Advanced Instructional School (AIS-RTA)

Representation Theory and its Applications

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Exercises in group theory, semisimple rings and modules, group rings and their use in representation theory (compiled by A.K. Bhandari)

Part I (Group Theory)

Exercise 1.1. Suppose that $|G| < \infty$, $|X| < \infty$. For $t \in G$, let $f(t) = |\{x \in X \,|\, tx = x\}|$. Let N be equal to the number of distinct G-orbits of X. Then show that

$$N = \frac{1}{|G|} \sum_{t \in G} f(t).$$

Exercise 1.2. Let G be a finite group of order n, $H \leqslant G$ is such that $[G : H]$ is the least prime divisor p of n. Then show that $H \trianglelefteq G$.

Exercise 1.3. If G is a finite group of order n, and H is a subgroup of G of index l such that n does not divide $l!$, then show that H contains a subgroup ($\neq \{e\}$) which is normal in G.

Exercise 1.4. Show that for every positive integer k, there exist a positive integer $N(k)$ such that for every finite group G with class number k, $|G| \leq N(k)$.

Exercise 1.5. Show that the group

$$\Gamma = SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) | \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \right\}$$

acts on the upper half plane $\hbar$ by

$$\gamma z = \frac{az+b}{cz+d} \ , \ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \ , \ z \in \hbar \ ,$$

and the kernel of this action is $\pm I_2$. Show that the subset $\mathcal{D}$ of $\hbar$ described by the conditions $-\frac{1}{2} \leq x < \frac{1}{2}$ when $|z| \geq 1$ and $x \leq 0$ when $|z| = 1$ is a Fundamental domain for the action of Γ on $\hbar$ in the sense that it meets every Γ orbit in $\hbar$ in exactly one point.

Exercise 1.6. Let G be a finite group and Let $k(G)$ denote the class number of G, i.e., the number of conjugacy classes of G. Then show that
(i) $k(G) = 2 \Leftrightarrow G \cong C_2$
(ii) $k(G) = 3 \Leftrightarrow$ either $G \cong C_3$ or $G \cong S_3$.

Exercise 1.7. Show that a group of order pq, p and q primes, $p < q$ is not simple .

Exercise 1.8. If G is a group of order pqr, $p > q > r$ primes, then G has a unique Sylow p subgroup . If, in addition, $p \not\equiv 1(\text{mod q})$, then G has a unique Sylow q-subgroup.

Exercise 1.9. (Frattini Argument) Let $H \trianglelefteq G$ and let P be a Sylow p-subgroup of H. Show that $G = H.N_G(P)$.

Exercise 1.10. Let $H \trianglelefteq G$ and let P be a Sylow p-subgroup of G. Then $P \cap H$ is a Sylow p-subgroup of H.

Exercise 1.11. Let $|G| = p^e m$, $p > m$, $(p, m) = 1$. Show that G has a normal Sylow p-subgroup and hence is not simple.

Exercise 1.12. Show that no group of order 108 is simple.

Exercise 1.13. Show that no group of order 300 is simple.

Exercise 1.14. Let H and K be groups and let $\phi : K \to \text{Aut}(H)$ be a homomorphism. Then show that the following are equivalent:
(i) The identity (set) map between $H \rtimes K$ and $H \times K$ is a group homomorphism (hence an isomorphism).
(ii) ϕ is the trivial homomorphism from K into Aut$H$.
(iii) $K \trianglelefteq H \rtimes K$.

Exercise 1.15. Let K be cyclic, H be an arbitrary group and φ_1 and φ_2 be homomorphisms from K into $\text{Aut}(H)$ such that $\varphi_1(K)$ and $\varphi_2(K)$ are conjugate subgroups of $\text{Aut}(H)$. Prove by constructing an explicit isomorphism that $H \rtimes_{\varphi_1} K \cong H \rtimes_{\varphi_2} K$ (in particular, if the subgroups $\varphi_1(K)$ and $\varphi_2(K)$ are equal in $\text{Aut}(H)$, then the resulting semidirect products are isomorphic.

Exercise 1.16. Prove that the order of a permutation in S_n, written in disjoint cyclic form is the l.c.m. of the lengths of the cycles.

Exercise 1.17. Show that a finite group with nontrivial cyclic Sylow 2-subgroup has a subgroup of index 2.

