Proximinal and Strongly Proximinal Subspaces of Finite codimension

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1 Proximinal subspaces of finite codimension

1.1 Basic definitions and results

Let X be a normed linear space. We will consider only normed linear spaces over $\mathbb{R}$ (Real line), though many of the results we describe hold good for n.l. spaces over $\mathbb{C}$ (the complex plane). The dual of X, the class of all bounded, linear functionals on X, is denoted by X^*. The closed unit ball of X is denoted by B_X and the unit sphere, by S_X. That is, $B_X = \{x \in X : \|x\| \leq 1\}$ and $S_X = \{x \in X : \|x\| = 1\}$.

For $x \in X$ and $r > 0$, we set $B(x,r) = \{y \in X : \|x-y\| < r\}$ and $B[x,r] = \{y \in X : \|x-y\| \leq r\}$. By a "subspace" we mean a closed subspace. If $A \subseteq X$, $\bar{A}$, denotes the norm closure of the set A. If $A \subseteq X^*$, by $\bar{A}^{\omega^*}$, we denote the weak* closure of A in X^*.

Let A be a non-empty subset of X and $x \in X$. The distance of x from A, denoted by $d(x,A)$, is given by

$$d(x,A) = \inf\{\|x-a\| : a \in A\}.$$

Define

$$P_A(x) = \{a \in A : \|x-a\| = d(x,A)\}.$$

Then $P_A(x)$ is a subset of X and can be empty. If A is convex, it is easily verified that $P_A(x)$ is a convex set, if non-empty.

Definition 1.1.1 Let A be a (non-empty) subset of normed linear space X. Then A is said to be *proximinal in* X if $P_A(x)$ is non-empty for each $x \in X$.

Note that $P_A(x) = \{x\}$, if $x \in A$. Thus $P_A(x) \neq \phi$ is a non-trivial condition only for $x \in X \backslash A$. Also, if $x \in \overline{A} \backslash A$, then $d(x, A) = 0$ and $P_A(x) = \phi$. Thus a proximinal set must be closed.

Definition 1.1.2 Let $A \subseteq X$, where X is normed linear space. If $P_A(x) = \{x\}$ for all $x \in X$, A is called a *Chebychev subset of* X.

Definition 1.1.3 Let $A \subseteq X$, where X is a normed linear space. If $x \in X$ and $a_0 \in P_A(x)$ then a_0 is called *a nearest element to x from A or a best approximation to x from A.*

Definition 1.1.4 Let $A \subseteq X$, where X is a normed linear space, and $x \in X$. A sequence $(a_n) = (a_n)_{n=1}^{\infty}$ in A is called a *minimizing sequence for x from A* if $\lim\limits_{n\to\infty} \|x - a_n\| = d(x, A)$.

It immediately follows from the definition of $d(x, A)$ that each $x \in X$ has at least one minimizing sequence from A. If a minimizing sequence (a_n) for x from A converges to a_0 in A, then

$$\|x - a_0\| = \lim_{n\to\infty} \|x - a_n\| = d(x, A)$$

and $a_0 \in P_A(x)$. This simple observation and its modifications form the basic idea of many proximinality proofs (see [FD]). We give two sample results below.

Proposition 1.1.5 *Let X be a finite dimensional normed linear space. and $A \in X$ be closed. Then A is proximinal in X.*

Proof: Pick any $x \in X$ and let (a_n) be a minimizing sequence for x. Let $d = d(x, A)$. Since $\lim\limits_{n\to\infty} \|x - a_n\| = d$, the sequence (a_n) is bounded. Now $\dim X < \infty$ implies that (a_n) has a convergent sequence, say, $(a_{n_k})_{k=1}^{\infty}$, that converges to some element a_0. Since A is closed, a_0 is in A and

$$\|x - a_0\| = \lim_{k\to\infty} \|x - a_{n_k}\| = \lim_{n\to\infty} \|x - a_n\| = d.$$

So $a_0 \in P_A(x)$ and A is proximinal in X.

