

# Notes on the Vershik-Okounkov approach to the representation theory of the symmetric groups

## 1 Introduction

These notes<sup>1</sup> contain an expository account of the beautiful new approach to the complex finite dimensional irreducible representations of the symmetric group, developed by Anatoly Vershik and Andrei Okounkov [4].

The main task of any representation theory of the symmetric groups is to explain the appearance of Young diagrams and Young tableaux in the theory in a natural way. The traditional approach (see [1, 2, 3]) is indirect and rests upon nontrivial auxiliary constructions. The presence of Young tableaux in the theory is justified only in the end, after the proof of the branching rule. The main steps of the Vershik-Okounkov approach are as follows:

- (i) A direct elementary argument shows that branching from  $S_n$  to  $S_{n-1}$  is simple, i.e., multiplicity free.
- (ii) Consider an irreducible  $S_n$ -module  $V$ . Since the branching is simple the decomposition of  $V$  into irreducible  $S_{n-1}$ -modules is canonical. Each of these modules, in turn, decompose canonically into irreducible  $S_{n-2}$ -modules. Iterating this construction we get a canonical decomposition of  $V$  into irreducible  $S_1$ -modules, i.e., one dimensional subspaces. Thus there is a canonical basis of  $V$ , determined upto scalars, and called the the Gelfand-Tsetlin basis (GZ-basis).

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(iii) Let  $Z_n$  denote the center of  $\mathbb{C}S_n$ . The Gelfand-Tsetlin algebra (GZ-algebra), denoted  $GZ_n$ , is defined to be the (commutative) subalgebra of  $\mathbb{C}S_n$  generated by  $Z_1 \cup Z_2 \cup \cdots \cup Z_n$ . It is shown that  $GZ_n$  consists of all elements in  $\mathbb{C}S_n$  that act diagonally in the GZ-basis in every irreducible representation. Thus  $GZ_n$  is a maximal commutative subalgebra of  $\mathbb{C}S_n$  and its dimension is equal to the sum of dimensions of the distinct inequivalent irreducible  $S_n$ -modules. It follows that any vector in the GZ-basis (in any irreducible representation) is uniquely determined by the eigenvalues of the elements of the GZ-algebra on this vector.

(iv) For  $i = 1, 2, \dots, n$  define  $X_i = (1, i) + (2, i) + \cdots + (i-1, i) \in \mathbb{C}S_n$ . The  $X_i$ 's are called the Young-Jucys-Murphy elements (YJM-elements) and it is shown that they generate the GZ-algebra. To a GZ-vector  $v$  (i.e., an element of the GZ-basis in some irreducible representation) we associate the tuple  $\alpha(v) = (a_1, a_2, \dots, a_n)$ , where  $a_i =$  eigenvalue of  $X_i$  on  $v$  and we let

$$\text{Spec}(n) = \{\alpha(v) : v \text{ is a GZ-vector}\}.$$

It follows from step (iii) that, for GZ-vectors  $u$  and  $v$ ,  $u = v$  iff  $\alpha(u) = \alpha(v)$  and thus  $\#\text{Spec}(n)$  is equal to the sum of the dimensions of the distinct irreducible inequivalent representations of  $S_n$ .

(v) In the final step we construct a bijection between  $\text{Spec}(n)$  and  $\text{Tab}(n)$  (= set of all standard Young tableaux on the letters  $\{1, 2, \dots, n\}$ ) such that tuples in  $\text{Spec}(n)$  whose GZ-vectors belong to the same irreducible representation go to standard Young tableaux of the same shape. This step is carried out inductively using an elementary analysis of the commutation relation

$$s_i^2 = 1, \quad X_i X_{i+1} = X_{i+1} X_i, \quad s_i X_i + 1 = X_{i+1} s_i.$$

where  $s_i =$  the Coxeter generator  $(i, i+1)$ .

For more detailed explanations about the underlying philosophy of this method as well as many remarks on extensions and analogies

with Lie theory, see the original papers [4, 5]. Our exposition closely follows [4] (we have added a few details here and there). In these notes we have restricted ourselves to only the most basic result, namely that the branching graph of the chain of symmetric groups is the Young graph. For the further development of the theory, see [4, 5].

### *Notation*

A *partition* is a finite sequence  $\mu = (\mu_1, \mu_2, \dots, \mu_r)$  of weakly decreasing positive integers. We say that  $\mu$  has  $r$  parts and that  $\mu$  is a partition of  $n = \mu_1 + \dots + \mu_r$ . Let  $\mathcal{P}(n)$  denote the set of all partitions of  $n$  and let  $\mathcal{P}_1(n)$  denote the set of all pairs  $(\mu, i)$ , where  $\mu$  is a partition of  $n$  and  $i$  is a part in  $\mu$ . A part is nontrivial if it is  $\geq 2$  and we let  $\#\mu$  denote the sum of the nontrivial parts of  $\mu$ . The number of all partitions of  $n$  is denoted  $p(n)$ .

Conjugacy classes in  $S_n$  are parametrized by partitions of  $n$ . Given a partition  $\mu \in \mathcal{P}(n)$ , let  $c_\mu \in \mathbb{C}S_n$  denote the sum of all partitions in  $S_n$  with cycle type  $\mu$ . It is well known that  $\{c_\mu : \mu \in \mathcal{P}(n)\}$  is a basis of the center of  $\mathbb{C}S_n$ .

For a permutation  $s \in S_n$  we denote by  $\ell(s)$  the number of inversions in  $s$ . It is well known that the  $s$  can be written as a product of  $\ell(s)$  Coxeter transpositions  $s_i = (i, i+1)$ ,  $i = 1, 2, \dots, n-1$  and that  $s$  cannot be written as a product of fewer Coxeter transpositions.

All our algebras are finite dimensional, over  $\mathbb{C}$ , and have units. Subalgebras contain the unit, and algebra homomorphisms preserve units. Given elements or subalgebras  $A_1, A_2, \dots, A_n$  of an algebra  $A$  we denote by  $\langle A_1, A_2, \dots, A_n \rangle$  the subalgebra of  $A$  generated by  $A_1 \cup A_2 \cup \dots \cup A_n$ .

## 2 Gelfand-Tsetlin bases and Gelfand-Tsetlin algebras

In this section we shall introduce Gelfand-Tsetlin bases (GZ bases) and Gelfand-Tsetlin algebras (GZ algebras) for an inductive chain of finite groups with simple branching.

Let

$$\{1\} = G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n \subseteq \cdots \quad (1)$$

be an inductive chain of finite groups. Denote by  $G_n^\wedge$  the set of equivalence classes of finite dimensional complex irreducible representations of  $G_n$ . Denote by  $V^\lambda$  the irreducible  $G_n$  module corresponding to  $\lambda \in G_n^\wedge$ . Define the following directed graph, called the *branching multigraph* or *Bratelli diagram* of this chain: its vertices are the elements of the set

$$\coprod_{n \geq 1} G_n^\wedge \quad (\text{disjoint union})$$

and two vertices  $\mu, \lambda$  are joined by  $k$  directed edges from  $\mu$  to  $\lambda$  whenever  $\mu \in G_{n-1}^\wedge$  and  $\lambda \in G_n^\wedge$  for some  $n$ , and the multiplicity of  $\mu$  in the restriction of  $\lambda$  to  $G_{n-1}$  is  $k$ . We call  $G_n^\wedge$  the  $n^{\text{th}}$  level of the branching multigraph. We write  $\mu \nearrow \lambda$  if there is an edge from  $\mu$  to  $\lambda$ .

For the rest of this section assume that the branching multigraph defined above is actually a graph, i.e., the multiplicities of all restrictions are 0 or 1. We say that the *branching or multiplicities are simple*.

