots, 1)$, we have $\mathbb{S}_{(1,\cdots,1)}(V) = \bigwedge^n(V)$ and $\chi_{\mathbb{S}_{(1,\cdots,1)}}(g) = E_n(x_1, \cdots, x_k)$, where the quantity on the right denotes the nth **elementry symmetric polynomial** in the variables x_i, $E_n(x_1, \cdots, x_k) = \sum_{1 \leq i_1 < \cdots < i_n \leq k} x_{i_1} \cdots x_{i_n}$. More generally, we have (this will be proved below)

$$\chi_{\mathbb{S}_\lambda(V)}(g) = S_\lambda(x_1, \cdots, x_k),$$

where $S_\lambda(x_1, \cdots, x_k)$ is the **Schur polynomial** given by

$$S_\lambda(x_1, \cdots, x_k) = \frac{\det(x_j^{\lambda_i + k - i})_{1 \leq i,j \leq k}}{\det(x_j^{k-i})_{1 \leq i,j \leq k}} = \frac{\det(x_j^{\lambda_i + k - i})}{\Delta},$$

where $\Delta = \prod_{i<j}(x_i - x_j)$ is the **discriminant**. One can also compute $S_\lambda(x_1, \cdots, x_k)$ by the **Jacobi-Trudy identity :**

$$S_\lambda = \det(H_{\lambda_i + j - i}),$$

where the H_j is the complete symmetric polynomial introduced above. If $\lambda_{p+1} = \cdots = \lambda_k = 0$, the above determinant is the same as the determinant of the upper left $p \times p$ corner. In terms of the elementary symmetric polynomials, we have

$$S_\lambda = \det(E_{\mu_i+j-i}),$$

where $\mu = (\mu_1, \cdots, \mu_l)$ is the conjugate partition of λ. We are interested in decomposing the tensor representation $V^{\otimes^n}$ of $\mathrm{GL}(V)$ and finding out what representations of S_n and $\mathrm{GL}(V)$ occur in the decomposition. Towards this we have,

Theorem 0.0.29. **1.** *Let $k = \dim(V)$. Then $\mathbb{S}_\lambda(V)$ is zero if $\lambda_{k+1} \neq 0$. If $\lambda = (\lambda_1, \cdots, \lambda_k),\ \lambda_1 \geq \cdots \geq \lambda_k \geq 0$ then*

$$\dim(\mathbb{S}_\lambda(V)) = S_\lambda(1, \cdots, 1) = \prod_{1 \leq i,j \leq k} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

Let m_λ be the dimension of the irreducible representation V_λ of S_n, corresponding to the partition λ of n. Then
2. $V^{\otimes^n} \simeq \bigoplus_\lambda \mathbb{S}_\lambda(V)^{\oplus m_\lambda}$.
3. *For any $g \in \mathrm{GL}(V)$, $\chi_{\mathbb{S}_\lambda(V)}(g) = S_\lambda(x_1, \cdots, x_k)$.*
4. *Each $\mathbb{S}_\lambda(V)$ is an irreducible representation of* $\mathrm{GL}(V)$.

Corollary 0.0.30. *Let $c \in \mathbb{C}[S_n]$ and*

$$\mathbb{C}[S_n].c = \bigoplus_\lambda V_\lambda^{\oplus r_\lambda}$$

as representations of S_n, then we have a decomposition of $\mathrm{GL}(V)$*-spaces*

$$V^{\otimes^n}.c = \bigoplus_\lambda (\mathcal{S}_\lambda(V))^{\oplus r_\lambda}.$$

We will now prepare for the proof of the above theorem. We need a few results from the theory of semisimple algebras for this. Let G be a finite group and let U be a **right**-module over $\mathbb{C}[G]$. Let $B = \mathrm{Hom}_{\mathbb{C}[G]}(U, U)$. Then B is a $\mathbb{C}$-algebra and U is a **left**-B module via the usual evaluation action. Let $f \in B,\ \ g \in G$. Then, for any $u \in U$ we have, $f(v)g = f(v.g)$, so that the actions of G and B **commute**. We can decompose U (as a represenrtation of G) into a direct sum of representations, isotypical of irreducible type U_i, i.e.

$$U = \bigoplus_i U_i^{n_i}.$$

By Schur's Lemma, we have

$$B = \mathrm{Hom}_{\mathbb{C}[G]}(U_i^{n_i}, U_i^{n_i}) = \bigoplus_i M_{n_i}(\mathbb{C}).$$

Note that, for any **left**-$\mathbb{C}[G]$ module W, the tensor product $U \otimes_{\mathbb{C}[G]} W$ is a **left** B-module, with B acting on the first factor. We need