Proposition 1.1.6 *Let X be a normed linear space and $A \subseteq X^*$ be a weak* closed set. Then A is proximinal in X^*.*

Proof: Pick any $f \in X^*$. The map $T : X^* \to \mathbb{R}$ given by $T(g) = \|f-g\|$, $g \in X^*$, is a weak* lower semi-continuous (l.s.c) on X^*. Let $d = d(f, A)$ and $D = A \cap B[f, d+1]$. Now $B[f, d+1] = f+(d+1)B_{X^*}$, is a weak* compact set by the Banach-Alaoglu's Theorem. Now A is weak* closed and so D is weak* compact. The l.s.c map T attains its infimum over D. If $T(g_0) = \inf_{g \in D} T(g)$, clearly $g_0 \in P_A(f)$.

Similarly it can be shown that if $A \subseteq X$ is weakly compact then A is proximinal in X. For further results in this direction, we refer to [12].

We observe that the above proximinality proofs use "compactness" in a crucial way. But the availability of compactness arguments in infinite dimensional spaces is rare. Thus it is necessary to look for methods, leading to proximinality proofs, that do not require compactness. In this regard, Functional Analysis and Geometry of Banach spaces, provide effective tools for some specified proximinal problems. More precisely, in these lectures, we will derive characterizations of proximinal (strongly proximinal) subspaces of finite codimension, using the methods of Functional Analysis and concepts from the Geometry of Banach spaces.

1.2 Garkavi's characterization

Let X be a normed linear space and $x \in X$. Throughout we identify x (and X) with its image $\hat{x}(\hat{X})$ under the canonical embedding of X into X^{**}. Let Y be a subspace of X. We denote by Q, the quotient map $Q : X \to X/Y$. We have $\|Q(x)\| = \|x + y\| = d(x, y)$.

Remark 1.2.1 *Let Y be a subspace of a normed linear space X. If $z \in X$ and $P_Y(z) \neq \phi$ then $\exists\ x \in X$ with $\|x\| = d(z, Y)$ and $x + Y = z + Y$.*

To see this, select $y_0 \in P_Y(x)$ and take $z = x - y_0$.

Let Y be a subspace of a normed linear space X. We denote $Y^\perp$, the annihilator of Y, given by

$$Y^\perp = \{f \in X^* : f \equiv 0 \text{ on } Y\}.$$

Recall that $(X/Y)^* \simeq Y^\perp$. That is, $(X/Y)^*$ is isometrically isomorphic to $Y^\perp$. If $T : (X/Y)^* \to Y^\perp$ is the natural isometric isomorphism, then $T^* : (Y^\perp)^* \to (X/Y)^{**}$ is an isometric isomorphism. If Y is of finite codimension, i.e. $\dim X/Y < \infty$, then X/Y is reflexive and $(X/Y)^{**} \cong X/Y$. In this case, $T^* : (Y^\perp)^* \to X/Y$ is an isometric isomorphism.

Remark 1.2.2 *Let Y be a subspace of a normed linear space X. If $x \in X$ and $\alpha \in \mathbb{R}$, $P_Y(\alpha x) = \alpha\ P_Y(x)$ and $d(\alpha x, Y) = |\alpha|\ d(x, Y)$. Thus for $x \in X \backslash Y$, $P_Y(x) \neq \phi \Leftrightarrow P_Y(\frac{x}{d}) \neq \phi$ where $d = d(x, Y)$. Thus, in order to prove Y is proximinal, it suffices to show that $P_Y(x) \neq \phi$ for $x \in X$ with $d(x, Y) = 1$.*

We now present a simple characterization of proximinal subspaces.

Proposition 1.2.1 *Let Y be a subspace of normed linear space X and $Q : X \to X/Y$ be the quotient map. Then Y is proximinal in X if and only if $Q(B_X) = B_{X/Y}$.*

Proof: Assume Y is proximinal. Since $Q(B_X) \subseteq B_{X/Y}$ we need to prove only the other inclusion. Consider $x \in X$ with $\|x + Y\| \leq 1$, and select any $y_0 \in P_Y(x)$. If $z = x - y_0$ then $\|z\| = \|x - y_0\| \leq 1$ and $Q(z) = x + Y$.

Now assume $Q(B_X) = B_{X/Y}$. To prove Y is proximinal, by above Remark, it suffices to show that $P_Y(x) \neq \phi$ for x with $d(x, Y) = 1$. For such an element x, by assumption, there exists z in B_X such that $z + Y = x + Y$. Let $y_0 = x - z$. Then $y_0 \in Y$, $\|x - y_0\| = \|z\| \leq 1 \leq d(x, Y)$. So $y_0 \in P_Y(x)$.