Consider the  $G_n$ -module  $V^\lambda$ , where  $\lambda \in G_n^\wedge$ . Since the branching is simple, the decomposition

$$V^\lambda = \bigoplus_{\mu} V^\mu,$$

where the sum is over all  $\mu \in G_{n-1}^\wedge$  with  $\mu \nearrow \lambda$ , is canonical. Iterating this decomposition we obtain a canonical decomposition of  $V^\lambda$  into irreducible  $G_1$ -modules, i.e., one-dimensional subspaces,

$$V^\lambda = \bigoplus_T V_T, \quad (2)$$

where the sum is over all possible chains

$$T = \lambda_1 \nearrow \lambda_2 \nearrow \cdots \nearrow \lambda_n, \quad (3)$$

with  $\lambda_i \in G_i^\wedge$  and  $\lambda_n = \lambda$ . By choosing a nonzero vector  $v_T$  in each one-dimensional space  $V_T$  above, we obtain a basis  $\{v_T\}$  of  $V^\lambda$ , called the *Gelfand-Tsetlin basis* (*GZ-basis*). From here till the end of Section 6 we consider the GZ-basis as fixed. In Section 7 we shall discuss an appropriate choice of these nonzero vectors (in the case of symmetric groups) so that all irreducibles are realized over  $\mathbb{Q}$ . By the definition of  $v_T$ , we have

$$\mathbb{C}[G_i] \cdot v_T = V^{\lambda_i}, \quad i = 1, 2, \dots, n.$$

Also note that chains in (3) are in bijection with directed paths in the branching graph from the unique element  $\lambda_1$  of  $G_1^\wedge$  to  $\lambda$ .

We have identified a canonical basis (upto scalars), namely the GZ-basis, in each irreducible representation of  $G_n$ . A natural question at this point is to identify those elements of  $\mathbb{C}[G_n]$  that act diagonally in this basis (in every irreducible representation). In other words, consider the algebra isomorphism

$$\mathbb{C}[G_n] \cong \bigoplus_{\lambda \in G_n^\wedge} \text{End}(V^\lambda), \quad (4)$$

given by

$$g \mapsto (V^\lambda \xrightarrow{g} V^\lambda : \lambda \in G_n^\wedge), \quad g \in G_n.$$

Let  $D(V^\lambda)$  consists of all operators on  $V^\lambda$  diagonal in the GZ-basis of  $V^\lambda$ . Our question can now be stated as: what is the image under the isomorphism (4) of the subalgebra  $\bigoplus_{\lambda \in G_n^\wedge} D(V^\lambda)$  of  $\bigoplus_{\lambda \in G_n^\wedge} \text{End}(V^\lambda)$ .

Let  $Z_n$  denote the center of the algebra  $\mathbb{C}[G_n]$  and set  $GZ_n = \langle Z_1, Z_2, \dots, Z_n \rangle$ . It is easy to see that  $GZ_n$  is commutative subalgebra of  $\mathbb{C}S_n$ . We call  $GZ_n$  the *Gelfand-Tsetlin algebra* (*GZ-algebra*) of the inductive family of group algebras.

**Theorem 2.1.**  *$GZ_n$  is the image of  $\bigoplus_{\lambda \in G_n^\wedge} D(V^\lambda)$  under the isomorphism (4) above, i.e.,  $GZ_n$  consists of all elements of  $\mathbb{C}[G_n]$  that act diagonally in the GZ-basis in every irreducible representation of  $G_n$ . Thus  $GZ_n$  is a maximal commutative subalgebra of  $\mathbb{C}[G_n]$  and its dimension is equal to  $\sum_{\lambda \in G_n^\wedge} \dim \lambda$ .*

**Proof** Consider the chain  $T$  from (3) above. For  $i = 1, 2, \dots, n$ , let  $p_{\lambda_i} \in Z_i$  denote the central idempotent corresponding to the representation  $\lambda_i \in G_i^\wedge$ . Define  $p_T \in GZ_n$  by

$$p_T = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_n}.$$

A little reflection shows that the image of  $p_T$  under the isomorphism (4) is  $(f_\mu : \mu \in G_n^\wedge)$ , where  $f_\mu = 0$ , if  $\mu \neq \lambda$  and  $f_\lambda$  is the projection on  $V_T$  (with respect to the decomposition (2) of  $V^\lambda$ .)

It follows that the image of  $GZ_n$  under (4) includes  $\bigoplus_{\lambda \in G_n^\wedge} D(V^\lambda)$ , which is a maximal commutative subalgebra of  $\bigoplus_{\lambda \in G_n^\wedge} \text{End}(V^\lambda)$ . Since  $GZ_n$  is commutative, the result follows.  $\square$ .

By a *GZ-vector* of  $G_n$  (determined upto scalars) we mean a vector in the GZ-basis of some irreducible representation of  $G_n$ . As an immediate consequence of the theorem above we get the following result.

**Corollary 2.2.** *(i) Let  $v \in V^\lambda$ ,  $\lambda \in G_n^\wedge$ . If  $v$  is an eigenvector (for the action ) of every element of  $GZ_n$ , then (a scalar multiple of)  $v$  belongs to the GZ-basis of  $V^\lambda$ .*

*(ii) Let  $v, u$  be two GZ-vectors. If  $v$  and  $u$  have the same eigenvalues for every element of  $GZ_n$ , then  $v = u$ .*

We shall find an explicit set of generators for the GZ-algebras of the symmetric groups in Section 4.

### 3 Simplicity of branching for symmetric groups

In this section we give a criterion for simple branching and use the theory of involutive algebras to verify this criterion in the case of symmetric groups.

**Theorem 3.1.** *Let  $M$  be a finite dimensional semisimple  $\mathbb{C}$ -algebra and let  $N$  be a semisimple subalgebra. Let  $Z(M, N)$  denote the centralizer of this pair consisting of all elements of  $M$  that commute with  $N$ . Then  $Z(M, N)$  is semisimple and the following conditions are equivalent:*

1. *The restriction of any finite dimensional complex irreducible representation of  $M$  to  $N$  is multiplicity free.*
2. *The centralizer  $Z(M, N)$  is commutative.*

**Proof** By Wedderburn's theorem we may assume, without loss of generality, that  $M = M_1 \oplus \cdots \oplus M_k$ , where each  $M_i$  is a matrix algebra. We write elements of  $M$  as  $(m_1, \dots, m_k)$ , where  $m_i \in M_i$ . For  $i = 1, \dots, k$ , let  $N_i$  denote the image of  $N$  under the natural projection of  $M$  onto  $M_i$ . Being the homomorphic image of a semisimple algebra,  $N_i$  itself is semisimple.

We have  $Z(M, N) = Z(M_1, N_1) \oplus \cdots \oplus Z(M_k, N_k)$ . By the double centralizer theorem each  $Z(M_i, N_i)$ , and thus  $Z(M, N)$ , is semisimple.

For  $i = 1, \dots, k$ , let  $V_i$  denote the set of all  $(m_1, \dots, m_k) \in M$  with  $m_j = 0$  for  $j \neq i$  and with all entries of  $m_i$  not in the first column equal to zero. Note that  $V_1, \dots, V_k$  are all the distinct inequivalent irreducible  $M$ -modules and that the decomposition of

$V_i$  into irreducible  $N$ -modules is identical to the decomposition of  $V_i$  into irreducible  $N_i$ -modules.

It now follows from the double centralizer theorem that  $V_i$  is multiplicity free as a  $N_i$ -module, for all  $i$  iff all irreducible modules of  $Z(M_i, N_i)$  have dimension 1, for all  $i$  iff  $Z(M_i, N_i)$  is abelian, for all  $i$  iff  $Z(M, N)$  is abelian.  $\square$

We shall now give an elegant proof that the restriction of an irreducible representation of  $S_n$  to  $S_{n-1}$  is multiplicity free. The proof exploits the additional structure of an involution that certain algebras possess (like  $g \mapsto g^{-1}$  in groups and  $A \mapsto A^*$  in matrices).