Lemma 0.0.31. *Let U be a finite dimensional (as a $\mathbb{C}$-vector space)* **right** $\mathbb{C}[G]$-*module. Then*
1. *For any $c \in \mathbb{C}[G]$, the natural map $U \otimes_{\mathbb{C}[G]} \mathbb{C}[G]c \longrightarrow Uc$ is an isomorphism of* **left** *B-modules.*
2. *Let $W = \mathbb{C}[G]c$ be an* **irreducible** *left $\mathbb{C}[G]$-module. Then $U \otimes_{\mathbb{C}[G]} W = Uc$ is an irreducible* **left** *B-module.*
3. *Let $W_i = \mathbb{C}[G]c_i$ be the distinct irreducible left $\mathbb{C}[G]$-modules,* $\mathrm{dim} W_i = n_i$. *Then*

$$U \simeq \bigoplus_i (U \otimes_{\mathbb{C}[G]} W_i)^{\oplus n_i} \simeq \bigoplus_i (Uc_i)^{\oplus n_i}$$

is the decomposition of U into irreducible left B-modules.

Proof. **1.** Let $A = \mathbb{C}[G]$. Then A is a semisimple $\mathbb{C}$-algebra and hence the (left) ideal Ac is a direct summand of A as a left A-module. We have a surjection $U \otimes_A A \longrightarrow U \otimes_A c$ (induced by $A \longrightarrow Ac,\ \ a \mapsto ac$) and an injection $U \otimes_A Ac \hookrightarrow U \otimes_A A$ (since Ac is a direct summand of A) of left B-modules, fitting in the commutative diagram

$$\begin{array}{lllll}
U \otimes_A A & \xrightarrow{.c} & U \otimes_A Ac & \hookrightarrow & U \otimes_A A \\
\downarrow & & \downarrow & & \downarrow \\
U & \xrightarrow{.c} & U.c & \hookrightarrow & U
\end{array}$$

In the above diagram, the vertical maps are the maps $u \otimes a \mapsto u.a$, coming from the right A-module structure on U. The horizontal maps in the first commutative square are surjective, as observed before. The ones in the second commutative diagram are injective and finally, the vertical maps on the outer edges of the diagram are isomorphisms. Now the commutativity of the diagram forces the middle vertical map to be an isomorphism (why?).

2. Let us first assume U is an irreducible A-module. Then, $B = Hom_A(U, U) = \mathbb{C}$. We will show that $\dim_{\mathbb{C}}(U \otimes_A W) \leq 1$. This would settle the assertion. We have $A \simeq \prod_{i=1}^r M_{n_i}(\mathbb{C})$, for some integers n_i and r. Since W is an irreducible (left) A-module, we can identify W with a simple left ideal of A. Identifying A with the product of matrix algebras, we see that W consists of r-tuples of matrices, all of which are zero except in one factor, and in this factor, we have matrices with all *columns* except one zero. Now, U is an irreducible *right* A-module. Hence, upto isomorphism, we may identify U with the r-tuples of matrices, all of which are zero except in one factor, and in this factor, we have matrices with all *rows* except one are zero. It is now clear that $U \otimes_A W = 0$ if the positions of the nonzero factors in U and W are different, and $U \otimes_A W$ consists of matrices which are zero except one column and one row in that factor, if the positions of the nonzero factor in U and W are same. Hence $\dim_{\mathbb{C}}(U \otimes_A W) \leq 1$, in particular $U \otimes_A W$ is irreducible. Let us now settle the general case. We decompose U into a direct sum of irreducible right A-modules as $U = \bigoplus_i U_i^{m_i}$. Then

$$U \otimes_A W = \bigoplus_i (U_i \otimes_A W)^{m_i}.$$

We may assume $U \otimes_A W \neq 0$. As observed above, W consists of r-tuple of matrices, all of which are zero except in one factor, and in this factor, all matrices have all columns zero except one, in he same fixed position. And each U_i consists of r-tuples of matrices, all of which are zero, except in one factor, and in this factor, all matrices have all rows zero except one, in the same fixed position. We have seen that $U_i \otimes_A W$ is zero unless the factor is the same for U_i and W, and $U_i \otimes_A W = \mathbb{C}$ when the factor is same. Hence $U \otimes_A W = \bigoplus_i (U_i \otimes_A W)^{m_i} = \mathbb{C}^{m_k}$ for some k. Now, $B = Hom_A(U, U) = \prod_i M_{m_i}(\mathbb{C})$. Hence $U \otimes_A W$ is simple as a left B-module.