We next derive a characterization of proximinal subspaces of finite codimension.

Theorem 1.2.2 *[14] (Garkavi). Let Y be a subspace of finite codimension in a normed linear space. X. Then Y is proximinal in X if and only if for each ϕ in $B_{(Y^\perp)^*}$ there exists $x \in B_X$ such that $\phi(f) = f(x)$ for all $f \in Y^\perp$.*

Proof: Since Y is finite codimension in X, $(Y^\perp)^*$ is isometrically isomorphic to X/Y. Thus if $\phi \in (Y^\perp)^*$ there exists $x \in X$ such that $\|\phi\| = \|x + Y\|$

and $\phi(f) = f(x+Y) = f(x) \; \forall \; f \in Y^{\perp}$. Now by Proposition 1.2.1, we have

$$\begin{aligned} Y \text{ is proximinal} \quad &\Leftrightarrow \quad Q(B_X) = B_{X/Y} \\ &\Leftrightarrow \quad Q(B_X) = B_{X/Y} \cong B_{(Y^{\perp})^*} \\ &\Leftrightarrow \quad \text{For each } \phi \in B_{(Y^{\perp})^*}, \\ &\qquad \text{there exists } x \text{ in } B_X \text{ such that} \\ &\qquad \phi(f) = f(x) \text{ for all } f \in Y^{\perp}. \end{aligned}$$

There are two useful corollaries that can be derived from the above characterization of Garkavi.

Corollary 1.2.3 *Let Y be a subspace of finite codimension in a normed linear space X. If Y is proximinal in X then every subspace of X such that $Z \supseteq Y$, is proximinal in X.*

Proof: We have $Z^{\perp} \subseteq Y^{\perp}$ and Z is of finite codimension in X. Thus it suffices to show that for any $\psi \in B_{(Z^{\perp})^*}$, there exists $x \in B_X$ such that $\psi(f) = f(x) \;\; \forall \; f \in Z^{\perp} \subseteq Y^{\perp}$.

Let $\psi \in B_{(Z^{\perp})^*}$. Now $Z \supseteq Y$ implies $Z^{\perp} \subseteq Y^{\perp}$. By the Hahn-Banach theorem, there exists $\phi \in B_{(Y^{\perp})^*}$ such that $\phi \mid_{Z^{\perp}} = \psi$. Since Y is proximinal, by Theorem 1.2.2 there exists $x \in B_X$ such that $\phi(f) = f(x) \; \forall \; f \in Y^{\perp}$. Now $\phi \mid_{Z^{\perp}} = \psi$ and so $\psi(f) = f(x) \; \forall \; f \in Z^{\perp} \subseteq Y^{\perp}$. Using Theorem 1.2.2, we conclude Z is proximinal in X.

Corollary 1.2.4 *If Y is a proximinal subspace of finite codimension in X then every $f \in Y^{\perp}$ is a norm attaining functional on X.*

Proof: By the above Corollary 1.2.3, every hyperplane containing Y is proximinal. If $H = \ker f, \; f \in X^*$, is a hyperplane in X then $H \supseteq Y$ if and only if $f \in Y^{\perp}$. Observe that H is proximinal in X if and only if f is a norm attaining functional on X. It is now clear that every f in $Y^{\perp}$ is a norm attaining functional on X.

Let $NA(X)$ denote the set of all norm attaining functionals on X. That is,

$$NA(X) = \{f \in X^* : \exists \; x \in B_X \text{ such that } f(x) = \|f\|\}.$$

Corollary 1.2.4 implies that $Y^{\perp} \subseteq NA(X)$, if Y is proximinal in X.

Example 1.2.1 *Let $X = c_0$. Then $f \in X^* \cong l_1$ is in $NA(X)$ if and only if $f(n) = 0$, except for a finite subset of $\mathbb{N}$. Thats, $\exists$ a finite subset $\wedge$ of $\mathbb{N}$ such that $f(n) = 0 \;\; \forall \; n \in \mathbb{N} \backslash \wedge$.*

Let $f = (f(n))_{n=i}^{\infty} \in l_1$. Then $f \in NA(X)$ if and only if $\exists \; x = (x(n))_{n=i}^{\infty}$ in B_{c_0} such that

$$f(x) = \sum_{n=1}^{\infty} f(n) \; x(n) = \|f\| = \sum_{n=1}^{\infty} |f(n)|.$$

This with $\sup\{|x(n)| : n \in \mathbb{N}\} \leq 1$ implies that $x(n) = \text{sgn } f(n)$ whenever $f(n) \neq 0$. Since $\lim_{n \to \infty} x(n) = 0$, this implies $\{n \in \mathbb{N} : f(n) \neq 0\}$ is a finite set.

