Advanced Instructional School on Representation Theory

Exercises

The exercises here are supplementry to the short course on S_n and $GL(V)$ duality (Schur-Weyl duality). We denote the group of permutations on n symbols by S_n and V for a vector space of dimension k usually over field $\mathbb{C}$.

Exercise 1. Let H be a subgroup of G. Then G acts on G/H by permutation action. Calculate the character of this representation. Let $H' = gHg^{-1}$, a conjugate of H. Then the permutation representation on G/H' is conjugate to the representation on G/H.

Exercise 2. The conjugacy classes of S_n are in one-one correspondance with partitions of n.

Exercise 3. Let V be a k-dimensional vector space. Define the n-tensors by $V^{\otimes n} = V \otimes \ldots \otimes V$ (n times). Then $T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$ is an algebra, called **tensor algebra** of V. Define **symmetric algebra** by

$$Sym(V) = \frac{T(V)}{\langle v_1 \otimes v_2 - v_2 \otimes v_1 \mid v_1, v_2 \in V \rangle}.$$

We also define nth symmetric power by

$$Sym^n(V) = \frac{V^{\otimes n}}{\langle v_1 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \mid v_i \in V, \sigma \in S_n \rangle}.$$

Prove that $Smy(V) \cong \bigoplus_{n=0}^{\infty} Sym^n(V)$. Define **exterior algebra** by

$$\Lambda(V) = \frac{T(V)}{\langle v \otimes v \mid v \in V \rangle}.$$

We also define nth exterior power by

$$\Lambda^n(V) = \frac{V^{\otimes n}}{\langle v_1 \otimes \cdots \otimes v_n \mid v_i \in V, v_i = v_j \text{ for some } i \neq j \rangle}.$$

Prove that $\Lambda^n(V) = 0$ for $n > k$ and $\Lambda(V) \cong \bigoplus_{n=0}^{\infty} \Lambda^n(V)$. What is the dimension of $\Lambda^n(V)$ and $\Lambda(V)$.

Hints: See Dummit and Foote, Abstract Algebra, section 11.5.

Exercise 4. The group $GL(V)$ acts naturally on V via $t.v = t(v)$ for $t \in GL(V)$ and $v \in V$. Verify that $GL(V)$ acts on $V^{\otimes n}, T(V), Smy(V)$, $Sym^n(V), \Lambda(V)$ and $\Lambda^n(V)$. Also verify that the action of $GL(V)$ on $\Lambda^k(V)$ is by det, i.e., for $t \in GL(V), \Lambda^k(t)(v_1 \wedge \cdots \wedge v_k) = \det(t) v_1 \wedge \cdots \wedge v_k$.

Exercise 5. Let V be a vector space of dimension k. Then the symmetric group S_n acts from right on $V^{\otimes n}$ by the action:

$$(v_1 \otimes v_2 \otimes \ldots \otimes v_n).\sigma = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \ldots \otimes v_{\sigma^{-1}(n)}$$

where $\sigma \in S_n$.

Exercise 6. Combining previous exercises, for a vector space of dimension k we get action of the group $GL(V) \times S_n$ on $V^{\otimes n}$ where $GL(V)$ acts from left and S_n acts from right. Check that the two actions commute with each other.

Exercise 7. Let $t \in GL(V)$. Calculate the trace of the element $Sym^n(t)$ acting on $Sym^n(V)$ and the the trace of $\Lambda^n(t)$ acting on $\Lambda^n(V)$.

Hint: You can assume the matrix of t is upper triangular. First calculate it for the diagonal matrix. Actually final answer is same as in this case.

Exercise 8. Let V be a k-dimenaional vector space. A flag of length r is a sequence $V_0 \subset V_1 \subset \ldots \subset V_r$ of subspaces of V such that $\dim V_i$ are distinct. Let $\tilde{\mathfrak{F}}(V)$ be the set of all flags of all lengths $r, 1 \leq r \leq k$ of V. The group $GL(V)$ acts on $\mathfrak{F}(V)$. What are the orbits and corresponding stablizer subgroups. The stablizer subgroups are called **parabolic subgroups** of $GL(V)$. If the flag is of length k then show that the stablizer subgroup, by fixing a basis, is T_k, the upper triangular matrices in GL_k (in general its a conjugate of this subgroup).

Exercise 9. Let V be the irreducible permutation (standard) representation of S_n. Prove that $\Lambda^r(V)$ is irreducible S_n module for all r.

Hint: See Fulton, Harris Proposition 3.12.

Exercise 10. The only one dimensional representations of S_n are the trivial representation and the sign representation (which maps transpositions to -1).

Exercise 11. Let ϕ be an irreducible representation and χ be a one dimensional representation. Then ϕ is irreducible if and only if $\phi \otimes \chi$ is so.

Notice that this can be used to obtain some new representations of S_n by tensoring with the sign representation.

Exercise 12.
(1) Show that every element of S_n is conjugate to its own inverse. In fact, more is true that every element of S_n is a product of two involutions (elements of which square is 1). Conclude that all characters of S_n are real valued.
(2) An element $x \in G$ is called rational if all of the generators of the subgroup $< x >$ are conjugate to each other. Prove that every element of S_n is rational. Conclude that each character of S_n takes value in $\mathbb{Q}$, in fact, it takes values in $\mathbb{Z}$.

Exercise 13. Write down all representations of $S_3, S_4, S_5, \ldots$.

Exercise 14. Write down the character table of D_8 (dihedral group with 8 elements) and Q_8, the quaternion group. Notice that the character table for both of these groups are same. Hence character theory fails to distinguish between these groups. Check that $\mathbb{C}D_8 \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \cong \mathbb{C}Q_8$. But $\mathbb{R}D_8 \cong \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus M_2(\mathbb{R})$ and $\mathbb{R}Q_8 \cong \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{H}$ where $\mathbb{H}$ is the division algebra of real quaternions. Hence $\mathbb{R}D_8 \ncong \mathbb{R}Q_8$.

Exercise 15. Let V be a G-module with character χ.
(1) Show that the $< 1, \chi >= \dim(V^G) = \dim(\mathrm{Hom}_{\mathbb{C}G}(1, V))$ where 1 denotes the trivial $\mathbb{C}G$ module and V^G is the subspace of V fixed under the action of G.
(2) If V is irreducible then show that $V \cong V^*$ (also called V is self-dual) if and only if there exists a nonzero bilinear form on V which is invariant under the action of G.
(3) Let V be self-dual and irreducible then the trivial representation occurs exactly once in $V \otimes V$. Write down the decomposition $V \otimes V \cong Sym^2(V) \oplus \Lambda^2(V)$ as G-module. If the trivial representation occurs in $Sym^2(V)$ (res. $\Lambda^2(V)$) we get a G-invariant symmetric (resp. skew-symmetric) bilinear form on V and the representation is orthogonal (resp. sympletic).

Hint: Check that $\chi_{Sym^2}(x) = \frac{1}{2}(\chi^2(x) + \chi(x^2))$ and $\chi_{\Lambda^2}(x) = \frac{1}{2}(\chi^2(x) - \chi(x^2))$ and calculate $< \chi^2, 1 >=< \chi_{Sym^2}, 1 > + < \chi_{\Lambda^2}, 1 >$.