3. Since $W_i = A.c_i$ are all the distinct irreducible (i.e. simple) left A-modules, we have $A \simeq \bigoplus_i W_i^{\oplus n_i}$ (recall $A = \mathbb{C}[G]$ and W_i are the irreducible representations of G) and hence we have an isomorphism

$$U \simeq U \otimes_A A \simeq U \otimes_A (\bigoplus_i W_i^{\oplus n_i}) \simeq \bigoplus_i (U \otimes_A W_i)^{\oplus n_i} \simeq \bigoplus_i (Uc_i)^{\oplus n_i},$$

and Uc_i are irreducible left B-modules, by **2**. □

Proof of theorem : To prove the theorem stated above, we will apply the above lemma to the right $\mathbb{C}[S_n]$-module $U = V^{\otimes^n}$. By the lemma, we know the decomposition of $V^{\otimes^n}$ as a B-module, where $B = \mathrm{Hom}_{\mathbb{C}[S_n]}(V^{\otimes^n}, V^{\otimes^n})$ is the algebra of all endomorphisms of $V^{\otimes n}$ which commute with all permutations of the factors of $V^{\otimes^n}$. Any endomorphism ϕ of V induces the endomorphism $\phi^{\otimes^n}$ of $V^{\otimes^n}$ and this commutes with the permutations of the factors of $V^{\otimes^n}$, as we have discussed before. We shall identify $\mathrm{End}_{\mathbb{C}}(V)$ as a subalgebra of $\mathrm{End}_{\mathbb{C}}(V^{\otimes^n})$ via $\phi \mapsto \phi^{\otimes^n}$. Hence B contains these induced endomorphisms of $V^{\otimes^n}$. More precisely,

Lemma 0.0.32. *$B = \mathrm{Hom}_{\mathbb{C}[G]}(V^{\otimes^n}, V^{\otimes^n})$ is spanned, as a vector subspace of $\mathrm{End}_{\mathbb{C}}(\mathrm{V}^{\otimes^n})$ by $\mathrm{End}_{\mathbb{C}}(\mathrm{V})$. A subspace of $V^{\otimes^n}$ is a B-submodule if and only if it is invariant under $\mathrm{GL}(V)$.*

Exercise : For a finite dimensional vector space V, show that

$$\mathrm{End}_{\mathbb{C}}(V^{\otimes^n}) = \mathrm{End}_{\mathbb{C}}(\mathrm{V})^{\otimes^n},$$

as representations of S_n.

Exercise : Let W be a finite dimensional vector space. Prove that $\mathrm{Sym}^r(W)$ is the subspace of $W^{\otimes^r}$ spanned by the vectors $w^r = w \otimes \cdots \otimes w$, as w runs through all vectors in W.

Proof. Since the $\mathbb{C}[G]$-linear endomorphisms of $V^{\otimes^n}$ are precisely those endomorphisms of $V^{\otimes^n}$ which commute with all the permutations of the factors, we see that $B = \mathrm{Sym}^n(\mathrm{End}_{\mathbb{C}}(V))$. Hence, by the above exercise, as a subspace of $\mathrm{End}_{\mathbb{C}}(\mathrm{V}^{\otimes^n})$, B is spanned by $\mathrm{End}_{\mathbb{C}}(V)$. For the second assertion, we note that a subspace S of $V^{\otimes^n}$ is a B-submodule if and only if S is invariant under the action of B. But B is spanned by $\mathrm{End}_{\mathbb{C}}(V)$. Hence S is a B-submodule if and only if S is invariant under $\mathrm{End}_{\mathbb{C}}(V)$. Now, $\mathrm{GL}(V)$ is a dense subspace of $\mathrm{End}_{\mathbb{C}}(V)$ in the usual topology and the actions are continuous (why?), hence S is a B-submodule if and only if S is invariant under $\mathrm{GL}(V)$. ◻

Now we are in a position to prove the theorem. Fisrt observe that $\mathcal{S}_\lambda(V) = Uc_\lambda$ for $U = V^{\otimes^n}$. With this, the proofs of the assertions **2** and **4** are immediate from the above two lemmas.