Conversely if $\{n \in \mathbb{N} : f(n) \neq 0\}$ is a finite set define

$$x(n) = \begin{cases} \text{sgn } f(n) & \text{if } \; f(n) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $x \in c_0$ and $\|x\|_\infty = 1$ and it is easy to see that $f(x) = \|f\|_1$.

A natural question that arises at this stage is whether the converse to Corollary 1.2.4 is valid. More precisely does $Y^\perp \subseteq NA(X)$ implies Y is proximinal? Phelps [13] gave an example, as early as 1963, that answers this question in the negative. In order to describe this example we require Garkavi's characterisation of proximinal subspace of finite codimension in $C(Q)$, where $C(Q)$ is the Banach space of continuous real valued functions defined on the compact Hausdroff space Q, with sup norm. We recall that $C(Q)^*$ is $M(Q)$, the space of all regular, bounded Borel measures on Q. If $\mu \in M(Q)$, the total variation of the measure μ over Q is $\|\mu\|$. Also by *supp*(μ), we denote the support of the measure μ, is the smallest closed subset Q outside which μ vanishes.

Theorem 1.2.5 *(Garkavi [15]) Let M be a subspace of finite dimension in $C(Q)$. Then M is proximinal if and only if the annihilator space $M^\perp$ satisfies the following three conditions:*

1. *supp*$(\mu^+) \cap$ *supp*$(\mu^-) = \emptyset$, *for every* $\mu \in M^\perp \backslash \{0\}$.
2. μ_2 *is absolutely continuous with respect to* μ_1 *on supp*(μ_1), *for every pair* $\mu_1, \mu_2 \in M^\perp \backslash \{0\}$.

3. *$supp(\mu^+)\backslash supp(\mu^-)$ is closed for each $\mu_1, \mu_2 \in M^\perp \backslash \{0\}$.*

We now give the example of Phelps [32] in which a subspace M of codimension 2 in any infinite dimensional $C(Q)$ space is constructed such that every hyperplane containing M is proximinal but M itself is not proximinal.

Example 1.2.2 *Select a sequence $(q_n)_{n=1}^{\infty}$ in Q with $q_n \neq q_m$ for $n \neq m$ which has a cluster point $q_0 \in Q$ with $q_0 \neq q_n$, for $n = 1, 2, \ldots$. Define $\mu_1, \mu_2 \in M(Q)$ by*

$$\begin{aligned} \mu_1 &= \sum_{n=1}^{\infty} \frac{1}{2^n}\, \delta_{q_n} + \delta_{q_0} \\ \mu_2 &= \sum_{n=1}^{\infty} \frac{1}{4^n}\, \delta_{q_n}, \end{aligned}$$

where

$$\delta_q(B) = \begin{cases} 1 & \text{if } q \in B \\ 0 & \text{if } q \notin B \end{cases} \quad \text{for any subset } B \text{ of } Q$$

Let $M = \{x \in C(Q) : \mu_i(x) = 0, \text{ for } i = 1, 2\}$. Then $M^\perp$ is the two dimensional subspace of $M(Q)$ generated by μ_1 and μ_2. If a is a scalar, we have

$$(\mu_1 + a\mu_2)q_n = \frac{1}{2^n} + \frac{a}{4^n} > 0 \text{ if } 2^n > -a.$$

This implies that for any $\mu \in M^\perp$, we have $supp(\mu^+) \cap supp(\mu^-) = \emptyset$ or, equivalently, every $\mu \in M^\perp$ attains its norm on $C(Q)$. Hence every hyperplane containing M is proximinal. However, M is not proximinal in $C(Q)$, since condition (2) of above theorem does not hold for μ_1 and μ_2.

1.3 Examples of $R(1)$ Spaces

In this section, we define $R(1)$ spaces and give two examples of $R(1)$ spaces.

