ONE HUNDRED EXERCISES IN BASIC REPRESENTATION THEORY

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Contents

These exercises were originally the weekly homework problems I assigned while teaching an introductory course on representation theory at the University of Iowa in Spring 2005. The prerequisites for this course was some basic algebra: groups, rings and modules, fields, linear algebra, tensor product, etc. The first ten or eleven weeks of this course was about the representation theory of finite groups and each week I assigned about ten exercises. They have been rearranged for the purposes of making a coherent list of exercises and also to make them independent of the sequence of lectures as in the original course. If these exercises are used to supplement a course, then I would assume that the basic definitions and results are being covered in the lectures. To help the reader, in each section I have mentioned the topics that are covered by the exercises in that section.

1. Warming up with some group theory

The class equation, conjugacy classes, dihedral groups D_n, symmetric groups S_n, alternating groups A_n, finite fields $\mathbb{F}_q$, general linear groups $\mathrm{GL}_n(\mathbb{F}_q)$.

Exercise 1. Let G be a finite group. Consider the action of G on itself by conjugation. This is given by $g \cdot x = gxg^{-1}$. The orbit of an element $x \in G$ under this action will be denoted by $\mathcal{O}(x)$, and is called the conjugacy class of x. The stabilizer of $x \in G$ under this action is the centralizer $Z_G(x)$ of x consisting of all elements of G which commute with x. Show that

$$|G| = \sum_{\mathcal{O}(x)} \frac{|G|}{|Z_G(x)|}$$

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where the summation runs over all distinct conjugacy classes. (This equation is the so-called *class equation.*) Let p be a prime number. Use the above formula to show that any group of order p^m has nontrivial center. Now show that a group of order p^2 is abelian. Can you find an nonabelian group of order p^3?

The dihedral group D_n is defined as $D_n = \{1, r, r^2, ..., r^{n-1}, s, sr, ..., sr^{n-1}\}$ where the elements s and r are subject to the relations $r^n = 1,\ s^2 = 1,\ srs = r^{-1}$.

Exercise 2. (1) Determine the center of D_n. (The answer depends on whether n is even or odd.)
(2) Determine the conjugacy classes of D_n. (It might help a little to know that, if n is even, then there are $\frac{n}{2} + 3$ conjugacy classes, and if n is odd then there are $\frac{n-1}{2} + 2$ conjugacy classes.)

The permutation group on n letters will be denoted S_n.

Exercise 3. Show that there is a homomorphism $\epsilon : S_n \to \{\pm 1\}$ such that $\epsilon(s) = -1$ for every transposition s as follows: Consider the discriminant polynomial in n variables defined as:

$$\Delta(X_1, ..., X_n) = \prod_{1 \leq i < j \leq n} (X_i - X_j).$$

For every $\sigma \in S_n$, define Δ^σ by the relation:

$$\Delta^\sigma(X_1, ..., X_n) = \Delta(X_{\sigma(1)}, ..., X_{\sigma(n)}).$$

Now show the following:

(1) $\Delta^\sigma = \pm\Delta$. Use this to define the sign $\epsilon(\sigma)$ by $\Delta^\sigma = \epsilon(\sigma)\Delta$.
(2) Show that $\epsilon(\sigma\tau) = \epsilon(\sigma)\epsilon(\tau)$ for all $\sigma, \tau \in S_n$.
(3) Show that $\epsilon(s) = -1$ for all transpositions $s \in S_n$.

The *alternating group* A_n on n letters is the kernel of ϵ and is the unique normal subgroup of S_n of index 2.

Exercise 4. If $n \geq 3$, show that the center of S_n is trivial. For $n \geq 4$, show that the center of A_n is trivial.

Exercise 5. Let n be a positive integer. An unordered partition of n is any expression of the form $n = n_1 + \cdots + n_k$, for some $k \geq 1$, with each n_i a positive integer; not paying attention to the order in which the n_i's are written. Show that the conjugacy classes of S_n are parametrized by unordered partitions of n. (*Hint: If $\sigma \in S_n$, then the cycle decomposition of σ gives an unordered partition of n, by counting the lengths of the cycles which show up. Now show that σ is conjugate to τ if and only if σ and τ determine the same unordered partition. Try to first see this when both σ and τ are n-cycles, to get a feel for the problem.)*

For a field F the group of all invertible $n \times n$ matrices will be denoted $\mathrm{GL}_n(F)$. The subgroup of all matrices of determinant 1 will be denoted by $\mathrm{SL}_n(F)$.

Exercise 6. (1) Let F be any field. Let $G = \mathrm{GL}_2(F)$. Let $V = F^2$ be a two dimensional vector space with the vectors written as column vectors. Consider the natural action of G on V given by matrix multilpication. Show that this action is *linear*, i.e., for all $g \in G$; all $v, w \in V$; and all scalars a, b we have $g \cdot (av + bw) = a\, g \cdot v + b\, g \cdot w$.

(2) Let $\mathbb{P}(V)$ be the projectivization of V which defined as the set of all nonzero vectors of V modulo scalar multiplication. One can also define $\mathbb{P}(V)$ as the set of all lines through the origin. Since $V = F^2$, we also denote $\mathbb{P}(V)$ by $\mathbb{P}^1(F)$. Show that the G-action on V naturally gives a G-action on $\mathbb{P}^1(F)$.

(3) Let B be the subgroup of G consisting of all upper triangular matrices in G. This B is called the *standard Borel subgroup* of G; after Armand Borel. Show that $\mathbb{P}^1(F)$ is isomorphic as a G-set to G/B. (A G-set is a set equipped with a G-action.)

Exercise 7. Consider a finite field $\mathbb{F}_q$ of q elements; $q = p^m$ where p is a prime number.

(1) Find the order of $\mathrm{GL}_n(\mathbb{F}_q)$.
(2) Calculate the highest power of p which divides the order of $\mathrm{GL}_n(\mathbb{F}_q)$.
(3) Show that the subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$ consisting of all upper triangular matrices with all the diagonal entries equal to 1, is a p-Sylow subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$. (Can you now find a nonabelian group of order p^3?)

Now do the same questions for $\mathrm{SL}_n(\mathbb{F}_q)$.

Exercise 8. Give two elements of $SL_2(\mathbb{R})$ which are conjugate to each other by an element of $GL_2(\mathbb{R})$, however, they are not conjugate by an element of $SL_2(\mathbb{R})$. As a hint, here is a suggestive matrix identity:

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & t^2 a \\ 0 & 1 \end{pmatrix}$$

The same question for $\mathrm{SL}_2(\mathbb{F}_q)$ and $\mathrm{GL}_2(\mathbb{F}_q)$, with $q = p^m$ and p an odd prime.