3. From the first lemma (the first assertion) above, we have an isomorphism of $\mathrm{GL}(V)$-modules,

$$\mathcal{S}_\lambda(V) \simeq V^{\otimes^n} \otimes_A V_\lambda,$$

recall that $V_\lambda = A.c_\lambda$. Similarly, for $U_\lambda = A.a_\lambda$ we have

$$a_\lambda(V^{\otimes^n}) = \mathrm{Sym}^{\lambda_1}(V) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_k}(V) \simeq V^{\otimes^n} \otimes_A U_\lambda.$$

By Young's rule, we have an isomorphism of A-modules $U_\lambda \simeq \bigoplus_\mu K_{\mu\lambda} V_\mu$ and hence an isomorphism of $\mathrm{GL}(V)$-modules

$$\mathrm{Sym}^{\lambda_1}(V) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_k}(V) \simeq \bigoplus_\mu K_{\mu\lambda} \mathcal{S}_\mu(V),$$

using the above with the isomorphism $\mathcal{S}_\mu(V) \simeq V^{\otimes^n} \otimes_A V_\mu$. The coefficients $K_{\mu\lambda}$ are called **Kostka numbers** and can be also defined as $K_{\mu\lambda} = [S_\mu]_{(\lambda_1, \cdots, \lambda_r)}$, the coefficient of $x_1^{\lambda_1}\cdots x_r^{\lambda_r}$ in the Schur polynomial S_μ corresponding to the partition μ.

Now, for $g\in \mathrm{GL}(V)$, the trace of g as an endomorphism of the left hand side of the above equation (using an exercise in the text) is $H_\lambda(x_1,\cdots,x_k)$, the product of the complete symmetric polynomials $H_{\lambda_i}(x_1,\cdots,x_k)$. Let us denote by $\mathcal{S}_\lambda(g)$ the endomorphism of $\mathcal{S}_\lambda(V)$ induced by an endomorphism g of V. Therefore, we have,

$$H_\lambda(x_1,\cdots,x_k)=\sum_\mu K_{\mu\lambda}\mathrm{Trace}(\mathcal{S}_\mu(g)).$$

On the other hand, one has

Lemma 0.0.33. *$H_\lambda=\sum_\mu K_{\mu\lambda}S_\mu$, where S_μ are the Schur polynomials defined above.*

Also, $K_{\mu\lambda}$ are all non-negative with $K_{\lambda\lambda}=1$ and $K_{\mu\lambda}=0$ if $\lambda>\mu$. Hence the matrix $(K_{\mu\lambda})$ is a triangular matrix with 1's on the diagonal, in particular, is invertible. Therefore, $\mathrm{Trace}(\mathcal{S}_\lambda(g))=S_\lambda(x_1,\cdots,x_k)$ as asserted.
1. If $\lambda=(\lambda_1,\cdots,\lambda_r)$ with $r>k$ and $\lambda_{k+1}\neq 0$, then, as computed above, the trace of $\mathcal{S}_\lambda(g)$ is $S_\lambda(x_1,\cdots,x_k,0,\cdots,0)$. Using the formula for S_λ in terms of the elementary symmetric polynomials, it follows that this is zero. In this case, evaluating this for $g=1$, we see that $\mathcal{S}_\lambda(V)=0$ (the trace of identity endomorpshism is zero on a complex vector space, then the space must be zero!). We have finally, by **3**,

$$\mathrm{dim}\mathcal{S}_\lambda(V)=S_\lambda(1,\cdots,1)=\prod_{i<j}\frac{\lambda_i-\lambda_j+j-i}{j-i}.$$

This completes the proof of the theorem.

Proof of the corollary : Let $c\in A=\mathbb{C}[S_n]$ and $Ac=\bigoplus_\lambda V_\lambda^{\oplus r_\lambda}$ as representations of S_n. By the lemma above, a subspace of $V^{\otimes n}$ is a left $B=\mathrm{End}_A(V^{\otimes n})$-module if and only if it is a left $\mathrm{GL}(V)$-module. Hence, we have, by the first lemma (first assertion), an isomorphism of left $\mathrm{GL}(V)$-modules

$$V^{\otimes n}.c\simeq V^{\otimes n}\otimes_A Ac\simeq\bigoplus_\lambda(V^{\otimes n}\otimes_A V_\lambda)^{\oplus r_\lambda}\simeq\bigoplus_\lambda(V^{\otimes n}\otimes_A Ac_\lambda)^{\oplus r_\lambda}$$

$$\simeq\bigoplus_\lambda(V^{\otimes n}.c_\lambda)^{\oplus r_\lambda}=\bigoplus_\lambda c_\lambda(V^{\otimes n})^{\oplus r_\lambda}=\bigoplus_\lambda\mathcal{S}_\lambda(V)^{\oplus r_\lambda}.$$

This completes the proof.

Proof of Theorem 0.0.14 : We wish to prove that the permutation characters ζ_λ of S_n, corresponding to the Young subgroups Y_λ, form a basis of the space of characters of S_n. To this, we observe the relation $\zeta_\lambda = \sum K_{\mu\lambda}\chi_\mu$. We have noted in the proof of the theorem above that the matrix $(K_{\mu\lambda})$ is an invertible matrix of coefficients. Hence we can express the χ_λ's in terms of the ζ_λ's. Since the characters ζ_λ's are as many in number as there are partitions of n, it follows that they form a basis for the space of characters of S_n. This finishes the proof of Theorem 0.0.14.