Definition 1.3.1 *[25] A Banach space X would be called $R(1)$ space, if whenever Y is a subspace of finite codimension in X, then Y is proximinal if $Y^\perp \subseteq NA(X)$.*

One of the well known example of a $R(1)$ space is the sequence space c_0. We now give a proof of this fact. We need the following lemma in the proof.

Lemma 1.3.2 *Let X be a normed linear space with a monotone basis $(e_i)_{i=1}^{\infty}$ and $(e_i^*)_{i=1}^{\infty}$ denote the corresponding biorthogonal functionals in X^*. If M is a subspace of finite codimension in X such that $M^{\perp}$ is contained in span of $\{e_1^*, e_2^*, \dots e_k^*\}$ for some finite k, then M is proximinal.*

Proof: Let M_k denote the closed subspace generated by the infinite set of elements $\{e_{k+1}, e_{k+2}, \dots\}$. Note that $M_k^{\perp} = span\{e_1^*, e_2^*, \dots, e_k^*\} \supseteq M^{\perp}$. Thus $M \subseteq M_k$. Let P denote the natural projection from X onto M_k. Let $m = \sum\limits_{i=k+1}^{\infty} m_i e_i$ and $x = \sum\limits_{i=1}^{\infty} x_i e_i$ (where m_i and x_i are scalars) denote arbitrary elements in M_k and $X \backslash M_k$. Since (e_i) is monotone basis, we have

$$\begin{aligned} \|x - m\| &= \left\| \sum_{i=1}^{k} x_i e_i + \sum_{i=k+1}^{\infty} (x_i - m_i) e_i \right\| \\ &\geq \left\| \sum_{i=1}^{k} x_i e_i \right\| = \|x - P(x)\|. \end{aligned}$$

Since $m \in M_k$ was chosen arbitrarily, this implies $P(x) \in P_{M_k}(x)$. Thus M_k is proximinal in X. Then by Corollary 1.2.3 M is also proximinal in X. □

Lemma 1.3.3 *The Banach space c_0 is a $R(1)$ space.*

Proof: Let Y be a subspace of finite codimension in c_0 such that $Y^{\perp} \subseteq NA(X)$. It suffices to show that Y is proximinal. Let $(e_i^*)_{i=1}^{\infty}$ is the natural basis of $l_1 \cong (c_0)^*$ and $\{f_1, \dots, f_m\} \subseteq l_1$ be a basis of the finite dimensional space $Y^{\perp}$. Each functional of the basis is norm attaining as $Y^{\perp} \subseteq NA(X)$. Using Example 1.2.1 we conclude that each f_i, $1 \leq i \leq m$ has only finite number of non zero coordinates and hence in the span of $\{e_1^*, e_2^*, \dots e_{k_i}^*\}$, for suitable positive integer k_i. If $k = \max\{k_i : 1 \leq i \leq m\}$, it is clear that $Y^{\perp}$ is contained in the span of $\{e_1^*, e_2^*, \dots e_k^*\}$. Note that the natural basis $\{e_1, e_2, \dots\}$ of c_0 is monotone basis. Thus, by Lemma 1.3.2, Y is proximinal in c_0. □

We now describe a class of Banach spaces which are $R(1)$ spaces. For a Banach space X, let $NA_1(X)$ denote the set $NA(X) \cap S(X^*)$. We also need the following definitions and facts in the sequel.

Let X be a normed linear space. and $x \in X$. If Y is a subspace of a X, then it can be shown using the Hahn-Banach theorem that

$$d(x, Y) = \sup\{f(x) : f \in S_{Y^\perp}\} = \|\hat{x} \mid_{Y^\perp} \|,$$

where $\hat{x}$ denote the image of x, under the canonical embedding of X into X^{**} and $\hat{x} \mid_{Y^\perp}$, the restriction of $\hat{x}$ to the subspace $Y^\perp$ of X^*.

For $x \in X$, set $J_{X^*}(x) = \{f \in X^* : \|f\| \text{ and } f(x) = \|x\|\}$. Note that $J_{X^*}(x)$ is a non-empty subset, by the Hahn-Banach theorem.

Definition 1.3.4 Let X be a normed linear space and $x \in S_X$. We say x is a *smooth point of* B_X (or X) if $J_{X^*}(x)$ is a singleton set. In this case, we also write the normed linear space X is smooth at x.