Recall the notion of an involutive algebra. Let  $F$  denote the real field or the complex field. If  $F = \mathbb{C}$  and  $\alpha \in \mathbb{C}$ , then  $\bar{\alpha}$  denotes the complex conjugate of  $\alpha$ , and if  $F = \mathbb{R}$  and  $\alpha \in \mathbb{R}$ , then  $\bar{\alpha} = \alpha$ .

An *involutive algebra* over  $F$  is an algebra  $A$  (over  $F$ ) together with a conjugate linear anti-automorphism of order 2, i.e., a bijective mapping  $x \mapsto x^*$  such that

$$(x + y)^* = x^* + y^*, (\alpha x)^* = \bar{\alpha}x^*, (xy)^* = y^*x^*, (x^*)^* = x,$$

for all  $x, y \in A$ , and  $\alpha \in F$ . We call  $x^*$  the *adjoint* of  $x$ .

An element  $x \in A$  is said to be *normal* if  $xx^* = x^*x$  and is said to be *hermitian* or *self-adjoint* if  $x = x^*$ .

Given an involutive algebra  $A$  over  $\mathbb{R}$ , we define an involutive algebra over  $\mathbb{C}$ , called the *\*-complexification* of  $A$  as follows: The elements of the complexification are ordered pairs  $(x, y) \in A \times A$ . We write  $(x, y)$  as  $x + iy$ . The algebra operations are the natural ones. For  $\alpha + i\beta \in \mathbb{C}$ ,  $x, x_1, x_2, y, y_1, y_2 \in A$  we have

$$\begin{aligned} (x_1 + iy_1) + (x_2 + iy_2) &= (x_1 + x_2) + i(y_1 + y_2) \\ (x_1 + iy_1)(x_2 + iy_2) &= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2) \\ (\alpha + i\beta)(x + iy) &= (\alpha x - \beta y) + i(\alpha y + \beta x) \\ (x + iy)^* &= x^* - iy^* \end{aligned}$$



By a *real* element of the complexification we mean an element of the form  $x = x + i0$ , for some  $x \in A$ . Note that the adjoint of a real element in the complexification is the same as its adjoint in  $A$ .

**Example** Let  $F$  be  $\mathbb{R}$  or  $\mathbb{C}$  and let  $G$  be a finite group. Then the group algebra  $F[G]$  is an involutive algebra under the involution  $(\sum_i \alpha_i g_i)^* = \sum_i \bar{\alpha}_i g_i^{-1}$ . It can also be easily checked that  $\mathbb{C}[G]$  is the  $*$ -complexification of  $\mathbb{R}[G]$ .

**Theorem 3.2.** *Let  $A$  be a complex involutive algebra.*

- (i) *An element  $x \in A$  is normal if and only if  $x = y + iz$ , for some self-adjoint  $y, z \in A$  that commute.*
- (ii)  *$A$  is commutative if and only if every element of  $A$  is normal.*
- (iii) *If  $A$  is the  $*$ -complexification of a real involutive algebra, then  $A$  is commutative if every real element of  $A$  is self-adjoint.*

**Proof** (i) (if)  $xx^* = (y + iz)(y^* - iz^*) = yy^* + izy^* - iyz^* + zz^* = y^2 + z^2$ , where in the last step we use the fact that  $y, z$  are commuting self-adjoints. Similarly we can show  $x^*x = y^2 + z^2$ .

(only if) Write  $x = y + iz$ , where  $y = \frac{1}{2}(x + x^*)$  and  $z = \frac{1}{2i}(x - x^*)$ . It is easily checked that  $y$  and  $z$  are self-adjoint. Now

$$yz = \frac{1}{4i}(x^2 - xx^* + x^*x - (x^*)^2) = \frac{1}{4i}(x^2 - (x^*)^2),$$

where in the last step we use the normality of  $x$ . Similarly,  $zy = \frac{1}{4i}(x^2 - (x^*)^2)$ .

(ii) (only if) This is clear.

(if) Let  $y, z \in A$  be two self-adjoint elements. Then

$$\begin{aligned} (y + iz)(y^* - iz^*) &= y^2 + z^2 + i(zy - yz) \\ (y^* - iz^*)(y + iz) &= y^2 + z^2 + i(yz - zy) \end{aligned}$$

Since  $y + iz$  is normal (by hypothesis) we have  $zy = yz$ . So any two self-adjoint elements of  $A$  commute.

Let  $x_1, x_2 \in A$ . Since  $x_1, x_2$  are normal we can write  $x_1 = y_1 + iz_1$  and  $x_2 = y_2 + iz_2$ , where  $y_1, y_2, z_1, z_2$  are self-adjoint (and all commute with each other, by the paragraph above). We now have

$$x_1x_2 = (y_1y_2 - z_1z_2) + i(z_1y_2 + y_1z_2) = x_2x_1.$$

(iii) Let  $y, z$  be real elements of  $A$ . Then  $yz$  is also real and  $yz = (yz)^* = z^*y^* = zy$ , so  $y$  and  $z$  commute. Let  $x = y + iz \in A$ , where  $y$  and  $z$  are real. By part (i)  $x$  is normal and by part (ii)  $A$  is commutative.  $\square$

**Theorem 3.3.** *The centralizer  $Z[\mathbb{C}[S_n], \mathbb{C}[S_{n-1}]]$  of the subalgebra  $\mathbb{C}[S_{n-1}]$  in  $\mathbb{C}[S_n]$  is commutative.*

**Proof** The involutive subalgebra  $Z[\mathbb{C}[S_n], \mathbb{C}[S_{n-1}]]$  of  $\mathbb{C}[S_n]$  is the  $*$ -complexification of  $Z[\mathbb{R}[S_n], \mathbb{R}[S_{n-1}]]$  (why?). It therefore suffices to show that every element of  $Z[\mathbb{R}[S_n], \mathbb{R}[S_{n-1}]]$  is self-adjoint.

Let

$$f = \sum_{\pi \in S_n} \alpha_\pi \pi, \quad \alpha_\pi \in \mathbb{R},$$

be an element of  $Z[\mathbb{R}[S_n], \mathbb{R}[S_{n-1}]]$ .

Fix a permutation  $\sigma \in S_n$ . Now  $\sigma$  and  $\sigma^{-1}$  have the same cycle structure and are thus conjugate. There is a well known procedure to produce a permutation in  $S_n$  conjugating  $\sigma$  to  $\sigma^{-1}$ . Namely, first write down in a straight line the permutation  $\sigma$  in cycle form. Below this line write down  $\sigma^{-1}$  in cycle form, the only requirement being that the lengths of the corresponding cycles match. The permutation in  $S_n$  taking an element of the top row to the corresponding element of the bottom row conjugates  $\sigma$  to  $\sigma^{-1}$ . It is clear that we can so write down the two permutations that the letter  $n$  matches with itself. Thus we can choose  $\tau \in S_{n-1}$  such that  $\sigma^{-1} = \tau\sigma\tau^{-1}$ .

Since  $\tau \in S_{n-1}$  we have  $\tau f = f\tau$  or

$$f = \tau f \tau^{-1} = \sum_{\pi \in S_n} \alpha_\pi (\tau \pi \tau^{-1}).$$

It follows that  $\alpha_\sigma = \alpha_{\sigma^{-1}}$ . Since  $\sigma$  was an arbitrary element in  $S_n$  it follows that  $f^* = f$ .  $\square$

We denote the centralizer  $Z[\mathbb{C}[S_n], \mathbb{C}[S_{n-1}]]$  by  $Z_{(n-1,1)}$ . In the next section we shall provide another proof that  $Z_{(n-1,1)}$  is commutative.

## 4 Young-Jucys-Murphy elements

From this section onwards we consider only the symmetric groups and we put  $G_n = S_n$ . Thus chains in the branching graph (as in Section 2) will refer to chains in the branching graph of the symmetric groups.