Tensor product of Weyl modules : Let V be a vector space and let λ be a partition of m and μ be a partition of μ. We can then construct the Weyl modules $\mathcal{S}_\lambda(V)$ and $\mathcal{S}_\mu(V)$, corresponding to the symmetric groups S_n and S_m respectively. It is natural to ask how $\mathcal{S}_\lambda \otimes \mathcal{S}_\mu$ decomposes as a representation of $S_n \times S_m$. We have,

Proposition 0.0.34. *$\mathcal{S}_\lambda \otimes \mathcal{S}_\mu \simeq \bigoplus_\nu \mathcal{S}_\nu(V)^{N_{\lambda\mu\nu}}$, where the sum is over all partitions ν of $n+m$ and $N_{\lambda\mu\nu}$ are multiplicities of the corresponding representations.*

Proof. We have

$$\mathcal{S}_\lambda(V) \otimes \mathcal{S}_\mu(V) = V^{\otimes^n} c_\lambda \otimes V^{\otimes^m} c_\mu = V^{\otimes^n} \otimes V^{\otimes^m}.(c_\lambda \otimes c_\mu) = V^{\otimes^{n+m}} c,$$

where

$$c = c_\lambda \otimes c_\mu \in \mathbb{C}[S_n] \otimes \mathbb{C}[S_m] = \mathbb{C}[S_n \times S_m] \subset \mathbb{C}[S_{n+m}].$$

This sets us up to apply the above corollary to this situation. Hence, we have the decomposition of $\mathrm{GL}(V)$-module

$$\mathcal{S}_\lambda(V) \otimes \mathcal{S}_\mu(V) = \bigoplus_\nu \mathcal{S}_\nu(V)^{\oplus^{N_{\lambda\mu\nu}}},$$

corresponding to the decomposition

$$\mathbb{C}[S_n \times S_m].c = \bigoplus_\nu V_\nu^{\oplus^{N_{\lambda\mu\nu}}}$$

of $\mathbb{C}[S_n \times S_m]$-module, where the sum is over all partitions ν of $n+m$. $\square$

It remains to compute the multiplicities $N_{\lambda\mu\nu}$. To determine these, we must know the decomposition of the character of the tensor product of Weyl modules. This is the tensor product of the characters of the factors. By the theorem above, the characters of the factors are given by the Schur polynomials. Hence, to compute the multiplicities $N_{\lambda\mu\nu}$ of the Weyl modules $\mathcal{S}_\nu$, we must write the product $S_\lambda S_\mu$ of Schur polynomials as a linear combination of Schur polynomials,

$$S_\lambda.S_\mu = \sum_\nu N_{\lambda\mu\nu} S_\nu.$$

The Schur-Weyl Duality

We will now take steps towards a decomposition of $V^{\otimes^n}$ as a representation of $\mathrm{GL}(V) \times S_n$. This will spell out exactly which irreducible representations of $\mathrm{GL}(V)$ occur and which irreducibles of S_n occur in the representation on $V^{\otimes^n}$. For this, we need a few more results from the theory of semisimple algebras.

Lemma 0.0.35. *Let R be a $\mathbb{C}$-algebra and M a semisimple R-module. Then the evaluation map*

$$\phi : \bigoplus_S \mathrm{Hom}_R(S, M) \otimes_{\mathbb{C}} S \longrightarrow M, \quad \sum (f \otimes s) \mapsto \sum f(s)$$

is an isomorphism of R-modules and of $\mathrm{End}_R(M)$-modules. Here S runs over a complete set of non-isomorphic simpe R-modules and on $S \otimes_{\mathbb{C}} \mathrm{Hom}_R(S, M)$ we use the action of B on S and of $\mathrm{End}_B(M)$ on $\mathrm{Hom}_B(S, M)$, given by $g.f = g \circ f$ for $g \in \mathrm{End}_R(M), \quad f \in \mathrm{Hom}_{\mathbb{C}}(S, M)$.

Proof. It is a simple checking that ϕ is a R-$\mathrm{End}_R(M)$ module map. To prove ϕ is an isomorphism, We may assume, without loss of generality (why?) that M is simple. The proof now is immediate from Schur's lemma. $\square$

Lemma 0.0.36. *Let R be a $\mathbb{C}$-algebra and M a finite dimensional semisimple R-module. Then there is an isomorphism of R-algebras*

$$\mathrm{End}_R(M) \simeq \prod_S \mathrm{End}_{\mathbb{C}}(\mathrm{Hom}_R(S, M)),$$

where S runs over a complete set of non-isomorphic simple R-modules.

