Resolution of singularities

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Resolution of singularities of algebraic varieties is considered to be one of the deepest theorems in algebraic geometry. It is classical for curves, has been proved by Abhyankar for surfaces (in any characteristic) and by Hironaka in general (in characteristic 0). The case of positive characteristic is, in general, still open. The problem of resolution of singularities can be formulated as follows. Given a reduced algebraic variety X construct a non–singular variety X' and a proper[1] birational[2] map $\pi : X' \to X$ such that π induces an isomorphism $X' \smallsetminus \pi^{-1}(\mathrm{Sing}(X)) \cong X \smallsetminus \mathrm{Sing}(X)$. The first proof that the resolution of singularities in characteristic zero is always possible was given by Hironaka in 1964. Some years ago Villamayor and Encinas and independently Bierstone and Milman gave constructive proofs for this theorem leading to algorithms which we implemented in Singular. With these algorithms one can compute the so–called embedded resolution. We start with a variety X embedded in a smooth n–dimensional variety W. The idea of the resolution process is to use a sequence of blowing ups.[3] The choice of the centers in this sequence of blowing ups is the crucial point in the resolution process. The center has to be chosen in such a way that after the blowing up the singularity of the strict transform together with the configuration of exceptional divisors[4] improves. The intersection of the exceptional divisors in the process of blowing ups should be as simple as possible.[5]

The center should be contained in the singular locus of X. The choice of the center for the next blowing up in the resolution process is guided by an invariant which is a vector of integers and locally defined at each point. The invariants at the points can be compared lexicographically and the center has to be chosen as the set of points with maximal value of the invariant. There are several possibilities to define such an invariant and every choice leads to different algorithms. The definition of such an invariant is rather complicated and includes the knowledge about the "history"of the resolution process. The invariant has values in a well-ordered set and its maximal value decreases under blowing up in the correct center which guarantees termination of the resolution process.

[1] In the classical topology *proper* means that the preimages of compact sets are compact again.

[2] The map is locally defined by rational functions and has an inverse of this type.

[3] Blowing ups are special birational maps replacing the points of a smooth subvariety (the center of the blowing up) by projective spaces. If the dimension of the variety is n and the dimension of the center is d then its points will be replaced by ¶$^{n-d-1}$.

[4] If $\pi : W' \to W$ is the blowing up with center $C \subseteq X \subseteq W$ then $\pi^{-1}(C)$ is the exceptional divisor. The closure of $\pi^{-1}(X \smallsetminus C)$ in W' is called the strict transform of X.

[5] The exceptional divisors should have normal crossings and should also have normal crossings with the resolved variety.