For  $i = 2, \dots, n$  define the following elements of  $\mathbb{C}S_n$ :

$$\begin{aligned} Y_i &= \text{Sum of all } i\text{-cycles in } S_{n-1}. \\ Y'_i &= \text{Sum of all } i\text{-cycles in } S_n \text{ containing } n. \end{aligned}$$

Note that  $Y_n = 0$ .

For  $(\mu, i) \in \mathcal{P}_1(n)$  define  $c_{(\mu,i)} \in \mathbb{C}S_n$  to be the sum of all permutations  $\pi$  in  $S_n$  satisfying

- type  $(\pi) = \mu$ .
- size of the cycle of  $\pi$  containing  $n = i$ .

Observe that each of  $Y_2, \dots, Y_{n-1}, Y'_2, \dots, Y'_n$  is equal to  $c_{(\mu,i)}$ , for suitable  $\mu$  and  $i$ .

**Lemma 4.1.** (i)  $\{c_{(\mu,i)} \mid (\mu,i) \in \mathcal{P}_1(n)\}$  is a basis of  $Z_{(n-1,1)}$ . It follows that  $\langle Y_2, \dots, Y_{n-1}, Y'_2, \dots, Y'_n \rangle \subseteq Z_{(n-1,1)}$ .

(ii)  $c_{(\mu,i)} \in \langle Y_2, \dots, Y_k, Y'_2, \dots, Y'_k \rangle$ , where  $k = \#\mu$ .

$$(iii) \ Z_{(n-1,1)} = \langle Y_2, \dots, Y_{n-1}, Y'_2, \dots, Y'_n \rangle.$$

$$(iv) \ Z_{n-1} = \langle Y_2, \dots, Y_{n-1} \rangle.$$

**Proof** (i) The first statement is left as an exercise for the reader (it is similar to the proof that  $\{c_\mu \mid \mu \in \mathcal{P}(n)\}$  is a basis of  $Z_n$ ). The second statement then follows from the observation made just before the statement of the lemma.

(ii) By induction on  $\#\mu$ . If  $\#\mu = 0$ , then  $c_{(\mu,i)}$  is the identity permutation and the result is clearly true. Assume the result whenever  $\#\mu \leq k$ . Consider  $(\mu, i) \in \mathcal{P}_1(n)$  with  $\#\mu = k + 1$ . Let the nontrivial parts of  $\mu$  be  $\mu_1, \dots, \mu_l$  (here we are not assuming that  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_l$ ). Consider the following two subcases:

(a)  $i = 1$ : Consider the product  $Y_{\mu_1} Y_{\mu_2} \cdots Y_{\mu_l}$ . Then we have, using part (i) (why?),

$$Y_{\mu_1} Y_{\mu_2} \cdots Y_{\mu_l} = \alpha_{(\mu,1)} c_{(\mu,1)} + \sum_{(\tau,1)} \alpha_{(\tau,1)} c_{(\tau,1)},$$

where  $\alpha_{(\mu,1)} \neq 0$  and the sum is over all  $(\tau, 1)$  with  $\#\tau < \#\mu$ . The result follows by induction.

(b)  $i > 1$ : Without loss of generality we may assume that  $i = \mu_1$ . Consider the product  $Y'_{\mu_1} Y_{\mu_2} \cdots Y_{\mu_l}$ . Then we have, using part (i) (why?),

$$Y'_{\mu_1} Y_{\mu_2} \cdots Y_{\mu_l} = \alpha_{(\mu,i)} c_{(\mu,i)} + \sum_{(\tau,j)} \alpha_{(\tau,j)} c_{(\tau,j)},$$

where  $\alpha_{(\mu,i)} \neq 0$  and the sum is over all  $(\tau, j)$  with  $\#\tau < \#\mu$ . The result follows by induction.

(iii) This follows from parts (i) and (ii).

(iv) This is similar to the proof of part (iii).  $\square$

For  $i = 1, \dots, n$  define the following elements of  $\mathbb{C}S_n$ :

$$\begin{aligned} X_i &= (1, i) + (2, i) + \dots + (i-1, i) \\ &= \text{Sum of all 2-cycles in } S_i - \text{Sum of all 2-cycles in } S_{i-1}. \end{aligned}$$

Note that  $X_1 = 0$ . Also note that  $X_i \notin Z_i$ . In fact,  $X_i$  is the difference of an element in  $Z_i$  and an element in  $Z_{i-1}$ , so  $X_i \in GZ_i \subseteq GZ_n$ . These elements have remarkable properties and are called the *Young-Jucys-Murphy* (YJM) elements.

**Theorem 4.2.** (i)  $Z_{(n-1,1)} = \langle Z_{n-1}, X_n \rangle$ .

(ii)  $GZ_n = \langle X_1, X_2, \dots, X_n \rangle$ .

**Proof** (i) Clearly  $Z_{(n-1,1)} \supseteq \langle Z_{n-1}, X_n \rangle$  (note that  $X_n = Y'_2$ ). To show the converse it is enough to show that  $Y'_2, \dots, Y'_n \in \langle Z_{n-1}, X_n \rangle$ . Since  $Y'_2 = X_n$  we have  $Y'_2 \in \langle Z_{n-1}, X_n \rangle$ .

Assume that  $Y'_2, \dots, Y'_{k+1} \in \langle Z_{n-1}, X_n \rangle$ . We shall now show that  $Y'_{k+2} \in \langle Z_{n-1}, X_n \rangle$ . We write  $Y'_{k+1}$  as

$$\sum_{i_1, \dots, i_k} (i_1, \dots, i_k, n),$$

where the sum is over all distinct  $i_1, \dots, i_k \in \{1, 2, \dots, n-1\}$ . In the following we use this summation convention implicitly.

Now consider the product  $Y'_{k+1}X_n \in \langle Z_{n-1}, X_n \rangle$ :

$$\left\{ \sum_{i_1, \dots, i_k} (i_1, \dots, i_k, n) \right\} \left\{ \sum_{i=1}^{n-1} (i, n) \right\}. \quad (5)$$

Take a typical element  $(i_1, \dots, i_k, n)(i, n)$  of this product. If  $i \neq i_j$ , for  $j = 1, \dots, k$  this product is  $(i, i_1, \dots, i_k, n)$ . On the other hand if  $i = i_j$ , for some  $1 \leq j \leq k$ , this product becomes  $(i_1, \dots, i_j)(i_{j+1}, \dots, i_k, n)$ . It follows that the element (5) above is equal to

$$\sum_{i, i_1, \dots, i_k} (i, i_1, \dots, i_k, n) + \sum_{i_1, \dots, i_k} \sum_{j=1}^k (i_1, \dots, i_j)(i_{j+1}, \dots, i_k, n), \quad (6)$$

where the first sum is over all distinct  $i, i_1, \dots, i_k \in \{1, 2, \dots, n-1\}$  and the second sum is over all distinct  $i_1, \dots, i_k \in \{1, 2, \dots, n-1\}$ . We can rewrite (6) as

$$Y'_{k+2} + \sum_{(\mu, i)} \alpha_{(\mu, i)} c_{(\mu, i)},$$

where the sum is over all  $(\mu, i)$  with  $\#\mu \leq k+1$ . By induction hypothesis and part (ii) of Lemma 4.1 it follows that  $Y'_{k+2} \in \langle Z_{n-1}, X_n \rangle$ .

(ii) The proof is by induction (the cases  $n = 1, 2$  being clear). Assume we have proved that  $GZ_{n-1} = \langle X_1, X_2, \dots, X_{n-1} \rangle$ . It remains to show that  $GZ_n = \langle GZ_{n-1}, X_n \rangle$ . The lhs clearly contains the rhs so it suffices to check that the lhs is contained in the rhs. For this it suffices to check that  $Z_n \subseteq \langle GZ_{n-1}, X_n \rangle$ . This follows from part (i) since  $Z_n \subseteq Z_{(n-1, 1)}$ .  $\square$

Note that Theorem 5(i) implies that  $Z_{(n-1, 1)}$  is commutative.