Proposition 1.3.5 *[25] Let X be a Banach space with X^* smooth at every point of $NA_1(X)$. Then X is a $R(1)$ space.*

Proof: Let M be a subspace of finite codimension in X with $M^\perp \subseteq NA(X)$. It is enough to show that M is proximinal. Let $\phi \in B_{(M^\perp)^*}$. We need to get x in B_X such that

$$\phi(g) = g(x), \text{ for all } g \in Y^\perp.$$

Without loss of generality, we assume $\|\phi\| = 1$. Select $f \in M^\perp$ such that $\|f\| = 1$ and $\phi(f) = 1$. Let ϕ_0 be a Hahn-Banach extension of ϕ to X^*. Then ϕ_0 is in X^{**} and $\|\phi_0\| = \|\phi\| = 1$.

We have $M^\perp \subseteq NA(X)$ and hence there exist an $y \in S_X$ such that $f(y) = 1$. It is now clear that, $\{\hat{y}, \phi_0\} \subseteq J_{X^{**}}(f)$. Since X^* is smooth at f, we have $\phi_0 = \hat{y}$. This implies that

$$\phi(g) = \phi_0(g) = g(x) = g(y), \quad \forall\, g \in M^\perp.$$

Thus the condition of Theorem 1.2.2 is satisfied and M is proximinal in X.

□

1.4 Polyhedral spaces

In the next section, we prove that any subspace of c_0 is a $R(1)$ space. For this, we need the notion of polyhedral spaces and some related facts, which we present below. We begin with the definition of an extreme point.

Let X be a real vector space. For x, y in X, we denote by $[x, y]$ and (x, y), the closed and open line segments joining x and y respectively.

Definition 1.4.1 Let E be a (real) vector space and C be a convex subset of E. A non-empty subset D of C is called an *extremal subset of* C if whenever x and y are in C, and $D \cap (x, y)$ is non-empty, then $D \supseteq [x, y]$. An element x in C is called an *extreme point of* C if, $\{x\}$ is an extremal subset of C or equivalently, x is not an interior point of any line segment in C.

We denote by $\text{ext}(C)$, the set of extreme points of the set C. We list the following well known facts about extreme points. For a subset A of a vector space E, the smallest convex set containing A is called the *convex hull* of A and is denoted by $co(A)$.

Theorem 1.4.2 *Let X be a finite dimensional normed linear space and $C \subseteq X$ be convex and compact. Then $ext(C) \neq \phi$ and $C = co(ext(C))$.*

Proposition 1.4.3 *Let X be a Banach space.*

(i) If $C \subseteq X$ is weakly compact and convex, then $ext(C) \neq \phi$.

(ii) If $C \subseteq X^*$ is weak* compact and convex then $ext(C) \neq \phi$.

Definition 1.4.4 A finite dimensional normed linear space X is called *polyhedral* if its closed unit ball B_X has only finite number of extreme points. That is, $\text{ext}(B_X)$ is a finite set.

For infinite dimensional normed linear spacepaces, there are many definitions of polyhedral spaces. (See). We use the one given in [9].

Definition 1.4.5 *[9]* A Banach space X is called *polyhedral* if every finite dimensional subspace of X is polyhedral.

A well known example of a polyhedral space is the sequence space c_0.

The notion of a quasi polyhedral point, introduced by Dan Amir and Frank Deutsch, is closely related to polyhedral spaces.

Definition 1.4.6 *[1]* Let X be a Banach space and $x \in S_X$. We say x is a quasipolyhedral (QP) point of X (or B_X) if there exists $\epsilon > 0$ such that $J_{X^*}(y) \subseteq J_{X^*}(x)$ for all $y \in S_X \cap B(x, \epsilon)$.

We then have

Lemma 1.4.7 *Let X be a finite dimensional normed linear space. Then*

(i) X is polyhedral if and only if every x in S_X is a quasi polyhedral point of X.

(ii) X is polyhedral if and only if X^* is polyhedral.

Remark 1.4.1 *Let X be a Banach space and let Y be a subspace of X. If $x \in S_Y$ and x is a quasi polyhedral point of X, then x is a quasi polyhedral point of Y. This follows from the fact that if $z \in S_Y$ then $J_{Y^*}(z) = J_{X^*}(z) \mid_Y = \{f \mid_Y : f \in J_{X^*}(z)\}$.*

Definition 1.4.8 Let X be a Banach space and $x \in S_X$. We say x *is an exposed point of* B_X if there exists $f \in J_{X^*}(x)$ such that $f(y) < 1$ for all $y \in B_X \backslash \{x\}$. In this case, we say the linear functional f *exposes the element* x.