Exercise 9. Let $A \in \mathrm{GL}_n(F)$. Let ${}^tA$ denote the transpose of A, which is defined as $({}^tA)_{i,j} = A_{j,i}$. Show that ${}^tA$ is conjugate to A. (*This exercise is not easy but is a very important result. Begin by showing that A and ${}^tA$ are conjugate by an element of* $\mathrm{GL}_n(\overline{F})$, *i.e., after going to the closure of F. One can see this using the Jordan canonical form of A. Then show that two elements of* $\mathrm{GL}_n(F)$ *which are conjugate by an element of* $\mathrm{GL}_n(\overline{F})$, *are necessarily conjugate by an element of* $\mathrm{GL}_n(F)$. *Look this up in Lang's Algebra* [4]. *In words, we would say, that two elements of* $\mathrm{GL}_n(F)$ *which are conjugate over the closure are conjugate rationally, i.e., over F itself.)*

The next exercise involves some elementary number theory, and eventually goes to show that $\mathbb{F}_q^*$ is cyclic. The set of all positive integers is denoted $\mathbb{N}$.

Exercise 10. (1) Euler φ-function. For every positive integer n, define $\varphi(n)$ to be the number of integers a, with $1 \leq a \leq n$ and a is relatively prime to n. This may be written as

$$\varphi(n) := |\{a \in \mathbb{N} \mid a \leq n,\ (a, n) = 1\}|.$$

Show that φ is a multiplicative function, i.e., $\varphi(mn) = \varphi(m)\varphi(n)$ for all relatively prime positive integers m and n.

(2) Show that $\varphi(p^a) = p^{a-1}(p-1)$ for any prime p and any $a \in \mathbb{N}$. Show that for all $n \in \mathbb{N}$ we have
$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$
where the product runs over all primes p dividing n.

(3) If $f : \mathbb{N} \to \mathbb{N}$ is a multiplicative function then show that the function $g : \mathbb{N} \to \mathbb{N}$ defined by
$$g(n) := \sum_{d|n} f(d)$$
is also multiplicative. (The summation runs over all positive divisors of n.)

(4) Show that $\sum_{d|n} \varphi(d) = n$.

(5) Consider $\mathbb{F}_q^*$–the multiplicative group of a finite field with q elements. For every positive divisor $d|(q-1)$, let $N(d)$ denote the number of elements of $\mathbb{F}_q^*$ of order d. Show that
   (a) $N(d) \leq \varphi(d)$ for all $d|(q-1)$.
   (b) $\sum_{d|(q-1)} N(d) = q - 1$.
   (c) Conclude that $N(d) = \varphi(d)$ for all $d|(q-1)$. Hence $N(q-1) \geq 1$, whence $\mathbb{F}_q^*$ is cyclic.

2. Basic notions of representation theory

Definition of a representation of a (finite) group, equivalence of representations, irreducible representation, permutation representation, regular representation, operations on representations: dual/contragredient, tensor product, symmetric square and exterior square.

Exercise 11. Let V be an n-dimensional complex vector space. Let $A : V \to V$ be a linear transformation. If we fix a basis $\{v_1, ..., v_n\}$ for V, then we can describe the matrix $[A] = [A_{i,j}]$ of A by the equations
$$A(v_j) = \sum_{i=1}^{n} A_{i,j} v_i.$$
Show that the map $A \to [A]$ gives an isomorphism from $\mathrm{GL}(V)$ to $\mathrm{GL}_n(\mathbb{C})$.

A degree 1 representation of a group G is a homomorphism $\pi : G \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*$, which is also called a character or a quasi-character of G. There is a slight ambiguity in terminology as, later, we will have character of a representation (for any degree). After some experience, this ambiguity causes no confusion, but up to that point we stick to the painful terminology of a *degree* 1 *representation.*

Exercise 12. Show that there are n distinct degree 1 representations of $\mathbb{Z}/n\mathbb{Z}$.

Exercise 13. Determine the commutator subgroup of D_n. Show that the number of distinct degree 1 representations of D_n is 4 if n is even, and is 2 if n is odd.

Exercise 14. Show that $[S_n, S_n] = A_n$. Conclude that for $n \geq 2$, S_n has exactly two degree 1 representations.

Exercise 15. Write down all the elements of A_4. Show that the commutator subgroup of A_4 is a subgroup isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$. Show that A_4 has three distinct degree 1 representations.

Exercise 16. For $n \geq 5$, show that A_n is simple, i.e., has no proper nontrivial normal subgroup. *(This was first observed by Galois. Look up Jacobson's Basic Algebra, vol-I* [3]*. Here is the rough sketch of the argument: Show that A_n is generated by 3-cycles; show that if a normal subgroup has one 3-cycle then it is all of A_n; show that every nontrivial normal subgroup has at least one 3-cycle, by showing that an element with the maximum number of fixed points has to be a 3-cycle.)* Conclude that for $n \geq 5$, A_n has no nontrivial degree 1 representation.

Exercise 17. Let F be a field. Consider the group $\mathrm{GL}_2(F)$. This exercise is to show that the commutator subgroup of $\mathrm{GL}_2(F)$ is $\mathrm{SL}_2(F)$. Define the following subgroups:

$$N(F) = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in F \right\} \text{ and } \overline{N}(F) = \left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \mid a \in F \right\}.$$

(1) Show that $N(F)$ and $\overline{N}(F)$ generate $\mathrm{SL}_2(F)$ by (verifying and then) using the following matrix identities:

$$\begin{aligned}
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \text{ Let } w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \\
\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} &= w^{-1} \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix} w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} w^{-1} \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix}, \quad x \neq 0; \\
\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}, \quad a \neq 0; \\
\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} w \begin{pmatrix} -c & -d \\ 0 & -c^{-1} \end{pmatrix}, \quad c \neq 0 \text{ and } ad - bc = 1.
\end{aligned}$$

(2) If $|F| > 2$ then show that the commutator subgroup of $\mathrm{GL}_2(F)$ contains $N(F)$ and $\overline{N}(F)$ by (verifying and then) using the following matrix identities:

$$\begin{aligned}
\begin{pmatrix} 1 & x^2 - 1 \\ 0 & 1 \end{pmatrix} &= \left[\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right], \quad x \neq 0; \\
\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} ty^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} ty^{-1} & 0 \\ 0 & 1 \end{pmatrix}^{-1}, \quad ty \neq 0; \\
\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} &= w \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} w^{-1};
\end{aligned}$$

(The commutator of a and b is denoted $[a, b] = aba^{-1}b^{-1}$.) Note that the hypothesis $|F| > 2$ assures us that there is an $x \neq 0$ such that $x^2 - 1 \neq 0$.

(3) Now show that if $|F| > 2$ then the commutator subgroup of $\mathrm{GL}_2(F)$ is $\mathrm{SL}_2(F)$. Hence show that $\mathrm{GL}_2(\mathbb{F}_q)$ has exactly $q-1$ distinct degree 1 representations if $q > 2$.
(4) Show that $\mathrm{GL}_2(\mathbb{F}_2) = \mathrm{SL}_2(\mathbb{F}_2) \simeq S_3$. Can you identify the commutator subgroup of $\mathrm{GL}_2(\mathbb{F}_2)$? How many degree 1 representations does $\mathrm{GL}_2(\mathbb{F}_2)$ have?

The matrix identities above do not come out of the blue; they really come from the theory of algebraic groups.