Proof. Note that there is a natural action of the R-algebra $A = \prod_S \mathrm{End}_{\mathbb{C}}(\mathrm{Hom}_R(S, M))$ on the R-module $\bigoplus_S S \otimes_{\mathbb{C}} \mathrm{Hom}_R(S, M)$ and this action commutes with the action of R. Hence there is a homomorphism $A \longrightarrow \mathrm{End}_R(M)$ of R-algebras, which is injective (check this). To prove our assertion, we just have to count dimensions of both these algebras. By the lemma above, M is isomorphic to a direct sum of $\dim_{\mathbb{C}}\mathrm{Hom}_R(S, M)$ many copies of each simple module S. Hence applying Schur's lemma gives,

$$\dim_{\mathbb{C}}\mathrm{End}_R(M) = \sum_S (\dim_{\mathbb{C}}\mathrm{Hom}_R(S, M))^2,$$

which can be seen to be the dimension of A. $\square$

Lemma 0.0.37. *Let R be a semisimple $\mathbb{C}$-algebra and M be a finite dimensional R-module. The natural map*

$$\phi : R \longrightarrow \mathrm{End}_{\mathrm{End}_R(M)}(M), \quad r \mapsto (x \mapsto rx)$$

is surjective.

Proof. Let us compute the kernel of ϕ. We have $r \in \ker(\phi)$ if and only if $rx = 0$ for all $x \in M$. Hence $I = \ker(\phi)$ is the **annihilator** of M. Since R is semisimple, R/I is also semisimple and M is an R/I module with $\mathrm{End}_R(M) = \mathrm{End}_{R/I}(M)$. Hence, by replacing R by R/I, we may assume that M is **faithful** and ϕ is injective. By the lemma above, $\mathrm{End}_R(M) \simeq \prod_S \mathrm{End}_{\mathbb{C}}(\mathrm{Hom}_R(S, M))$. Since M is faithful, $\mathrm{Hom}_R(S, M) \neq 0$ for all S. By Schur's lemma, these must be precisely the simple $\mathrm{End}_R(M)$-modules. By an earlier lemma, M is isomorphic to $\bigoplus_S \mathrm{Hom}_R(S, M) \otimes_{\mathbb{C}} S$, hence M is isomorphic to the direct sum of $\dim(S)$ many copies of the simple module $\mathrm{Hom}_R(S, M)$ for each S. Hence

$$\dim_{\mathbb{C}} \mathrm{End}_{\mathrm{End}_R(M)}(M) = \sum_S (\dim_{\mathbb{C}}(S))^2,$$

which is the dimension of R, since S runs over all non-isomorphic simple R-modules. Hence ϕ is an isomorphism. $\square$

Lemma 0.0.38. *Let R be a $\mathbb{C}$-algebra and $c \in R$ be such that $c^2 = \alpha c$ for some nonzero scalar $\alpha \in \mathbb{C}$. For any R-module M we have an isomorphism*

$$\mathrm{Hom}_R(Rc, M) \simeq cM$$

of $\mathrm{End}_R(M)$-modules.

Proof. First we note that cM is an $\mathrm{End}_R(M)$-submodule of M. This is because, for $f \in \mathrm{End}_R(M)$ and $cx \in cM$, $f(cx) = cf(x) \in cM$. Without loss of generality, we may assume c is an idempotent (why?). Consider the $\mathrm{End}_R(M)$-linear map $\psi : \mathrm{Hom}_R(Rc, M) \longrightarrow cM$, $f \mapsto f(c)$. This has the map $cM \longrightarrow \mathrm{Hom}_R(Rc, M)$, $x \mapsto (r \mapsto rx)$ as the inverse, which is also $\mathrm{End}_R(M)$-linear. We leave the details to the reader. $\square$

The following result is the heart of this course, it connects the representation theory of the symmetric group with that of the general linear group and is absolutely fundamental in understanding representations of the general linear group and the special linear group.