Exercise 1.18. Let G be a finite group and let α be an automorphism of G which leaves only the unit element of G fixed. Show that the map $x \mapsto x^{-1}x^{\alpha}$ is a permutation of G. If for some $c \in G$, c^{α} is conjugate to c, show that there is a conjugate of c fixed by α, and deduce that $c = 1$. If, moreover, $\alpha^2 = 1$, show that $xx^{\alpha} = 1$ for all $x \in G$ and deduce that G is abelian.

Exercise 1.19. (W. Gaschütz) Let G be a finite group and let α be an automorphism of order prime to $|G|$. If α maps each conjugacy class into itself, show that the subgroup H of elements fixed by α meets each conjugacy class and deduce that $\alpha = 1$.

Exercise 1.20. Let G be a finite group. Show that the automorphisms which map each conjugacy class to itself form a normal subgroup of $\text{Aut}(G)$ whose order contains only prime divisors occurring in $|G|$.

Exercise 1.21. Show that a ring with n elements, where n is square-free is commutative.

Part II (Semisimplicity and Wedderburn-Artin Theory)

Exercise 2.1. Let I be an ideal of a ring R. Prove that $M_n(I)$ is an ideal of the full matrix ring $M_n(R)$ and such that $M_n(R)/M_n(I) \cong M_n(R/I)$.

Exercise 2.2. Let R be a ring without nonzero nilpotent elements. Prove that every idempotent of R is central.

Exercise 2.3. (Dieudonné) Let $R = \mathbb{Z} < x, y > / < y^2, yx >$. Then show that R is left Noetherian bur not right Noetherian.

Exercise 2.4. Let D be a division ring. Show that

$$L_i = \begin{array}{c} \begin{array}{ccccccc} & & & \overbrace{}^{i^{th}\ column} & & & \end{array} \\ \begin{pmatrix} 0 & . & 0 & D & 0 & . & 0 \\ 0 & . & 0 & D & 0 & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & . & 0 & D & 0 & . & 0 \end{pmatrix} \end{array}$$

is a minimal left ideal of $M_n(D)$.

Exercise 2.5. Let R be a ring and let $M_n(R)$ be the ring of $n \times n$ matrices over R. Show that any ideal I of $M_n(R)$ has the form $M_n(\mathcal{I})$ for a uniquely determined ideal $\mathcal{I}$ of R. In particular, if R is a simple ring , then so is $M_n(R)$.

Exercise 2.6. Let $R_1, \ldots, R_r$ be (left) semisimple rings. Then their direct product $R = R_1 \times \ldots, R_r$ is also a simisimple ring.

Exercise 2.7. In a semisimple ring R, let $L = Re$ be a left ideal with generating idempotent e. Then show that L is a minimal left ideal if and only if eRe is a skewfield.

Exercise 2.8. (Hopkins, Levitzki) Let R be a (left) Artinian ring. Then R is also (left) Noetherian.

Exercise 2.9. Show that for a ring R, the center of $M_n(R)$ consists of diagonal matrices rI_n, where $r \in \mathcal{Z}(R)$.

Exercise 2.10. A ring R is called Dedekind finite if $ab = 1$ in R implies that $ba = 1$. Give example of a nonDedekind- finite ring.

Exercise 2.11. Let A be an algebra over a field k such that every element of A is algebraic over k.
(a) Show that A is Dedekind-finite.
(b) Show that a left 0-divisor of A is also a right 0-divisor.
(c) Show that a nonzero element of A is a unit if and only if it is not a zero divisor.
(d) Let B be a subalgebra of A, and $b \in B$. Show that b is a unit in B if and only if it is a unit in A.

Exercise 2.12. Let x, y be elements in a ring R such that $Rx = Ry$. Show that there exists a right R-module isomorphism $f : xR \to yR$ such that $f(x) = y$

Exercise 2.13. Let R be the upper triangular ring

$$\begin{pmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix} .$$

(1) Show that every right 0-divisor in R is a left 0-divisor.
(2) Using (1) show that R is not isomorphic to R^{op}.

Exercise 2.14. Is any subring of a (left) semisimple ring (left) semisimple? can any ring be embedded as a subring of a (left) semisimple ring?