If G acts on a set X, then the corresponding *permutation representation* is denoted (π_X, V_X). The vector space V_X has a basis $\{e_x : x \in X\}$ and the action is $\pi_X(g)(e_x) = e_{gx}$. If we take $X = G$, and the action as left multiplication, then the permutation representation is called the *(left) regular representation.*

Exercise 18 (Regular representation)**.** Let G be a finite group. Equip G with the normalized counting measure, i.e., if $A \subset G$, then the measure of A is $|A|/|G|$. Since G is finite, any function on G is square integrable. Hence, $L^2(G)$ is the vector space of all complex valued functions on G. Given functions $f, g : G \to \mathbb{C}$, their L^2 inner product is simply

$$\langle f, g\rangle = \frac{1}{|G|}\sum_{x\in G} f(x)\overline{g(x)}.$$

(1) Define a representation π of G on $L^2(G)$ by: $(\pi(x)f)(y) = f(x^{-1}y)$ $\forall x, y \in G$, $\forall f \in L^2(G)$. Check that this is indeed a representation.
(2) Show that the above representation is equivalent to the left regular representation.
(3) Show that the trivial representation is a subrepresentation of $(\pi, L^2(G))$.
(4) Show that the L^2 inner product is G-invariant, i.e.,
$$\langle \pi(x)f, \pi(x)g\rangle = \langle f, g\rangle,\ \forall x \in G,\ \forall f, g \in L^2(G).$$
(5) Describe a G-complement of the trivial representation inside $L^2(G)$.

Exercise 19. Show that every finite group G has a faithful representation, i.e., there is a representation (π, V) such that homomorphism $\pi : G \to \mathrm{GL}(V)$ is injective. (*One can rephrase this as "every finite group is a linear group", meaning a subgroup of* $\mathrm{GL}_n(\mathbb{C})$ *for some* n.)

Exercise 20. Let $n > 1$. Show that the permutation representation of S_n, for the natural action of S_n on $\{1, 2, ..., n\}$, is not irreducible, by showing that the trivial representation is a subrepresentation.

Exercise 21. Let G be a finite group. Let χ_1 and χ_2 be two degree 1 representations. Let $\pi = \chi_1 \oplus \chi_2$ be their direct sum. Determine $\pi \otimes \pi$, $\mathrm{Sym}^2(\pi)$ and $\wedge^2(\pi)$.

Exercise 22. Let (π, V) be a representation of a finite group G. Define $\det(\pi)$ to be the degree one representation defined as $\det(\pi)(g) = \det(\pi(g))$. Suppose V is two dimensional, i.e., π is a degree two representation, then show that

$$\det(\pi) \simeq \wedge^2(\pi).$$

Exercise 23. Let (π, V) be a representation of G. Let (π^*, V^*) be the dual representation. Fix a basis $\{v_1, ..., v_n\}$ for V and let $\{f_1, ..., f_n\}$ be the dual basis, i.e, $f_i(v_j) = \delta_{i,j}$. Let $g \in G$. Show that with respect to these bases, we have

$$\pi^*_{i,j}(g) = \pi_{j,i}(g^{-1}).$$

In words, the matrix of $\pi^*(g)$ is the transpose–inverse of the matrix of $\pi(g)$.

Exercise 24. Let V and W be two finite dimensional vector spaces. Show that

$$\mathrm{Hom}_{\mathbb{C}}(V, W) \simeq V^* \otimes W.$$

Exercise 25. Suppose (π, V) and (ρ, W) are finite dimensional representations of a finite group G. Define a representation of G on $\mathrm{Hom}_{\mathbb{C}}(V, W)$ by the formula

$$(g \cdot \phi)(v) = \rho(g)\phi(\pi(g^{-1})v)$$

for all $g \in G$, $\phi \in \mathrm{Hom}_{\mathbb{C}}(V, W)$ and $v \in V$. Check that this indeed defines a representation. Denote this representation as $\mathrm{Hom}(\pi, \rho)$. Show that

$$\mathrm{Hom}(\pi, \rho) \simeq \pi^* \otimes \rho.$$

If you came up with a good isomorphism for the previous exercise, then that same isomorphism should give you an equivalence of these representations.

Exercise 26. Let V_1 and V_2 be two finite dimensional vector spaces. Show that

$$\mathrm{Sym}^2(V_1 \oplus V_2) \simeq \mathrm{Sym}^2(V_1) \oplus \mathrm{Sym}^2(V_2) \oplus (V_1 \otimes V_2).$$

Formulate and prove such a decomposition result for $\mathrm{Sym}^2(\pi_1 \oplus \pi_2)$ for two representations π_1 and π_2 of G.

Exercise 27. Let V_1 and V_2 be two finite dimensional vector spaces. Show that

$$\wedge^2(V_1 \oplus V_2) \simeq \wedge^2(V_1) \oplus \wedge^2(V_2) \oplus (V_1 \otimes V_2).$$

Formulate and prove such a decomposition result for $\wedge^2(\pi_1 \oplus \pi_2)$ for two representations π_1 and π_2 of G.

Exercise 28. Let V_1, V_2, W be three finite dimensional vector spaces. Show that

(1) $\mathrm{Hom}_{\mathbb{C}}(V_1 \oplus V_2, W) = \mathrm{Hom}_{\mathbb{C}}(V_1, W) \oplus \mathrm{Hom}_{\mathbb{C}}(V_2, W)$.
(2) $\mathrm{Hom}_{\mathbb{C}}(W, V_1 \oplus V_2) = \mathrm{Hom}_{\mathbb{C}}(W, V_1) \oplus \mathrm{Hom}_{\mathbb{C}}(W, V_2)$.
(3) Now suppose that all the three vector spaces are representation spaces for a finite group G. Then show that
   (a) $\mathrm{Hom}_G(V_1 \oplus V_2, W) = \mathrm{Hom}_G(V_1, W) \oplus \mathrm{Hom}_G(V_2, W)$.
   (b) $\mathrm{Hom}_G(W, V_1 \oplus V_2) = \mathrm{Hom}_G(W, V_1) \oplus \mathrm{Hom}_G(W, V_2)$.

In such problems, the isomorphisms that you cook up have to be *canonical*!

3. Character theory

Schur's lemma, character of a representation, matrix coefficients, Schur's orthogonality, Burnside's theorem, character table, isotypic components.

The character of a representation (π, V) is denoted χ_π. Recall that $\chi_\pi(g) = \mathrm{Tr}_V(\pi(g))$ where Tr_V is trace of an endomorphism of V. Given functions $f, g : G \to \mathbb{C}$, their L^2 inner product is simply

$$\langle f, g\rangle = \frac{1}{|G|}\sum_{x\in G} f(x)\overline{g(x)}.$$

Exercise 29. (Multiplicity of the trivial representation.) Let (π, V) be any representation of a group G. Let V^G be the subspace consisting of G-fixed vectors. Let $\mathbb{1}$ be the trivial representation of G. Show that the multiplicity of $\mathbb{1}$ in π is given by

$$\dim(\mathrm{Hom}_G(\mathbb{1}, \pi)) = \dim(V^G) = \langle \mathbb{1}, \chi_\pi\rangle.$$

Exercise 30. Let G be a finite group. Let π be a representation of G and let σ be an irreducible representation of G. Show that the multiplicity of σ in π is the dimension of $\mathrm{Hom}_G(\sigma, \pi)$. Show that π is irreducible if and only if $\dim(\mathrm{Hom}_G(\pi, \pi)) = 1$.