Theorem 0.0.39. (Schur-Weyl duality) *The images of* $\mathbb{C}[S_n]$ *and* $\mathbb{C}[\mathrm{GL}(V)]$ *in* $\mathrm{End}_{\mathbb{C}}(V^{\otimes^n})$ *are centralisers of each other.*

Proof. To prove that the image of $\mathbb{C}[GL(V)]$ in $\mathrm{End}_{\mathbb{C}}(V^{\otimes^n})$ is the centraliser of the image of $\mathbb{C}[S_n]$, recall that $\mathrm{End}_{\mathbb{C}[S_n]}(V^{\otimes^n})$ is spanned by the tensors $f^{\otimes^n}$, $f \in \mathrm{GL}(V)$, after identifying $\mathrm{End}_{\mathbb{C}}(V^{\otimes^n})$ with $\mathrm{End}_{\mathbb{C}}(V)^{\otimes^n}$. Also $\mathrm{End}_{\mathbb{C}[S_n]}(V^{\otimes^n})$ is a semisimple $\mathbb{C}$-algebra (follows from Maschke's theorem). We have shown that (see lemma above) $\mathbb{C}[S_n]$ maps onto $\mathrm{End}_A(V^{\otimes^n})$, where $A = \mathrm{End}_{\mathbb{C}[S_n]}(V^{\otimes^n})$. Now, the image of $\mathrm{GL}(V)$ spans A (see ?). Hence, it follows that $\mathrm{End}_A(V^{\otimes^n}) = \mathrm{End}_{\mathbb{C}[\mathrm{GL}(V)]}(V^{\otimes^n})$. Therefore $\mathbb{C}[S_n]$ maps onto $\mathrm{End}_{\mathbb{C}[\mathrm{GL}(V)]}(V^{\otimes^n})$. Hence the image of $\mathbb{C}[S_n]$ in $\mathrm{End}_{\mathbb{C}}(V^{\otimes^n})$ is the centraliser of the image of $\mathbb{C}[\mathrm{GL}(V)]$. $\square$

Decomposition of tensors : We finally, using the results derived in the course, will analyse the tensor representation $V^{\otimes^n}$ of $\mathrm{GL}(V) \times S_n$ and decompose it into irreducibles for this group. We record a result, which follows from Theorem 0.0.39 and Lemma 0.0.38.

Lemma 0.0.40. *The subalgebra of* $\mathrm{End}_{\mathbb{C}}(V^{\otimes^n})$ *of endomorphisms commuting with the image of* $\mathbb{C}[S_n]$ *is* $\mathrm{End}_{\mathbb{C}[S_n]}(V^{\otimes^n})$*. Furthermore,* $c_\lambda(V^{\otimes^n}) \simeq \mathrm{Hom}_{\mathbb{C}[S_n]}(\mathbb{C}[S_n]c_\lambda, V^{\otimes^n})$*.*

We now prove

Theorem 0.0.41. *There is a canonical decomposition*

$$V^{\otimes^n} \simeq \bigoplus_\lambda \mathcal{S}_\lambda(V) \otimes_{\mathbb{C}} V_\lambda$$

of $\mathrm{GL}(V)$*-*S_n *modules, where the notations are as before.*

Proof. Considering $V^{\otimes^n}$ as a $\mathbb{C}[S_n]$-module, we have, by Lemma 0.0.35,

$$V^{\otimes^n} = \bigoplus_\lambda \mathrm{Hom}_{\mathbb{C}[S_n]}(\mathbb{C}[S_n]c_\lambda, V^{\otimes^n}) \otimes_{\mathbb{C}} \mathbb{C}[S_n]c_\lambda.$$

By Lemma 0.0.40 we have therefore,

$$V^{\otimes^n} \simeq \bigoplus_\lambda \mathcal{S}_\lambda(V) \otimes_{\mathbb{C}} V_\lambda.$$

This decomposition is compatible with the actions of $\mathrm{GL}(V)$ and S_n and is therefore a decomposition of $\mathrm{GL}(V) \times S_n$-modules. $\square$

Remarks : Noe that, in the theorem above, the sum is over partitions λ of n with at most $\dim(V)$ parts (explain why). The multiplicity of V_λ in $V^{\otimes^n}$ equals $\dim(\mathcal{S}_\lambda(V))$ and the mupltiplicity of $\mathcal{S}_\lambda(V)$ is $\dim(V_\lambda)$.

For the proof of the Schur-Weyl duality, we have used the so called **Double centraliser theorem** implicitly. This theorem is of central importance in representation theory. We end this course with proving this theorem. Let V be a finite dimensional vector space. For any subset $S \subset \mathrm{End}(V)$, define the **commutant** or centraliser of S in $\mathrm{End}(V)$ by

$$\mathrm{Comm}(S) = \{x \in \mathrm{End}(V) | xs = sx \text{ for all } x \in S\}.$$

Then $\mathrm{Comm}(S)$ is a subalgebra of $\mathrm{End}(V)$containing I_V, the identity automorphism of V. Let $\mathcal{A}$ be a **semisimple** subalgebra of $\mathrm{End}(V)$ with $I_V \in \mathcal{A}$. Let $\mathcal{B} = \mathrm{Comm}(\mathcal{A})$. Then $\mathcal{A} \otimes \mathcal{B}$ is an algebra with the product induced by