Exercise 2.15. Determine which of the following are semisimple $\mathbb{Z}$-modules ($\mathbb{Z}_n$ means $\mathbb{Z}/n\mathbb{Z}$)

$$\mathbb{Z} \ , \ \mathbb{Q} \ , \ \mathbb{Q}/\mathbb{Z} \ , \ \mathbb{Z}_4 \ , \ \mathbb{Z}_6 \ , \ \mathbb{Z}_6 \ , \ \mathbb{Z}_{12} \ , \ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$$

$$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \cdots \quad , \quad \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \cdots \quad .$$

Exercise 2.16. What are semisimple $\mathbb{Z}$-modules? Characterize them in terms of their structure as abelian groups.

Exercise 2.17. Let R be the (commutative) ring of all real valued continuous functions on $[0,1]$. Is R a semisimple ring.

Exercise 2.18. Show that for a semisimple module M over a ring R, the following conditions are equivalent: (1) M is finitely generated
(2) M is Noetherian
(3) M is Artinian
(4) M is a finite direct sum of simple modules.

Exercise 2.19. Show that, if M is a simple module over a ring R, then as an abelian group, M is isomorphic to a direct sum of copies of Q, or a direct sum of copies of $\mathbb{Z}_p$, for some prime p.

Exercise 2.20. Show that if R is semisimple, so is $M_n(R)$.

Exercise 2.21. Let R be a domain. Show that if $M_n(R)$ is semisimple, then R is a division ring.

Exercise 2.22. Let R be a domain. If R has a minimal left ideal, show that R is a division ring (in particular, a left artinian domain must be a division ring).

Exercise 2.23. Let R be a semisimple ring.
(1) Show that any ideal of R is a sum of simple components
(2) Using (1), show that any quotient ring of R is semisimple.
(3) Show that a simple artinian ring S is isomorphic to a simple component of R if and only if there is a surjective ring homomorphism from R to S.

Exercise 2.24. Show that the center of a simple ring is a field and the center of a semisimple ring is a direct sum of fields.

Exercise 2.25. Let M be a finitely generated left R-module and let $E = \mathrm{End}(\ _RM)$. Show that if R is semisimple (respectively simple artinian), then so is E.

Exercise 2.26. Let M be left R-module and $E = \mathrm{End}(\,_RM)$. If $_RM$ is a semisimple R-module, show that M_E is a semisimple E-module.

Exercise 2.27. Let M be a left R-module and let $E = \mathrm{End}(\,_RM)$. If M_E is a semisimple E module, is $_RM$ necessarily a semisimple R-module.

Exercise 2.28. Let R be a simple ringthat is finite dimensional over its center k (observe that k is a field). Let M be a finitely generated left A-module and let $E = \mathrm{End}(\,_RM)$. Show that

$$(\dim_k M)^2 = (\dim_k R)(\dim_k E)\ .$$

Exercise 2.29. Let R be simple and finite dimensional over its center k. Show that R is isomorphic to a matrix algebra over its center k if an only if R has nonzero left ideal $\mathfrak{I}$ with $(\dim_k \mathfrak{I})^2 \leq \dim_k R$.

Exercise 2.30. (a) Let R, S be rings such that $M_n(R) \cong M_n(S)$. Does this imply that $m = n$ and $R \cong S$.
(b) Let us call a ring A a matrix ring if $A \cong M_m(R)$ for some $m \geq 2$and for some ring R. True or False: "A homomorphic image of a matrix ring is also a matrix ring".

Exercise 2.31. Let R be any semisimple ring.
(1) Show that R is Dedekind-finite, i.e., $ab = 1 \Rightarrow ba = 1$ in R.
(2) If $a \in R$ is such that $I = aR$ is an ideal in R, then $I = Ra$.
(3) Every element $a \in R$ can be written as a unit times an idempotent.

Exercise 2.32. Let R be an n^2-dimensional algebra over k. Show that $R \cong M_n(k)$ (as k-algebra)if and only if R is simple and has an element r whose minimal polynomial over k has the form $(x - a_1)(x - a_2)\cdots(x - a_n)$ where $a_1, \ldots, a_n \in k$.

Exercise 2.33. True or false:" If I is a minimal left ideal in a ring R, then $M_n(I)$ is a minimal left ideal of $M_n(R)$"?

Exercise 2.34. Let R be J-semisimple domain and let a be a nonzero central element of R. Show that the intersection of all maximal left ideals not containing a is zero.

Exercise 2.35. Show that if $f : R \to S$ is a surjective ring homomorphism, then $f(J(R)) \subseteq J(S)$. Give an example to show that $f(J(R))$ may be smaller than $J(S)$.