Exercise 31. If π is an irreducible representation of G which is self-dual ($\pi \simeq \pi^*$) then show that the trivial representation occurs exactly once in $\pi\otimes\pi$. (Since $\pi\otimes\pi = \mathrm{Sym}^2(\pi)\oplus\wedge^2(\pi)$, we say that π is *orthogonal* if $\mathbb{1}$ occurs in $\mathrm{Sym}^2(\pi)$ and we say π is *symplectic* if $\mathbb{1}$ occurs in $\wedge^2(\pi)$.)

Exercise 32. If π is an irreducible representation of a group G such that the trivial representation occurs in $\pi \otimes \pi$ then does π have to be self-dual?

Exercise 33. Let G be a finite group and let (π, V) be a representation of G. Let $\epsilon : G \to \mathbb{C}^*$ be a homomorphism. Show that $\chi_{\pi\otimes\epsilon}(g) = \epsilon(g)\chi_\pi(g)$. Show that π is irreducible if and only if $\pi \otimes \epsilon$ is irreducible.

Exercise 34. Let G_1 and G_2 be finite groups. Let $\{\pi_1, ..., \pi_h\}$ be all the irreducible mutually inequivalent representations of G_1 and let $\{\sigma_1, ..., \sigma_k\}$ be those of G_2. Let the degree of π_i be n_i and the degree of σ_j be m_j.

(1) Show that each $\pi_i \otimes \sigma_j$ is an irreducible representation of $G_1 \times G_2$.
(2) We know that $n_1^2 + \cdots + n_h^2 = |G_1|$ and $m_1^2 + \cdots + m_k^2 = |G_2|$. Hence show that

$$\sum_{i=1}^{h}\sum_{j=1}^{k} n_i^2 m_j^2 = |G_1 \times G_2|.$$

Deduce that every irreducible representation of $G_1 \times G_2$ is of the form $\pi_i \otimes \sigma_j$.

Exercise 35. ($L^2(G)$ as a $G \times G$-representation.) Let G be a finite group. Consider the space $L^2(G)$ of all functions on G. Consider the action of $G \times G$ on $L^2(G)$ given by

$$((x_1, x_2) \cdot \phi)(y) = \phi(x_1^{-1} y x_2), \quad \forall x_1, x_2, y \in G, \ \forall \phi \in L^2(G).$$

Let χ be the character of this representation of $G \times G$.

(1) Show that the character value $\chi(x, y)$ for $x, y \in G$ is given by

$$\chi(x, y) = \begin{cases} 0 & \text{if } y \notin \mathcal{O}(x), \\ |Z_G(x)| & \text{if } y \in \mathcal{O}(x). \end{cases}$$

(2) Let π_1 and π_2 be two representations of G. Show that

$$\langle \chi_{\pi_1 \otimes \pi_2}, \chi \rangle_{G \times G} = \langle \chi_{\pi_1}, \overline{\chi_{\pi_2}} \rangle_G.$$

The left hand side it is the inner product of functions on $G \times G$, and in the right hand side the inner product is on G.

(3) Conclude that as a $G \times G$-representation we have the decomposition

$$L^2(G) = \bigoplus_{\pi \in \widehat{G}} \pi \otimes \pi^*.$$

(Here π^* is the contragredient representation of π.)

Exercise 36. (Some classical results for group actions.) Let G be a finite group acting on a finite set X. For $g \in G$, and $x \in X$, define

$$\begin{aligned} X^g &:= \{x \in X : gx = x\}, \quad \text{the fixed point set of } g, \\ \mathrm{Stab}_G(x) &:= \{g \in G : gx = x\}, \quad \text{the stabilizer in } G \text{ of } x, \\ G \cdot x &:= \{gx : g \in G\}, \quad \text{the orbit of an element } x. \end{aligned}$$

(1) Show that $\sum_{x \in X} |\mathrm{Stab}_G(x)| = \sum_{g \in G} |X^g|$.
(*Hint: show that both sides are equal to* $|\{(g, x) \in G \times X : gx = x\}|$.)

(2) Show that $|G \cdot x| = |G| / |\mathrm{Stab}_G(x)|$.

(3) (Burnside) Show that the number of G-orbits in X is the average number of fixed points, i.e.,

$$\text{Number of } G\text{-orbits in } X = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

Exercise 37. (Permutation representation.) Let a finite group G act on a finite set X. Let (π_X, V_X) be the corresponding permutation representation. Show that the character of π_X is given by: $\chi_{\pi_X}(g) = |X^g|$. (In words, the character at g is the number of fixed points of g in X. This, according to Fulton-Harris [1], is *the original fixed point theorem.*) Show that the dimension of the subspace of G-fixed vectors in V_X is equal to the number of G-orbits of X, or in other words, the multiplicity of the trivial representation in π_X is the number of G-orbits in X. For every G-orbit explicitly construct a G-fixed vector in V. Show that if X has at least two elements, then π_X is not irreducible.

The next three exercises give a nice recipe to construct irreducible representations of a group G by considering doubly transitive actions. As applications of this recipe we will look at the *standard representation of* S_n as well as the *Steinberg representation of* $\mathrm{GL}_2(\mathbb{F}_q)$.

Exercise 38. Let a finite group G act on a finite set X. Consider the action of G on $X \times X$ given by $g \cdot (x, y) = (g \cdot x, g \cdot y)$. Show that $\pi_{X \times X} \simeq \pi_X \otimes \pi_X$. *(Hint: Calculate the character of both sides.)*

Exercise 39. Let a finite group G act on a finite set X. The action is said to be doubly transitive if for every $x, y, z, t \in X$ with $x \neq y$ and $z \neq t$, there is a $g \in G$ such that $g \cdot x = z$ and $g \cdot y = t$. Show that the action of G on X is doubly transitive if and only if G has two orbits in $X \times X$, namely the diagonal $\{(x, x) \mid x \in X\}$ and its complement in $X \times X$.

Exercise 40. Let a finite group G act transitively on a finite set X. Show that the trivial representation occurs once in π_X. Let $\pi_X = \mathbb{1} \oplus \sigma_X$. Now suppose that the action is doubly transitive, then show that σ_X is irreducible. *(Hint: Consider the representation $\pi_{X \times X}$ and conclude that the trivial representation occurs once in $\sigma_X \otimes \sigma_X$. In terms of characters this means that $\langle \mathbb{1}, \chi^2_{\sigma_X} \rangle = 1$. Observe that χ_{σ_X} takes on only integer values, and hence only real values. Conclude that $\langle \chi_{\sigma_X}, \chi_{\sigma_X} \rangle = \langle \mathbb{1}, \chi^2_{\sigma_X} \rangle = 1$.)*

Exercise 41. Let $n \geq 2$. Consider the natural action of S_n on $\{1, 2, ..., n\}$. Let π be the corresponding permutation representation. Show that $\pi = \mathbb{1} \oplus \sigma$ and σ is irreducible. Hence, S_n always has an irreducible representation of degree $n - 1$. This representation σ is called the *standard representation* of S_n.