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb,'$$

$\mathcal{A}$ is isomorphic to the subalgebra $\mathcal{A} \otimes I_V$ and $\mathcal{B}$ to the subalgebra $I_V \otimes \mathcal{B}$. We have seen a decomposition

$$V \simeq \bigoplus_{i=1}^{r} V_i \otimes U_i,$$

where V_i run over all the irreducible $\mathcal{A}$-modules, $V_i \not\simeq V_j$ when $i \neq j$ and $U_i = \mathrm{Hom}_{\mathcal{A}}(V_i, V)$. We have then $\mathcal{A} \simeq \bigoplus_{i=1}^r \mathrm{End}(V_i)$. Hence, respecting the above decomposition of V, we have

$$\mathcal{A} \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{A}) \simeq \bigoplus_{i=1}^{r} \mathrm{End}(V_i) \otimes I_{U_i}.$$

We have

Theorem 0.0.42. (Double Centraliser Theorem) *Let V be a finite dimensional vector space and $\mathcal{A} \subset \mathrm{End}(V)$ a semisimple subalgebra. Then $\mathcal{B} = \mathrm{Comm}(\mathcal{A})$ is semisimple and $\mathrm{Comm}(\mathcal{B}) = \mathcal{A}$. Moreover, relative to the decomposition of V as above, we have*

$$\mathcal{B} \simeq \bigoplus_{i=1}^{r} I_{V_i} \otimes \mathrm{End}(U_i).$$

Proof. Let us prove the assertion about $\mathcal{B}$ first. We continue to have the decomposition of V as $V \simeq \bigoplus_{i=1}^r V_i \otimes U_i$. Then it is straight forward that

$$\bigoplus_{i=1}^{r} I_{V_i} \otimes \mathrm{End}(U_i) \subset \mathcal{B}.$$

Let $\pi_i : V \longrightarrow V_i \otimes U_i$ be the projections corresponding to the decomposition of V. Then $\pi_i \in \mathcal{A}$ by the decomposition of $\mathcal{A}$ as above. For $T \in \mathcal{B}$ we have (using the definition of $\mathcal{B}$ as the commutant of $\mathcal{A}$),

$$\pi_i T = T\pi_i, \quad T = \bigoplus_{i=1}^{r} T_{|V_i \otimes U_i}.$$

Therefore, we would be done if we proved the assertion about $\mathcal{B}$ for $r = 1$. This would be settled in the lemma below. Now, using the assertion on $\mathcal{B}$, it is clear that $\mathcal{B}$ is semisimple. Now, applying the theorem to $\mathcal{B}$, we see that $\mathcal{A} = \mathrm{Comm}(\mathcal{B})$. $\square$

Lemma 0.0.43. *Let V and U be finite dimensional vector spaces and $T \in \mathrm{End}(V \otimes U)$ be such that*

$$T(X \otimes I_U) = (X \otimes I_U)T$$

for all $X \in \mathrm{End}(V)$. Then there exists $Y \in \mathrm{End}(U)$ such that $T = I_V \otimes Y$.

Proof. Let U^* be the dual of U. For $f \in U^*$, define a linear map $B_f : V \otimes U \longrightarrow V$ by

$$B_f(v \otimes u) = f(u)v, \quad u \in U, \; v \in V.$$

Given $u \in U$, define a linear map $A_u : V \longrightarrow V \otimes U$ by $A_u(v) = v \otimes u$. Let $S_{f,u} = B_f \circ T \circ A_u$. Then $S_{f,u} \in \mathrm{End}(V)$ and satisfies

$$S_{f,u}Xv = B_f T(Xv \otimes u) = B_f(X \otimes I_U)T(v \otimes u) = XB_fT(A_uv) = XS_{f,u}v$$

for all $X \in \mathrm{End}(V)$. We can now apply Schur's lemma to this situation to find $c(f,u) \in \mathbb{C}$ such that $S_{f,u} = c(f,u)I_V$. Now note that $c(\ ,\ ) : U^* \times U \longrightarrow \mathbb{C}$ is bilinear and nondegenerate. Hence we have an induced isomorphism $U \simeq U^{**}, \;\; u \mapsto c(\ ,u)$. But $U^{**} \simeq U$, hence we have $Y \in \mathrm{End}(V)$ such that $c(f,u) = f(Yu)$ for all $u \in U, \; f \in U^*$. Spelling out everything we have, $B_fT(v \otimes u) = f(Yu)v = B_f(v \otimes Yu)$, for all $f \in U^*$. Hence $T(v \otimes u) = v \otimes Yu$ for all $u \in U, \; v \in V$, showing thereby $T = I_V \otimes Y$.

$\square$

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