The GZ-basis in the case of symmetric groups is also called the *Young basis*, since it was first considered by A. Young (though not as a global basis as here). By Corollary 2.2(i) the Young (or GZ) vectors are common eigenvectors for  $GZ_n$ . Let  $v$  be a Young vector (for  $S_n$ ). Define

$$\alpha(v) = (a_1, \dots, a_n) \in \mathbb{C}^n,$$

where  $a_i$  = eigenvalue of  $X_i$  on  $v$ . We call  $\alpha(v)$  the *weight* of  $v$  (note that  $a_1 = 0$  since  $X_1 = 0$ ). Set

$$\text{Spec}(n) = \{\alpha(v) : v \text{ is a Young vector}\}.$$

It follows from Corollary 2.2(ii) that

$$\dim GZ_n = \#\text{Spec}(n) = \sum_{\lambda \in S_n^\wedge} \dim \lambda.$$

By definition of Young vectors, the set  $\text{Spec}(n)$  is in natural bijection with chains  $T$  as in (3). Given  $\alpha \in \text{Spec}(n)$  we denote by  $v_\alpha$  the Young vector with weight  $\alpha$  and by  $T_\alpha$  the corresponding chain in the branching graph. Similarly, given a chain  $T$  as in (3) we denote the corresponding weight vector  $\alpha(v_T)$  by  $\alpha(T)$ . Thus we have 1-1 correspondences

$$T \mapsto \alpha(T), \quad \alpha \mapsto T_\alpha$$

between chains (3) and  $\text{Spec}(n)$ .

There is a natural equivalence relation  $\sim$  on  $\text{Spec}(n)$ : for  $\alpha, \beta \in \text{Spec}(n)$ ,

$$\begin{aligned} \alpha \sim \beta &\Leftrightarrow v_\alpha \text{ and } v_\beta \text{ belong to the same irreducible } S_n\text{-module} \\ &\Leftrightarrow T_\alpha \text{ and } T_\beta \text{ end in the same vertex.} \end{aligned}$$

Clearly we have  $\#(\text{Spec}(n)/\sim) = \#S_n^\wedge$ .

## 5 Action of Coxeter generators on the Young basis and the algebra $H(2)$

The Young vectors are a simultaneous eigenbasis for the GZ-algebra. Let us now consider the action of the full algebra  $\mathbb{C}S_n$  on the Young basis. The action of the Coxeter generators  $s_i = (i, i+1)$ ,  $i = 1, \dots, n-1$  on the Young basis is "local" in the sense of the following lemma.

**Lemma 5.1.** *Let  $T$  be a chain*

$$\lambda_1 \nearrow \lambda_2 \nearrow \dots \nearrow \lambda_n, \quad \lambda_k \in S_k^\wedge \tag{7}$$

*and let  $1 \leq i \leq n-1$ . Then  $s_i \cdot v_T$  is a linear combination of vectors  $v_{T'}$ , where  $T'$  runs over chains of the form*

$$\lambda'_1 \nearrow \lambda'_2 \nearrow \dots \nearrow \lambda'_n, \quad \lambda'_k \in S_k^\wedge \tag{8}$$

such that  $\lambda'_k = \lambda_k$ , for  $k \neq i$ . Moreover, the coefficients of this linear combination depend only on  $\lambda_{i-1}, \lambda_i, \lambda_{i+1}$  and the choice of the scalar factors for the vectors in the Young basis. That is, the action of  $s_i$  affects only the  $i$ th level and depends only on levels  $i-1, i$ , and  $i+1$  of the branching graph.

**Proof** Clearly, for  $j \geq i+1$ , we have  $s_i \cdot v_T \in V^{\lambda_j}$ . For  $j \leq i-1$ , the action of  $s_i$  on  $V^{\lambda_{i+1}}$  is  $S_j$ -linear (since  $s_i$  commutes with all elements of  $S_j$ ). Thus  $s_i \cdot v_T$  belongs to the  $V^{\lambda_j}$ -isotypical component of  $V^{\lambda_{i+1}}$ . This proves the first part of the lemma. The moreover part is left as an exercise for the reader.  $\square$

We shall now give explicit formulas for the action of the Coxeter generators  $s_i$  on the Young basis vectors  $v_T$  in terms of the weights  $\alpha(T)$ . For this purpose, it will be useful to know the relations satisfied by the  $s_i$ 's and the  $X_j$ 's since the  $v_T$ 's are eigenvectors for  $X_1, \dots, X_n$ . The following relations are easily verified.

$$s_i X_j = X_j s_i, \quad j \neq i, i+1, \quad (9)$$

$$s_i^2 = 1, \quad X_i X_{i+1} = X_{i+1} X_i, \quad s_i X_i + 1 = X_{i+1} s_i. \quad (10)$$

(The preceding lemma can also be proved using the commutation relation (9) above. We leave this as an exercise). Let  $T$  be as in (7) above and let  $\alpha(T) = (a_1, a_2, \dots, a_n)$ . Let  $V$  be the subspace of  $V^{\lambda_{i+1}}$  generated by  $v_T$  and  $s_i \cdot v_T$  (note that  $V$  is at most two dimensional). The relation (10) shows that  $V$  is invariant under the actions of  $s_i, X_i$ , and  $X_{i+1}$  and a study of this action will enable us to write down a formula for  $s_i \cdot v_T$ .

We are thus led to consider the algebra  $H(2)$  generated by elements  $H_1, H_2$ , and  $s$  subject to the following relations:

$$s^2 = 1, \quad H_1 H_2 = H_2 H_1, \quad s H_1 + 1 = H_2 s. \quad (11)$$

Note that  $H_2$  can be excluded because  $H_2 = s H_1 s + s$ , but it is most useful to include  $H_2$  in the list of generators.



**Lemma 5.2.** (i) *All irreducible representations of  $H(2)$  are at most two dimensional.*

(ii) *For  $i = 1, 2, \dots, n-1$  the image of  $H(2)$  in  $\mathbb{C}S_n$  obtained by setting  $s = s_i = (i, i+1)$ ,  $H_1 = X_1$ , and  $H_2 = X_{i+1}$  is semisimple, i.e., the subalgebra  $M$  of  $\mathbb{C}S_n$  generated by  $s_i, X_i$ , and  $X_{i+1}$  is semisimple.*

**Proof** (i) Let  $V$  be an irreducible  $H(2)$  module. Since  $H_1$  and  $H_2$  commute they have a common eigenvector  $v$ . Let  $W$  be the space spanned by  $v$  and  $s \cdot v$ . Then  $\dim W \leq 2$  and (11) shows that  $W$  is a submodule of  $V$ . Since  $V$  is irreducible it follows that  $W = V$ .

(ii) Let  $Mat(n)$  denote the algebra of complex  $n! \times n!$  matrices, with rows and columns indexed by permutations in  $S_n$ . Consider the left regular representation of  $S_n$ . Writing this in matrix terms gives an embedding of  $\mathbb{C}S_n$  into  $Mat(n)$ . Note that the matrix in  $Mat(n)$  corresponding to a transposition in  $S_n$  is real and symmetric. Since  $X_i$  and  $X_{i+1}$  are sums of transpositions the matrices in  $Mat(n)$  corresponding to them are also real and symmetric. It follows that the subalgebra  $M$  is closed under the matrix  $*$  operation ( $A \mapsto (\bar{A})^t$ ). A standard result on finite dimensional  $\mathbb{C}^*$ -algebras shows that  $M$  is semisimple.  $\square$

We can now explicitly describe the action of the Coxeter generators on the Young basis in terms of transformation of weights. In the following it is convenient to parametrize Young vectors by elements of  $\text{Spec}(n)$  rather than by chains  $T$  (recall that these sets are in natural bijection).