Exercise 2.36. If an ideal $I \subseteq R$ is such that $J(R/I) = 0$, show that $I \supseteq J(R)$ (therefore $J(R)$ is the smallest ideal $I \subseteq R$ such that R/I is J-semisimple.

Exercise 2.37. Let $\{\mathfrak{A}_\mathrm{i}\}_{i\in\mathcal{I}}$ be ideals in R, and let $\mathfrak{A} = \cap_i \mathfrak{A}_i$. True or false:" If each $R/\mathfrak{A}_\mathrm{i}$ is J-semisimple, then so is $R/\mathfrak{A}$"?

Exercise 2.38. Show that for direct product of rings ΠR_i,

$$J(\Pi R_i) = \Pi J(R_i)$$

Exercise 2.39. For a triangular ring $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ (where M is (R, S)-bimodule), show that

$$J(T) = \begin{pmatrix} J(R) & M \\ 0 & J(S) \end{pmatrix}.$$

Apply this to compute $J(T_n(k))$, where $T_n(k)$ is the ring of $n \times n$ upper triangular matrices over a ring k.

Exercise 2.40. Let $GL_n(R)$ denote the group of units of $M_n(R)$. Show that for any ideal $I \subseteq J(R)$, the natural map $GL_n(R) \to GL_n(R/I)$ is surjective.

Exercise 2.41. Show that $J(M_n(R)) = M_n(J(R))$.

Exercise 2.42. Using the definition of $J(R)$ as the intersection of the maximal left ideals, show directly that $J(R)$ is an ideal.

Exercise 2.43. If R is a commutative Noetherian ring, then as a consequence of Krull intersection theorem $\cap_{n=1}^{\infty}(J(R))^n = (0)$. Show that this need not be true for noncommutative right Noetherian rings.

Part III (Group Rings and Representation Theory)

Exercise 3.1. Let G be a group and let $\Delta(G)$ be the augmentation ideal of $\mathbb{Z}G$. Then show that $G/G^{'} \cong \Delta(G)/\Delta(G)^2$.

Exercise 3.2. Suppose that a field F contains a primitive n^{th} root of unity and $\mathrm{Char} F \nmid |G|$, where n is the exponent of G. For any $\chi \in \mathrm{Hom}(G, F^*)$, put

$$e_\chi = \frac{1}{|G|}\sum_{g\in G}\chi(g^{-1})g \quad .$$

Then $\{e_\chi | \chi \in \mathrm{Hom}(G, F^*)\}$ is the full set of primitive idempotents of FG, which is also an orthogonal F-basis for FG.

Exercise 3.3. Let G be a finite group and let $\mathrm{Char}(K) \,|\, |G|$. Prove that KG is not semisimple by showing that the ideal generated by $\widehat{G} = \sum_{g\in G} g$ is not a direct summand.

Exercise 3.4. Let V be a kG-module and let H be a subgroup of G of finite index n not divisible by Char k. Modify the proof of Maschke's theorem to show that: If V is semisimple as a kH-module, then V is semisimple as a kG-module.

Exercise 3.5. Let G be a finite group whose order is a unit in a ring k and let $W \subseteq V$ be a left kG-module. Show that if W is a direct summand of V as k-module, then W is a direct summand of V as kG-module.

Exercise 3.6. Let k be a commutative ring and G be any group. If kG is left Noetherian (respectively left Artinian), show that kG is right Noetherian (respectively right Artinian).

Exercise 3.7. Give an example of pair of finite groups G, G' such that, for some field K, $KG \cong KG'$ as K-algebras, but $G \ncong G'$ as groups.

Exercise 3.8. Let k be a field such that $\mathrm{Char} k$ is prime to $|G|$. Show that the following two statements are equivalent:
(a) Each irreducible kG-module has k-dimension 1.
(b) G is abelian, and k is a splitting field for G.

Exercise 3.9. Let $G \cong S_3$. Show that $\mathbb{Q}G \cong \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus M_2(\mathbb{Q})$ and compute the central idempotent of $\mathbb{Q}G$ into its simple components. Compute similarly the decomposition of $\mathbb{Q}G_1, \mathbb{Q}G_2$, where G_1 is the Klein 4-group, and G_2 is the Quaternion group of order 8.

Exercise 3.10. Let $R = kG$, where k is a field and G is any group. Let I be the ideal of R generated by $ab - ba$ for all $a, b \in R$. Show that $R/I \cong k(G/G')$ as k-algebra, where G' is the commutator subgroup of G. Also show that $I = \sum_{a \in G'}(a-1)kG$.