Exercise 42. Consider the natural action of $\mathrm{GL}_2(\mathbb{F}_q)$ on $\mathbb{P}^1(\mathbb{F}_q)$. Let π be the corresponding permutation representation. Show that $\pi = \mathbb{1} \oplus \sigma$ and σ is irreducible. This representation σ is called the *Steinberg representation* of $\mathrm{GL}_2(\mathbb{F}_q)$. Show that the Steinberg representation has degree q.

Exercise 43 (Central Character)**.** Let G be a finite group and let Z be its center. Let (π, V) be an irreducible representation of G. Let $z \in Z$. Show that $\pi(z) \in \mathrm{Hom}_G(\pi, \pi)$. Conclude using Schur's lemma that for each $z \in Z$, there is a scalar, denoted $\omega_\pi(z)$, such that $\pi(z) = \omega_\pi(z) 1_V$. Show that ω_π is a homomorphism from Z to $\mathbb{C}^*$. It is called the *central character* of the irreducible representation π.

Exercise 44. Let π be an irreducible representation of a finite group G. Let the central character of π be ω_π. Show that the representations π^*, $\pi \otimes \pi$, $\mathrm{Sym}^2(\pi)$ and $\wedge^2(\pi)$ all admit central characters. Describe these central characters in terms of ω_π.

Exercise 45 (Schur's Lemma for Simple modules)**.** Let R be a commutative ring with identity. Let M be a simple R-module. Recall that M is said to be simple if it has no proper nontrivial submodule. Consider the ring $\mathrm{End}_R(M)$ of endomorphisms of M. (An endomorphism of M is any map $f : M \to M$ such that $f(r_1 m_1 + r_2 m_2) = r_1 f(m_1) + r_2 f(m_2)$ for all $r_i \in R$ and all $m_i \in M$. The set of all endomorphisms forms a ring under usual addition and composition.) Show that the ring $\mathrm{End}_R(M)$ is a division ring, i.e., every nonzero endomorphism of M is invertible.

Show by an example that the converse need not be true, i.e., there is an R-module M which is not simple but for which $\mathrm{End}_R(M)$ is a division ring.

Exercise 46. Show that if G is a group which has an abelian subgroup of index m, then any irreducible representation of G has degree at most m. Conclude that an irreducible representation of a dihedral group has degree at most 2.

Exercise 47. Let G be a finite group and let χ denote the character of the regular representation of G. Show that

$$\chi(g) = \begin{cases} |G| & \text{if } g = 1, \\ 0 & \text{if } g \neq 1. \end{cases}$$

In words, the character of the regular representation is the order of the group times the Dirac delta function at the identity.

Exercise 48. Let G be a finite group and let $\{\pi_1, ..., \pi_h\}$ be all the irreducible, mutually inequivalent, representations of G. Let d_{π_i} denote the degree of π_i. Let $1 \neq g \in G$. Show that

$$\sum_{i=1}^{h} d_{\pi_i} \chi_{\pi_i}(g) = 0.$$

Exercise 49. Let G be a finite group and let $\{\pi_1, ..., \pi_h\}$ be all the irreducible, mutually inequivalent, representations of G. Recall that the characters $\{\chi_{\pi_1}, ..., \chi_{\pi_h}\}$ forms an orthonormal basis for the space $C(G)$ of class functions on G. Let $x \in G$. Consider the class function f_x defined by: f_x takes the value 1 on the conjugacy class $\mathcal{O}(x)$ of x and vanishes outside $\mathcal{O}(x)$. (So f_x is the characteristic function of the conjugacy class of x.)

(1) Let $f_x = \sum_{i=1}^{h} a_i \chi_{\pi_i}$. Then show that $a_i = \frac{|\mathcal{O}(x)| \overline{\chi_{\pi_i}(x)}}{|G|} = \frac{\overline{\chi_{\pi_i}(x)}}{|Z_G(x)|}$.

(2) Show that

$$\sum_{i=1}^{h} |\chi_{\pi_i}(x)|^2 = \frac{|G|}{|\mathcal{O}(x)|} = |Z_G(x)|.$$

(3) If x and y are not conjugate then show that

$$\sum_{i=1}^{h} \chi_{\pi_i}(x) \overline{\chi_{\pi_i}(y)} = 0.$$

Observe that this exercise is a playoff between two bases for $C(G)$. First is the set of characters of irreducible representations of G, and the second is the set of characteristic functions of conjugacy classes. Is the second one an orthonormal basis? If not, modify it to make it orthonormal.

Exercise 50 (Finite abelian groups)**.** Let G be a finite abelian group. Let $\widehat{G}$ denote the set of all homomorphisms $\chi : G \to \mathbb{C}^*$. Then show that $\widehat{G}$ is a finite abelian group, with the product of χ_1 and χ_2 being the homomorphism $g \mapsto \chi_1(g)\chi_2(g)$. Consider the map $G \to \widehat{\widehat{G}}$ given by $g \mapsto e_g$ where $e_g(\chi) = \chi(g)$. Show that this gives an isomorphism between G and $\widehat{\widehat{G}}$. Show that this isomorphism is *natural*. Use the structure theory of finite abelian groups to show that G is isomorphic to $\widehat{G}$–however this isomorphism is not natural. (This is the analogue of a finite dimensional vector V space being naturally isomorphic to its double dual V^{**}. Note that V is also isomorphic to V^*–but only after fixing bases for V and V^*–however, the isomorphism is not natural.)

Exercise 51. Let G be a finite group. Suppose we know all but one rows of the character table of G, explain how you can calculate all the entries of the remaining row. (We follow the convention that rows are parametrized by characters of irreducible representations of G, and the columns are parametrized by conjugacy classes of G.)

Exercise 52. Let G be a finite group. Consider one row of the character table of G. Suppose you know all but one entries in this row, explain how you can calculate the remaining entry.

Exercise 53. Let π be an irreducible representation of a finite group G. If the degree of π is at least 2, then show that its character χ_π vanishes at at least one element of G. *(Hint: First do this in the case all character values are integers. Next, do a "Galois averaging" to reduce to the integral case.)*

Exercise 54. Let (π_1, V) be a representation of G_1 and (π_2, V) be a representation of G_2. (Both have the same representation space.) Assume that the action of G_1 on V commutes with action of G_2 on V, i.e.,

$$\pi_1(g_1)\pi_2(g_2) = \pi_2(g_2)\pi_1(g_1), \quad \forall g_1 \in G_1,\ \forall g_2 \in G_2.$$

Show that every isotypic component of G_1 in V is G_2-stable, and similarly every isotypic component of G_2 in V is G_1-stable.

4. Induced representation

Definition(s) of an induced representation, Frobenius reciprocity, Mackey theory, character of an induced representation, subgroups of index 2.

Exercise 55 (Some exercises on double cosets)**.** (1) Let $H, K < G$. Show that

$$|HgK| = \frac{|H||K|}{|H \cap gKg^{-1}|} = \frac{|H||K|}{|g^{-1}Hg \cap K|}.$$

(2) If $H \lhd G$ then $HgH = Hg = gH$ for all $g \in G$.