**Theorem 5.3.** *Let  $T$  be a chain as in (7) and let  $\alpha(T) = (a_1, \dots, a_n) \in \text{Spec}(n)$ . Consider the Young vector  $v_\alpha = v_T$ . Then*

(i)  $a_i \neq a_{i+1}$ , for all  $i$ .

(ii)  $a_{i+1} = a_i \pm 1$  if and only if  $s_i \cdot v_\alpha = \pm v_\alpha$ .

(iii) For  $i = 1, \dots, n-2$  the following statements are not true:

$$a_i = a_{i+1} + 1 = a_{i+2}, \quad a_i = a_{i+1} - 1 = a_{i+2}.$$

(iv) If  $a_{i+1} \neq a_i \pm 1$  then

$$\alpha' = s_i \cdot \alpha = (a_1, \dots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \dots, a_n)$$

belongs to  $\text{Spec}(n)$  and  $\alpha' \sim \alpha$ . Moreover the vector

$$v = \left( s_i - \frac{1}{a_{i+1} - a_i} \right) \cdot v_\alpha$$

is a scalar multiple of  $v_{\alpha'}$ . Thus, in the basis  $\{v_\alpha, v\}$  the actions of  $X_i, X_{i+1}$ , and  $s_i$  are given by the matrices

$$\begin{bmatrix} a_i & 0 \\ 0 & a_{i+1} \end{bmatrix}, \begin{bmatrix} a_{i+1} & 0 \\ 0 & a_i \end{bmatrix}, \text{ and } \begin{bmatrix} \frac{1}{a_{i+1}-a_i} & 1 - \frac{1}{(a_{i+1}-a_i)^2} \\ 1 & \frac{1}{a_i-a_{i+1}} \end{bmatrix}$$

respectively.

**Proof** (i) Suppose first that  $v_\alpha$  and  $s_i \cdot v_\alpha$  are linearly dependent. Then, since  $s_i^2 = 1$ , we have  $s_i \cdot v_\alpha = \pm v_\alpha$ . The relation (10) (equivalent to  $s_i X_i s_i + s_i = X_{i+1}$ ) now shows  $a_{i+1} = a_i \pm 1$ .

Now assume that  $v_\alpha$  and  $s_i \cdot v_\alpha$  are linearly independent and let  $V$  be the subspace of  $V^{\lambda_{i+1}}$  they span. Then, as checked before,  $V$  is invariant under the action of the algebra  $M$  and the matrices for the actions of  $X_i, X_{i+1}$  and  $s_i$  in the basis  $\{v_\alpha, s_i \cdot v_\alpha\}$  of  $V$  are

$$\begin{bmatrix} a_i & -1 \\ 0 & a_{i+1} \end{bmatrix}, \begin{bmatrix} a_{i+1} & 1 \\ 0 & a_i \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

respectively. The action of  $X_i$  on  $V^{\lambda_{i+1}}$  is diagonalizable and thus, since  $V$  is  $X_i$ -invariant, the action of  $X_i$  on  $V$  is also diagonalizable. This implies  $a_i \neq a_{i+1}$ .

(ii) The if part has already been proved in part (i) above. Let us now prove the only if part. Suppose  $a_{i+1} = a_i + 1$  (the case  $a_{i+1} = a_i - 1$  is similar). Assume that  $v_\alpha$  and  $s_i \cdot v_\alpha$  are linearly independent and let  $V$  be the subspace of  $V^{\lambda_{i+1}}$  they span. Now  $V$  is an  $M$ -module and  $M$  is semisimple. But it can be easily checked that there is only one 1-dimensional subspace of  $V$ , namely the space spanned by  $s_i \cdot v_\alpha - v_\alpha$ , that is invariant under the action of  $M$ , a contradiction. Thus  $v_\alpha$  and  $s_i \cdot v_\alpha$  are linearly dependent and this implies that  $s_i \cdot v_\alpha = v_\alpha$ .

(iii) Suppose  $a_i = a_{i+1} - 1 = a_{i+2}$  (the proof in the other case is the same). By part (ii) we have  $s_i \cdot v_\alpha = v_\alpha$  and  $s_{i+1} \cdot v_\alpha = -v_\alpha$ . Consider the Coxeter relation

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},$$

and let both sides act on  $v_\alpha$ . The lhs yields  $-v_\alpha$  and the rhs yields  $v_\alpha$ , a contradiction.

(iv) By part (ii) we have that  $v_\alpha$  and  $s_i \cdot v_\alpha$  are linearly independent. For  $j \neq i, i+1$  we can easily check using commutativity of  $X_j$  and  $s_i$  that  $X_j \cdot v = a_j v$ . Similarly, using (10), we can easily check that  $X_i \cdot v = a_{i+1} v$  and  $X_{i+1} \cdot v = a_i v$ . It follows from Corollary 2.2(i) that  $\alpha' \in \text{Spec}(n)$  and from Corollary 2.2(ii) that  $v$  is a scalar multiple of  $v_{\alpha'}$ . Clearly  $\alpha' \sim \alpha$  as  $v \in V^{\lambda_n}$ . The matrix representations of  $s_i, X_i$  and  $X_{i+1}$  are easily verified.  $\square$

If  $\alpha = (a_1, \dots, a_n) \in \text{Spec}(n)$  and  $a_i \neq a_{i+1} \pm 1$ , we say that the transposition  $s_i$  is *admissible* for  $\alpha$ . Note that if  $\alpha \in \text{Spec}(n)$  is obtained from  $\beta \in \text{Spec}(n)$  by a sequence of admissible transpositions then  $\alpha \sim \beta$ .

In the next section we shall show that  $\text{Spec}(n)$  consists of integral vectors. This given, the matrix for the action of  $s_i$  from part (iv) of Theorem 5.3 suggests that if we choose the GZ-basis  $\{v_T\}$  appropriately then all irreducible representations of  $S_n$  are defined over  $\mathbb{Q}$ . We shall show how to do this in Section 7.

## 6 Content vectors and Young tableaux

In the Vershik-Okounkov theory Young tableaux are related to irreducible  $S_n$  representations via their content vectors. Let us define these first.

**Definition 6.1.** Let  $\alpha = (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$ . We say that  $\alpha$  is a content vector if

- (i)  $a_1 = 0$ .
- (ii)  $\{a_i - 1, a_i + 1\} \cap \{a_1, a_2, \dots, a_{i-1}\} \neq \emptyset$ , for all  $i > 1$ .
- (iii) if  $a_i = a_j = a$  for some  $i < j$  then  $\{a-1, a+1\} \subseteq \{a_{i+1}, \dots, a_{j-1}\}$  (i.e., between two occurrences of  $a$  there should also be occurrences of  $a-1$  and  $a+1$ ).

Condition (ii) in the definition above can be replaced (in the presence of conditions (i) and (iii)) by condition (ii') below.

(ii') For all  $i > 1$ , if  $a_i > 0$  then  $a_j = a_i - 1$  for some  $j < i$  and if  $a_i < 0$  then  $a_j = a_i + 1$  for some  $j < i$ .

The set of all content vectors of length  $n$  is denoted  $\text{Cont}(n) \subseteq \mathbb{Z}^n$ .

**Theorem 6.1.**  $\text{Spec}(n) \subseteq \text{Cont}(n)$ .

**Proof** Let  $\alpha = (a_1, \dots, a_n) \in \text{Spec}(n)$ . Clearly  $a_1 = 0$  as  $X_1 = 0$ . We verify conditions (ii) and (iii) by induction on  $n$ . Since  $X_2 = (1, 2)$  we have  $a_2 = \pm 1$  and thus condition (ii) is verified (and condition (iii) does not apply). Now assume  $n \geq 3$ .

If  $a_{n-1} = a_n \pm 1$  there is nothing to prove, so assume this does not hold. Then the transposition  $(n-1, n)$  is admissible for  $\alpha$  and, by Theorem 5.3(iv),  $(a_1, \dots, a_{n-2}, a_n, a_{n-1}) \in \text{Spec}(n)$ . Now

$(a_1, \dots, a_{n-2}, a_n) \in \text{Spec } (n-1)$  and by the induction hypothesis  $\{a_n - 1, a_n + 1\} \cap \{a_1, \dots, a_{n-2}\} \neq \emptyset$ . Thus condition (ii) is verified.