Exercise 3.11. Let G be a finite group such that, for some field k, kG is a finite direct product of k-division algebras. Show that every subgroup H of G is normal.

Exercise 3.12. For finite abelian groups G and H, show that $\mathbb{R}G \cong \mathbb{R}H$ as $\mathbb{R}$-algebras if and only if $|G| = |H|$ and $|G/G^2| = |H/H^2|$.

Exercise 3.13. Show that for any two groups G, H, there exists a (nonzero) ring R such that $RG \cong RH$ as rings.

Exercise 3.14. Let $G = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group of order 6. It is known that, over $\mathbb{C}$, G has four 1-dimensional representations and a unique irreducible 2-dimensional representation D. Construct D explicitly and compute the character table for G.

Exercise 3.15. Let G be the dihedral group of order $2n$ generated by two elements r, s such that $r^n = 1 = s^2$ and $srs^{-1} = r^{-1}$. Let $\theta = 2\pi/n$.
(1) For any integer h, $0 \le h \le n$, show that

$$D_h(r) = \begin{pmatrix} \cos h\theta & -\sin h\theta \\ \sin h\theta & \cos h\theta \end{pmatrix} \quad , \quad D_h(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

defines a real representation of G.
(2) Show that over $\mathbb{C}$, D_h is equivalent to D'_h defined by

$$D'_h(r) = \begin{pmatrix} e^{-ih\theta} & 0 \\ 0 & e^{ih\theta} \end{pmatrix} \quad , \quad D'_h(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad .$$

(3) For $n = 2m + 1$, show that $D_1, \ldots, D_m$ give all irreducible representations of G (over $\mathbb{R}$ or $\mathbb{C}$) with dimensions > 1. For $n = 2m$, show the same for $D_1, \ldots, D_{m-1}$.
(4) Construct the character table for G.
(5) Verify that the two (nonisomorphic) nonabelian groups of order 8 have the same character table.

Exercise 3.16. For any finite group G and any field k, is it true that any irreducible representation of G over k is afforded by a minimal left ideal of kG (the answer is yes, but is hard to show).

Exercise 3.17. If a finite group G has at most three irreducible complex representations, show that $G \cong \{1\}, C_2, C_3$ or S_3.

Exercise 3.18. Suppose the character table of a finite group G has the following two rows:

	g_1	g_2	g_3	g_4	g_5	g_6	g_7
μ	1	1	1	ω^2	ω	ω^2	ω
ν	2	-2	0	-1	-1	1	1

where $\omega = e^{2\pi i/3}$. Determine the rest of the character table.

Exercise 3.19. (Littlewood's formula) Let $e = \sum_{g \in G} a_g g \in kG$ be an idempotent, where k is a field and G is a finite group. Let χ be the character of G afforded by the kG-module $kG.e$. Show that for any $h \in G$,

$$\chi(h) = |C_G(h)| \sum_{g \in C} a_g \quad .$$

Exercise 3.20. Let G be a group of order 21 generated by two elements a and b with relations $a^7 = 1, b^3 = 1, bab^{-1} = a^2$.
(1) Construct the irreducible complex representations of G and compute its character table.
(2) Construct the irreducible rational representations of G and determine the Wedderburn decomposition of $\mathbb{Q}G$.
(3) How about $\mathbb{R}G$ and real representations of G.

Exercise 3.21. Write the Wedderburn-decomposition of the rational group algebras of following groups:
(1) Generalized quaternion group of order $4p$:

$$Q_{4p} =< x, t | x^p = t^4 = 1, txt^{-1} = x^{-1} >$$

(2) Dihedral group of order $2p$, p odd :

$$D_{2p} =< x, y | x^p = 1 = y^2, x^y = x{-}1 >$$

(3) A_4
(4) The dihedral group of order $2m$, $m \geq 4$:

$$D_{2m} =< a, b | a^m = b^2, a^b = a^{-1} >$$

(5) The generalized quaternion group of order $2^k, k \geq 3$:

$$Q_{2^k} =< a, b | a^{2^{k-1}} = 1, a^{2^{k-2}} = b^2, a^b = a^{-1} >$$

(6) $G = S_n$
(7) $G =< a, b, c | a^3 = 1, b^3 = 1, c^2 = 1, ab = ba, ac = ca^{-1}, bc = cb{-}1 >$, $(|G| = 8)$.