(3) If $H < G$ then show that there is a set of elements $z_1, ..., z_r \in G$, $r = [G : H]$, which are representatives for both the set of right cosets and also the set of left cosets, i.e., G is the disjoint union of the Hz_i and also of the z_iH. *(Hint: Consider both left and right cosets inside the double coset HgH and find a common set of representatives for these cosets. Doing this for every double coset does the job. Look this up in Jacobson's Basic Algebra* [3]*.)*

Exercise 56. Let (π, V) be a representation of a group G. Let $g \in G$. Show that the representation (π^g, V) is equivalent to (π, V), where π^g is defined by: $\pi^g(x) = \pi(g^{-1}xg)$ for all $x \in G$.

Exercise 57. Give an example of a group G, a normal subgroup H, an irreducible representation π of H, an element $g \in G$, such that π^g is not equivalent to π.

Exercise 58. Let H be a subgroup of a group G. Show that $\mathrm{Ind}_H^G(\mathbb{1})$ is equivalent to the permutation representation of G for the natural action of G on $H\backslash G$. (Here $H\backslash G$ is the space of right cosets.) Show that if H is a proper subgroup then $\mathrm{Ind}_H^G(\mathbb{1})$ is not irreducible, by showing that the trivial representation of G is a subrepresentation. What is the multiplicity of the trivial representation in $\mathrm{Ind}_H^G(\mathbb{1})$.

Exercise 59. Let H be a subgroup of a finite group G. Observe that the action of G on $H\backslash G$ is transitive. Use Mackey theory to show that the action of G on $H\backslash G$ is doubly transitive if and only if G has exactly two (H, H)-double cosets.

Exercise 60. Let S_n be the permutation group on n letters. Let S_{n-1} be the subgroup fixing one of these n letters. Show that $\mathrm{Ind}_{S_{n-1}}^{S_n}(\mathbb{1})$ is the direct sum of the trivial representation and the standard representation of S_n.

Exercise 61 (Formula for character of an induced representation)**.** Let H be a subgroup of a finite group G. Let (σ, W) be a representation of H. Let χ be the character of the induced representation $\mathrm{Ind}_H^G(\sigma)$. Show that

$$\chi(g) \;=\; \frac{1}{|H|} \sum_{t\in G,\; t^{-1}gt\in H} \chi_\sigma(t^{-1}gt) \;=\; \frac{|Z_G(g)|}{|H|} \sum_{x\in H\cap\mathcal{O}_G(g)} \chi_\sigma(x).$$

This formula, due to Frobenius, expresses the character of an induced representation in terms of the character of the inducing representation. *(Hint: Construct explicitly a basis $\{f_{i,j}\}$ for the induced representation starting from a basis of $\{w_i\}$ of W and a set of coset representatives $G = \cup Hx_j$. Now calculate the trace for the action of an element of $g \in G$ in the induced space using this basis.)*

Exercise 62. Consider the dihedral group D_n. (Recall our notation that D_n is generated by r and s, with $r^n = s^2 = 1$ and $srs = r^{-1}$.) Let C_n denote the cyclic subgroup generated by r. For $0 \leq m \leq n-1$, define the character $\theta_m : C_n \to \mathbb{C}^*$ by $\theta_m(r) = e^{2\pi i m/n}$. Describe the decomposition of the representation

$$\mathrm{Ind}_{C_n}^{D_n}(\theta_m).$$

(Hint: Show that this induced representation is irreducible unless $\theta_m^2 = \mathbb{1}$. This condition is satisfied exactly when $m = 0$, or if n is even and $m = n/2$; and in these cases the induced representation breaks up into two pieces; now identify these two pieces.)

Exercise 63. Let H be a subgroup of index 2 in G. Then H is necessarily a normal subgroup. Let ϵ be the nontrivial character of G which factors through G/H:

$$\epsilon : G \to G/H \simeq \{\pm 1\} \subset \mathbb{C}^*.$$

Let π be an irreducible representation of G.

(1) If $\pi \simeq \pi \otimes \epsilon$ then show that $\pi|_H = \pi_1 \oplus \pi_2$, where each π_i is an irreducible representation of H, and $\pi_2 \simeq \pi_1^g$ for any $g \in G - H$,
(2) If $\pi \not\simeq \pi \otimes \epsilon$, then π restricted to H is irreducible.

5. Group algebra, integrality and rationality

Group algebra, semisimplicity, fields other than $\mathbb{C}$, degree of an irreducible representation divides order of the group.

Exercise 64. Show that the group algebra $\mathbb{C}[G]$ is isomorphic to the convolution algebra $L^1(G)$, by exhibiting explicit isomorphisms in both directions. (For $L^1(G)$, as with $L^2(G)$, we equip G with the counting measure.)

Exercise 65 (Non-algebraically closed fields)**.** Let $\mathrm{SO}(2)$ denote the special orthogonal group of order 2, defined as the group of all 2-by-2 matrices A with real entries such that $A \cdot {}^tA = I$ and $\det(A) = 1$. Show that

$$\begin{aligned}\mathrm{SO}(2) &= \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R},\ a^2 + b^2 = 1 \right\} \\ &= \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} :\ \theta \in \mathbb{R} \right\} \simeq S^1\end{aligned}$$

where S^1 is the *circle group* of all complex numbers of absolute value 1.

Let μ_n denote the group of n-th roots of unity. Consider the two dimensional $\mathbb{R}$-representation given by the inclusions

$$\mu_n \subset \mathrm{SO}(2) \subset \mathrm{GL}_2(\mathbb{R}).$$

If $n \geq 3$ then show that this representation of μ_n is irreducible. *Moral: Over a non-algebraically closed field, such as $\mathbb{R}$, an abelian group, such as μ_n, can have an irreducible representation of degree bigger than* 1.

Exercise 66 (Fields of positive characteristics)**.** Let p be a prime. Let G be the additive group of the finite field $\mathbb{F}_p$ of p elements. Consider the two dimensional $\mathbb{F}_p$-representation of G given by

$$x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_p).$$

Check that it is indeed a representation. Let $V = \mathbb{F}_p^2$ be the representation space. Show that V has a unique one dimensional subspace stable under G. Hence this one dimensional subspace can not possibly have a G-complement. *Moral: Over a field of positive characteristic, such as $\mathbb{F}_p$, the complete reducibility theorem need not be true.*

Exercise 67 (Infinite groups)**.** Let G be the additive group of real numbers $\mathbb{R}^+$. Consider the two dimensional representation of G given by

$$t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}).$$

Check that it is indeed a representation. Let $V = \mathbb{C}^2$ be the representation space. Show that V has a unique one dimensional subspace stable under G. Hence this one dimensional subspace can not possibly have a G-complement. *Moral: For an infinite group, such as $\mathbb{R}^+$, the complete reducibility theorem need not be true.*

Exercise 68. Let $M_d(\mathbb{C})$ be the algebra of all d-by-d matrices over $\mathbb{C}$. Show that the center of this algebra consists of all scalar matrices. Hence $Z(M_d(\mathbb{C})) \simeq \mathbb{C}$.