Now assume that  $a_i = a_n = a$  for some  $i < n$ . We may assume that  $i$  is the largest possible index, i.e.,  $a$  does not occur between  $a_i$  and  $a_n$ :  $a \notin \{a_{i+1}, \dots, a_{n-1}\}$ . Now assume that  $a - 1 \notin \{a_{i+1}, \dots, a_{n-1}\}$ . We shall derive a contradiction (the case where  $a + 1 \notin \{a_{i+1}, \dots, a_{n-1}\}$  is similar).

By induction hypothesis the number  $a+1$  occurs in  $\{a_{i+1}, \dots, a_{n-1}\}$  at most once (for, if it occurred twice, then by the induction hypothesis  $a$  would also occur contradicting our assumption). Thus there are two possibilities:

$$(a_i, \dots, a_n) = (a, *, \dots, *, a) \text{ or } (a_i, \dots, a_n) = (a, *, \dots, *, a+1, *, \dots, *, a),$$

where  $*$  stands for a number different from  $a - 1, a, a + 1$ .

In the first case we can apply a sequence of admissible transpositions to infer that  $(\dots, a, a, \dots) \in \text{Spec } (n)$ , contradicting Theorem 5.3(i) and in the second case we can apply a sequence of admissible transpositions to infer that  $(\dots, a, a + 1, a, \dots) \in \text{Spec } (n)$ , contradicting Theorem 5.3(iii).  $\square$

If  $\alpha = (a_1, \dots, a_n) \in \text{Cont } (n)$  and  $a_i \neq a_{i+1} \pm 1$ , we say that the transposition  $s_i$  is *admissible* for  $\alpha$ . We define the following equivalence relation on  $\text{Cont } (n)$ :  $\alpha \approx \beta$  if  $\beta$  can be obtained from  $\alpha$  by a sequence of (zero or more) admissible transpositions.

We are finally ready to introduce Young tableaux into the picture. Recall the definition of the *Young graph*  $\mathbb{Y}$ : its vertices are Young diagrams, and two vertices  $\mu$  and  $\tau$  are joined by a directed edge from  $\mu$  to  $\tau$  if and only if  $\mu \subseteq \tau$  and  $\tau - \mu$  is a single box. In this case we write  $\mu \nearrow \tau$ . The *content*  $c(B)$  of a box  $B$  of a Young diagram is its  $x$ -coordinate  $-$  its  $y$ -coordinate (our convention for drawing Young diagrams is akin to drawing matrices with  $x$ -axis running downwards and  $y$  axis running to the left). By  $\text{Tab } (\tau)$  we denote the set of all paths in  $\mathbb{Y}$  from the unique partition of 1 to

$\tau$ . Such paths are called *standard Young tableaux*. Given a path  $T \in \text{Tab}(\tau)$ , written as

$$\tau_1 \nearrow \tau_2 \nearrow \cdots \nearrow \tau_n = \tau,$$

a convenient way to represent it is to take the Young diagram of  $\tau$  and write the numbers  $1, 2, \dots, n$  in the boxes  $\tau_1, \tau_2 - \tau_1, \dots, \tau_n - \tau_{n-1}$  respectively. Set

$$\text{Tab}(n) = \cup_{\tau} \text{Tab}(\tau),$$

where the union is over all partitions  $\tau$  of  $n$ .

Let  $T_1 \in \text{Tab}(n)$  and assume that  $i$  and  $i + 1$  do not appear in the same row or column of  $T_1$ . Then exchanging  $i$  and  $i + 1$  in  $T_1$  produces another standard Young tableaux  $T_2 \in \text{Tab}(n)$ . We say that  $T_2$  is obtained from  $T_1$  by an *admissible transposition*. For  $T_1, T_2 \in \text{Tab}(n)$ , define  $T_1 \approx T_2$  if  $T_2$  can be obtained from  $T_1$  by a sequence of (zero or more) admissible transpositions (it is easily seen that  $\approx$  is an equivalence relation).

The proof of the following combinatorial lemma is left as an exercise.

**Lemma 6.2.** *Let  $\Phi : \text{Tab}(n) \longrightarrow \text{Cont}(n)$  be defined as follows. Given*

$$T = \tau_1 \nearrow \tau_2 \nearrow \cdots \nearrow \tau_n \in \text{Tab}(n),$$

*Define*

$$\Phi(T) = (c(\tau_1), c(\tau_2 - \tau_1), \dots, c(\tau_n - \tau_{n-1})).$$

*Then  $\Phi$  is a bijection which takes  $\approx$ -equivalent standard Young tableaux to  $\approx$ -equivalent content vectors.*

**Lemma 6.3.** *Let  $T_1, T_2 \in \text{Tab}(n)$ . Then  $T_1 \approx T_2$  if and only if the Young diagrams of  $T_1$  and  $T_2$  have the same shape.*

**Proof** The only if part is obvious. To prove the if part we proceed as follows. Let  $\mu = (\mu_1, \mu_2, \dots, \mu_r)$  be a partition of  $n$ . Define the

following element  $R$  of  $\text{Tab}(\mu)$ : in the first row write down the numbers  $1, 2, \dots, \mu_1$  (in increasing order), in the second row write down the numbers  $\mu_1 + 1, \mu_1 + 2, \dots, \mu_1 + \mu_2$  (in increasing order) and so on till the last row. We show that any  $T \in \text{Tab}(\mu)$  satisfies  $T \approx R$ . This will prove the if part. Consider the last box of the last row of  $\mu$ . Let  $i$  be written in this box of  $T$ . Exchange  $i$  and  $i + 1$  in  $T$  (which is clearly an admissible transposition). Now repeat this procedure with  $i + 1$  and  $i + 2$ , then  $i + 2$  and  $i + 3$ , and finally  $n - 1$  and  $n$ . At the end of this sequence of admissible transpositions we have the number  $n$  written in the last box of the last row of  $\mu$ . Now repeat the same procedure for  $n - 1, n - 2, \dots, 2$ .  $\square$

Let us make a remark about the proof of Lemma 6.3. Let  $s$  denote the permutation that maps  $R$  to  $T$ . Then the proof shows that  $R$  can be obtained from  $T$  by a sequence of  $\ell(s)$  admissible transpositions. Thus  $T$  can be obtained from  $R$  by a sequence of  $\ell(s)$  admissible transpositions.

We can now present one of the central results in the representation theory of the symmetric groups.

**Theorem 6.4.** (i)  $\text{Spec}(n) = \text{Cont}(n)$  and the equivalence relations  $\sim$  and  $\approx$  coincide.

(ii) The map  $\Phi^{-1} : \text{Spec}(n) \longrightarrow \text{Tab}(n)$  is a bijection and, for  $\alpha, \beta \in \text{Spec}(n)$  we have  $\alpha \sim \beta$  if and only if  $\Phi^{-1}(\alpha)$  and  $\Phi^{-1}(\beta)$  have the same Young diagram.

(iii) The branching graph of the chain of symmetric groups is the Young graph  $\mathbb{Y}$ .

**Proof** We have

- $\text{Spec}(n) \subseteq \text{Cont}(n)$ .
- If  $\alpha \in \text{Spec}(n)$ ,  $\beta \in \text{Cont}(n)$ , and  $\alpha \approx \beta$  then it is easily seen that  $\beta \in \text{Spec}(n)$  and  $\alpha \sim \beta$ . It follows that given an

$\sim$ -equivalence class  $\mathcal{A}$  of  $\text{Spec } (n)$  and an  $\approx$ -equivalence class  $\mathcal{B}$  of  $\text{Cont } (n)$ , either  $\mathcal{A} \cap \mathcal{B} = \emptyset$  or  $\mathcal{B} \subseteq \mathcal{A}$ .

- $\#(\text{Spec } (n) / \sim) = p(n)$ , since the number of irreducible  $S_n$ -representations is equal to the number of conjugacy classes in  $S_n$ ,
- $\#(\text{Cont } (n) / \approx) = p(n)$ , by Lemmas 6.2 and 6.3.

The four statements above imply part (i). Parts (ii) and (iii) are now clear.  $\square$

## 7 Young's seminormal and orthogonal forms

We now discuss the choice of the scalar factors in the Young basis  $\{v_T\}$ , so that all irreducible representations of  $S_n$  are defined over  $\mathbb{Q}$ .

Given the results of Section 6, we can now parametrize irreducible  $S_n$ -modules by partitions of  $n$  and we can parametrize the Young basis vectors in an  $S_n$ -irreducible (parametrized by  $\mu \in \mathcal{P}(n)$ ) by standard Young tableaux of shape  $\mu$ .

Fix a partition  $\mu$  of  $n$  and consider the irreducible  $S_n$ -module  $V^\mu$ . Let  $R$  be the tableau defined in the proof of Lemma 6.3. Choose any nonzero vector  $v_R \in V_R$ . Now consider a tableau  $T \in \text{Tab } (\mu)$ . Let  $s$  be the permutation that maps  $R$  to  $T$ . Define

$$v_T = p_T(s \cdot v_R),$$

and define  $\ell(T) = \ell(s)$ . Recall, from Section 2, that  $p_T$  denotes the projection onto  $V_T$ . We will now show that  $v_T \neq 0$ . It will then follow that  $\{v_T : T \in \text{Tab } (n)\}$  is a basis of  $V^\mu$ .



Before proceeding further we observe the following: let  $T \in \text{Tab}(\mu)$  and choose a nonzero vector  $v \in V_T$ . Theorem 5.3(iv) shows that, if  $s_j$  is an admissible transposition for  $T$ , then  $s_j \cdot v$  is the sum of a nonzero rational multiple of  $v$  and a nonzero vector in  $V_{s_j \cdot T}$ . If  $s_j$  is not admissible for  $T$ , then Theorem 5.3(ii) shows that  $s_j \cdot v = \pm v$ .

Fix  $T \in \text{Tab}(\mu)$  and let  $s$  be the permutation that maps  $R$  to  $T$ . Let  $k = \ell(T)$ . The proof of Lemma 6.3 shows that we can write

$$T = s \cdot R = c_1 c_2 \cdots c_k \cdot R,$$

where each  $c_j$  is an admissible transposition (w.r.t  $c_{j+1} \cdots c_k \cdot R$ ). It now follows (why?) from the definition of  $v_T$ , the observation above, and the fact that  $s$  cannot be written as a product of fewer than  $k$  Coxeter transpositions that

$$s \cdot v_R = v_T + \sum_Q u_Q,$$

where  $v_T \neq 0$ ,  $u_Q \in V_Q$  and the sum is over all  $Q \in \text{Tab}(\mu)$  with  $\ell(Q) < \ell(T)$ .

**Theorem 7.1.** *Consider the basis  $\{v_T : T \in \text{Tab}(n)\}$  of  $V^\mu$  defined above. Fix  $T \in \text{Tab}(n)$  and let  $\alpha(T) = (a_1, \dots, a_n)$ . Let  $s_i$  be a Coxeter generator. The action of  $s_i$  on  $v_T$  is as follows.*

- *If  $i$  and  $i + 1$  are in the same column of  $T$  then  $s_i$  leaves  $v_T$  unchanged.*
- *If  $i$  and  $i + 1$  are in the same row of  $T$  then  $s_i$  multiplies  $v_T$  by  $-1$ .*
- *Suppose  $i$  and  $i + 1$  are not in the same row or column of  $T$ . Let  $S = s_i \cdot T$  (i.e, swap  $i$  and  $i + 1$  in  $T$ ).*

If  $\ell(S) = \ell(T) + 1$  then the action of  $s_i$  in the two dimensional subspace with ordered basis  $\{v_T, v_S\}$  is given by the matrix

$$\begin{bmatrix} \frac{1}{a_{i+1}-a_i} & 1 - \frac{1}{(a_{i+1}-a_i)^2} \\ 1 & \frac{1}{a_i-a_{i+1}} \end{bmatrix}$$

If  $\ell(S) = \ell(T) - 1$  then the action of  $s_i$  in the two dimensional subspace with ordered basis  $\{v_T, v_S\}$  is given by the transpose of the matrix above.

Thus, all irreducible representations of  $S_n$  are defined over  $\mathbb{Q}$ .

**Proof** Items (i) and (ii) in the statement of the theorem follow from Theorem 5.3(ii).

Let us now consider item (iii). Assume that  $\ell(S) = \ell(T) + 1$ .

We have (why?), using the expression for  $s \cdot v_R$  above,

$$\begin{aligned} s_i \cdot v_T &= s_i \cdot (s \cdot v_R - \sum_Q u_Q) \\ &= s_i \cdot (c_1 c_2 \cdots c_k \cdot v_R - \sum_Q u_Q) \\ &= v_S + \sum_{Q'} u_{Q'} \end{aligned}$$

where  $u_{Q'} \in V_{Q'}$  and the sum is over all  $Q' \in \text{Tab}(\mu)$  with  $\ell(Q') \leq \ell(T)$ . It now follows (why?) from Theorem 5.3(iv) that

$$s_i \cdot v_T = \frac{1}{a_{i+1} - a_i} v_T + v_S$$

The expression for  $s_i \cdot v_S$  can be obtained by applying  $s_i$  to both sides of the equation above.

If  $\ell(S) = \ell(T) - 1$  then we write  $T = s_i \cdot S$  and switch  $T$  and  $S$  in the formulas for  $v_T, v_S$  above along with switching  $a_i$  and  $a_{i+1}$ . Doing this is equivalent to transposing the matrix.  $\square$

Another way to prove the theorem above is to directly verify the Coxeter relations (we leave this as an exercise).

The basis and the action described above is called Young's *semi-normal form* of  $V^\mu$ . Now let us consider Young's *orthogonal form* for  $V^\mu$ . This is defined over  $\mathbb{R}$ . Since  $V^\mu$  is irreducible there is a unique (upto scalars) (why?)  $S_n$ -invariant inner product on  $V^\mu$ . Pick one such inner product and normalize the vectors  $\{v_T\}$  defined above (we use the same notation for the normalized vectors). Note that the vectors  $\{v_T\}$  are orthonormal (why?). The following result now follows from the previous result.

**Theorem 7.2.** *Consider the orthonormal basis  $\{v_T : T \in \text{Tab}(n)\}$  of  $V^\mu$  defined above. Fix a chain  $T$  and let  $\alpha(T) = (a_1, \dots, a_n)$ . Let  $s_i$  be a Coxeter generator and put  $r = a_{i+1} - a_i$ . The action of  $s_i$  on  $v_T$  is as follows.*

- *If  $i$  and  $i + 1$  are in the same column of  $T$  then  $s_i$  leaves  $v_T$  unchanged.*
- *If  $i$  and  $i + 1$  are in the same row of  $T$  then  $s_i$  multiplies  $v_T$  by  $-1$ .*
- *Suppose  $i$  and  $i + 1$  are not in the same row or column of  $T$ . Let  $S = s_i \cdot T$  (i.e, swap  $i$  and  $i + 1$  in  $T$ ).*

*The action of  $s_i$  in the two dimensional subspace with ordered basis  $\{v_T, v_S\}$  is given by the matrix*

$$\begin{bmatrix} r^{-1} & \sqrt{1 - r^{-2}} \\ \sqrt{1 - r^{-2}} & -r^{-1} \end{bmatrix}$$

The number  $r$  is called the *axial distance*. It is the difference of the contents of the corresponding boxes in the Young diagram.

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