Let $\mathrm{GL}_d(\mathbb{C})$ be the group of all d-by-d invertible matrices over $\mathbb{C}$. Show that the center of this group consists of all nonzero scalar matrices. Hence $Z(\mathrm{GL}_d(\mathbb{C})) \simeq \mathbb{C}^*$.

Exercise 69. Show that $A = M_d(\mathbb{C})$ is a simple algebra, by showing that a nonzero two sided ideal is necessarily all of A. One can see simplicity of A also as: For $1 \leq k \leq d$, if I_k consists of all matrices with arbitrary entries in the k-th column and zeros elsewhere, then show that I_k is a simple left ideal. Show that all the I_k's are isomorphic as left ideals of A. Show that any simple left ideal of A is isomorphic to I_k for some $1 \leq k \leq d$.

Exercise 70. Let $(\pi_1, V_1), ..., (\pi_h, V_h)$ be all the irreducible representations of a finite group G. For each i, let $\widetilde{\pi_i} : \mathbb{C}[G] \to \mathrm{End}(V_i)$ be the corresponding algebra homomorphism. Use Schur's lemma to show that $\widetilde{\pi_i}(Z(\mathbb{C}[G])) = \mathbb{C}1_{V_i}$. Conclude that

$$\widetilde{\pi} = \prod_{i=1}^{h} \widetilde{\pi_i} : Z(\mathbb{C}[G]) \to \prod_{i=1}^{h} \mathbb{C}$$

is an isomorphism from the center of the group algebra to h-copies of $\mathbb{C}$.

Exercise 71. Let G be a group of order p^2. Use the fact that the degree of an irreducible representation divides the order of the group to show that every irreducible representation of G is necessarily one dimensional. Hence conclude that G must be abelian.

Exercise 72. Let G be a finite group and let Z be its center. Let (π, V) be an irreducible representation of G of degree d_π. The following series of exercises is due to John Tate which shows that the degree of π divides the index $[G : Z]$ of the center in the whole group.

(1) Consider $\pi^{(m)}$ defined as $\pi \otimes \pi \otimes \cdots \otimes \pi$ (m times). Observe that it is irreducible as a representation of $G \times \cdots \times G$.
(2) The representation $\pi^{(m)}$ is trivial on the subgroup
$$\{(z_1, ..., z_m) \in Z \times \cdots \times Z \mid z_1 z_2 ... z_m = 1\}.$$
Hence conclude that d_π^m divides $|G|^m/|Z|^{m-1}$.
(3) Conclude that d_π divides $|G|/|Z| = [G : Z]$.

6. Concrete examples: S_n and A_n (for small n)

Character tables of A_3, S_3, A_4, S_4, A_5 and S_5.

Exercise 73. Let G be S_3, A_4 or S_4. Let π and π' be two irreducible representations of G. Using the character table of G, verify that $\langle \chi_\pi, \chi_{\pi'} \rangle = 1$ or 0 according as π and π' are equivalent or not. At least do this for enough cases to get a feel for what this exercise entails.

Exercise 74. Show that there is only one way to solve
$$24 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2$$
with $x_1 = 1$ and $x_2, ..., x_5 \geq 1$ are all integers. (This exercise gives the degrees of irreducible representations of S_4 once we know that S_4 has five conjugacy classes.)

Exercise 75. Let τ be the unique degree 2 irreducible representation of S_4. Let σ be the standard representation of S_4 and let ϵ be the sign homomorphism of S_4. Using the character tables of S_4 and A_4,

(1) Describe the decomposition of τ as a representation of A_4.
(2) Show that $\tau \simeq \tau \otimes \epsilon$ as S_4-representations.
(3) Show that $\sigma \simeq \sigma \otimes \epsilon$ as A_4-representations. However they are inequivalent as S_4-representations.

Exercise 76. Let σ be the standard representation of S_n. For $n \geq 4$ show that σ restricted to A_n is irreducible. *(Hint: Show that the action of A_n on $\{1, 2, ..., n\}$ is $(n-2)$-transitive. Hence for $n \geq 4$ it is doubly transitive.)*

Exercise 77. Let σ be the standard representation of S_n and let ϵ be the sign homomorphism. If $n \geq 4$, then show that σ and $\sigma \otimes \epsilon$ are inequivalent as representations of S_n, but are equivalent as representations of A_n.

Exercise 78. Let σ be the standard representation of S_5. Describe the decomposition of $\mathrm{Sym}^2(\sigma)$ into irreducible representations. *(Hint: Use character theory to show that* $\mathrm{Sym}^2(\sigma)$ *is a direct sum of three distinct irreducible representations; now show that the trivial representation and* σ *both occur in* $\mathrm{Sym}^2(\sigma)$*; finally, identify the third piece using the character table.)*

Exercise 79. Let σ be the standard representation of S_5. Show that the representation $\wedge^2(\sigma)$ is irreducible as an S_5 representation, and that it decomposes into the direct sum of two irreducible representations (ρ_1 and ρ_2, say) upon restricting to A_5. How can we calculate the character values of ρ_1 and ρ_2?

Exercise 80. Let $G = S_n$ for $n = 3, 4$ or 5. Let π be an irreducible representation of G. Use the character table of G to show that π is self-dual, i.e., $\pi \simeq \pi^*$. *(Hint: Observe that the character values are all integers, and hence real numbers.)* If you know enough about the representation theory of S_n, do this exercise for all n. The character values are always integers.

Exercise 81. Give an example of an irreducible representation of A_n for some n for which the character values are not all integers.

Exercise 82. Let τ be one of the degree 5 irreducible representation of S_5. Show that τ remains irreducible on restricting to A_5. Write down the character values of $\tau|_{A_5}$.

7. $\mathrm{GL}_2(\mathbb{F}_q)$

Interesting subgroups, conjugacy classes, degree one representations, principal series representations, Steinberg representation, cuspidal representations.

Exercise 83. Consider the embedding of $\mathbb{C}$ into $M_{2\times 2}(\mathbb{R})$ given by

$$x + iy \mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

Show that this is a homomorphism of $\mathbb{R}$-algebras.

Exercise 84. Let $F = \mathbb{F}_q$. Let E/F be a degree n extension. Show that we can embed E^* into $\mathrm{GL}_n(\mathbb{F}_q)$. Given two elements $x, y \in E^*$, analyze when they are conjugate in $\mathrm{GL}_n(\mathbb{F}_q)$. *The image of* E^* *in* $\mathrm{GL}_n(\mathbb{F}_q)$ *is called an anisotropic torus.*

Exercise 85. Let $F = \mathbb{F}_q$. Let E/F be the quadratic extension. Let $\theta \in E$ such that $E = F[\theta]$. Let $s = \mathrm{Trace}_{E/F}(\theta)$ and let $\delta = \mathrm{Norm}_{E/F}(\theta)$. Show that

$$x + \theta y \mapsto \begin{pmatrix} x & -y\delta \\ y & x + ys \end{pmatrix}$$

is an embedding $E \to M_{2\times 2}(F)$ of F-algebras.

Exercise 86 (Bruhat decomposition)**.** Let B be the standard Borel subgroup of $G = \mathrm{GL}_2(F)$ consisting of upper triangular matrices. Show that

$$G = B \cup BwB, \quad \text{the union is disjoint.}$$

Show that any element in the double coset BwB may be uniquely expressed as an element of BwN. (Here N is the subgroup of B consisting of all elements with 1s on the diagonal and $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ is the Weyl group element.)

Exercise 87. Study the representation theory of the *Shalika subgroup* of $\mathrm{GL}_2(\mathbb{F}_q)$ which is defined as:

$$S = \left\{ \left(\begin{array}{cc} a & b \\ 0 & a \end{array} \right) \mid a \in \mathbb{F}_q^*, b \in \mathbb{F}_q \right\}.$$

Show that S is abelian and write down all the $q(q-1)$ degree 1 representations of S.

Exercise 88. Study the representation theory of the *mirabolic subgroup* of $\mathrm{GL}_2(\mathbb{F}_q)$ which is defined as:

$$M = \left\{ \left(\begin{array}{cc} a & b \\ 0 & 1 \end{array} \right) \mid a \in \mathbb{F}_q^*, b \in \mathbb{F}_q \right\}.$$

Show that M has $(q-1)$ degree 1 representations and one irreducible representation of degree $(q-1)$. This latter may be constructed as: Take a nontrivial character $\psi : N \to \mathbb{C}^*$. Show that $\mathrm{Ind}_N^M(\psi)$ is irreducible, using Mackey's criterion; the same proof will also show that $\mathrm{Ind}_N^M(\psi)$ is independent of ψ, as long as ψ is nontrivial.

Exercise 89. Study the representation theory of the *Borel subgroup* B of $\mathrm{GL}_2(\mathbb{F}_q)$. Let Z be the center of B. Show that $B = ZM$. Using Schur's lemma, show that the study of irreducible representations of B reduces to that of M. Show that B has $(q-1)^2$ degree 1 representations and has $(q-1)$ irreducible representations each of degree $(q-1)$.

Exercise 90. Let T be the subgroup of $G = \mathrm{GL}_2(\mathbb{F}_q)$ consisting of all diagonal matrices. Describe the normalizer of T in G. Show that $N_G(T)/T = \mathbb{Z}/2\mathbb{Z}$.

Exercise 91. Show that the character of an irreducible principal series representation of $G = \mathrm{GL}_2(\mathbb{F}_q)$ vanishes on any element conjugate to an element of the anisotropic torus.

Exercise 92. Let B be the standard Borel subgroup of $G = \mathrm{GL}_2(\mathbb{F}_q)$. Let χ, χ_1 and χ_2 be characters of $\mathbb{F}_q^*$. Show that

$$\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2) \otimes \chi \simeq \mathrm{Ind}_B^G(\chi_1\chi \otimes \chi_2\chi).$$

(There is a more general statement one can prove: Let H be a subgroup of a finite group G. Let π be a representation of G and let σ be a representation of H. Prove that $\mathrm{Ind}_H^G(\sigma) \otimes \pi \simeq \mathrm{Ind}_H^G(\sigma \otimes \pi|_H)$. In words, one can say that tensoring commutes with induction.)

Exercise 93. Write down the character values of the Steinberg representation of $\mathrm{GL}_2(\mathbb{F}_q)$. *(Hint: The Steinberg representation is obtained from cutting out the trivial representation from the permutation representation for the action of* $\mathrm{GL}_2(\mathbb{F}_q)$ *on* $\mathbb{P}^1(\mathbb{F}_q)$*.)*

Exercise 94. (1) Let G be any finite group and let H be a subgroup. Let $g \in G$. Let σ be a representation of H. Show that $\mathrm{Ind}_H^G(\sigma)^g \simeq \mathrm{Ind}_{gHg^{-1}}^G(\sigma^g)$, i.e., conjugating commutes with inducing.
(2) Let $\overline{B}$ be the subgroup of all lower triangular matrices in $G = \mathrm{GL}_2(\mathbb{F}_q)$. In constructing the principal series representations, show that it does not matter whether we use B or $\overline{B}$.

Exercise 95. Determine the central characters of all the irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$. If π is an irreducible representation with central character ω_π, and $\chi : \mathbb{F}_q^* \to \mathbb{C}^*$ is a homomorphism, then show that the central character of $\pi \otimes \chi$ is $\omega_\pi \chi^2$. (Here $\pi \otimes \chi$ is the representation $g \mapsto \chi(\det(g))\pi(g)$.)

Exercise 96. Let π be an irreducible representation of $G = \mathrm{GL}_2(\mathbb{F}_q)$. Let π^* be the dual (also called the contragredient) of π. Define a representation π' of G by the formula $\pi'(g) = \pi({}^t g^{-1})$. Show that $\pi' \simeq \pi^*$. *(Hint: The transpose of a matrix is conjugate to the matrix itself.)*

Exercise 97. Let π be an irreducible representation of $G = \mathrm{GL}_2(\mathbb{F}_q)$. Let ω_π be the central character of π. Show that $\pi^* \simeq \pi \otimes \omega_\pi^{-1}$. *(Hint: Use the matrix identity:*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} (ad-bc)^{-1} & 0 \\ 0 & (ad-bc)^{-1} \end{pmatrix} w \begin{pmatrix} a & c \\ b & d \end{pmatrix} w^{-1}$$

where w *is the Weyl group element.)*

Exercise 98. Let N be the subgroup of $G = \mathrm{GL}_2(\mathbb{F}_q)$, consisting of all upper triangular elements with 1s on the diagonal. Show that an irreducible representation of G is not cuspidal if and only if it has a nonzero vector fixed by N. *(Hint: You can use character theory for this. You can also do this using Frobenius reciprocity.)*

Exercise 99. Let ψ be a nontrivial character of N. Let π be an irreducible representation of $G = \mathrm{GL}_2(\mathbb{F}_q)$ of degree > 1. Show that π occurs exactly once in the induced representation $\mathrm{Ind}_N^G(\psi)$. *This result is called multiplicity one for Whittaker models. (Hint: Use Frobenius reciprocity, degree considerations, and character theory over* N*.)*

Exercise 100. Show that the restriction to the Mirabolic subgroup M of any cuspidal representation of $\mathrm{GL}_2(\mathbb{F}_q)$ is the unique irreducible representation of degree $q-1$. *This result is part of Kirillov theory. (Hint: Frobenius reciprocity and comparison of degrees.)*

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References

[1] W. Fulton and J. Harris, *Representation theory.* A first course. Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer-Verlag, New York, 1991.
[2] M. Isaacs, *Character theory of finite groups.* Dover Publications, Inc., New York, 1994.
[3] N. Jacobson, *Basic algebra. Volumes I and II.* Second edition. W. H. Freeman and Company, New York, 1985 and 1989.
[4] S. Lang, *Algebra.* Revised third edition. Graduate Texts in Mathematics, 211. Springer-Verlag, New York, 2002.
[5] J.-P. Serre, *Linear representations of finite groups.* Graduate Texts in Mathematics, Vol. 42. Springer-Verlag, New York-Heidelberg, 1977.

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