Lemma 1.4.9 *Let X be a finite dimensional polyhedral space. Then every extreme point of B_X is an exposed point of B_X.*

Remark 1.4.2 *Let X be a finite dimensional polyhedral space, Y a subspace of X and $x \in S_Y$, a quasi polyhedral point of X. Then by Remark 1.4.1,*

x is quasi polyhedral point of Y. This with (i) of Lemma 1.4.7 implies that Y is polyhedral. Thus any subspace of finite dimensional polyhedral space is again polyhedral. It now follows from Definition 1.4.5 that subspace of a polyhedral space is polyhedral. Equivalently, being polyhedral is a hereditary property.

For more details on polyhedral spaces and related topics, we refer the reader to [9] and [6].

1.5 Subspaces of c_0

In this section we prove that any subspace of c_0 is a $R(1)$ space. We need the following result giving an useful sufficient condition for proximinality of subspaces of finite codimension.

Proposition 1.5.1 *[20] Let X be a Banach space and Y be a subspace of finite codimension in X satisfying $Y^{\perp} \subseteq NA(X)$. Further assume that every point of $S_{Y^{\perp}}$ is a quasi polyhedral point of $Y^{\perp}$ (or equivalently by Lemma 1.4.7, $Y^{\perp}$ is polyhedral). Then Y is proximinal in X.*

Proof: Since the finite dimension space $Y^{\perp}$ is polyhedral, by Lemma 1.4.7 $(Y^{\perp})^*$ is also polyhedral. By Lemma 1.4.9 every extreme point of $B_{(Y^{\perp})^*}$ is an exposed point. Now select any extreme point ϕ of $B_{(Y^{\perp})^*}$. Let $f \in S_{Y^{\perp}}$ expose ϕ. Since $Y^{\perp} \subseteq NA(X)$, f is norm attaining and there exists $x \in X$ such that $\|x\| = f(x) = \|f\| = 1$. Now let $\psi \in \hat{x} \mid_{Y^{\perp}}$. Clearly, $\psi \in B_{(Y^{\perp})^*}$ and $\psi(f) = f(x) = 1$. Since f exposes ϕ, we must have $\psi = \phi$. Hence $\phi(g) = \psi(g) = g(x)$, for all $g \in Y^{\perp}$ and $\|\phi\| = \|x\| = 1$. Thus Garkavi criterion of Theorem 1.2.2 is satisfied for every extreme point of $B_{(Y^{\perp})^*}$. Now consider any element ψ of $B_{(Y^{\perp})^*}$. By Theorem 1.4.2, ψ is in the convex hull of $\text{ext}(B_{(Y^{\perp})^*})$. Thus there exist a finite subset $\phi_1, \phi_2, \dots \phi_k$ of $\text{ext}(B_{(Y^{\perp})^*})$ and positive scalar $\lambda_1, \lambda_2, \dots \lambda_k$ satisfying $\sum_{i=1}^{k} \lambda_i = 1$ such that $\psi = \sum_{i=1}^{k} \lambda_i \phi_i$. By the above argument, there exists a finite subset $\{x_1, x_2, \dots x_k\}$ of S_X such that $\hat{x}_i \mid_{Y^{\perp}} = \phi_i$, for $1 \leq i \leq k$. Let $x = \sum_{i=1}^{k} \lambda_i x_i$.

Clearly $x \in B_X$ and $\hat{x}\mid_{Y^\perp} = \psi$. Hence Garkavi's criterion of Theorem 1.2.2 holds for every element of $B_{(Y^\perp)^*}$ and it follows that Y is proximinal in X.

$\square$

The following corollary is an easy consequence of the above result and Remark 1.4.1. If X is a normed linear space, we set $NA_1(X) = NA(X) \cap S_{X^*}$.

Corollary 1.5.2 *Let X be a Banach space such that every functional in $NA_1(X)$ is a quasi polyhedral point of X^*. If Y is a subspace of finite codimension in X with $Y^\perp \subseteq NA(X)$, then $Y^\perp$ is polyhedral and Y is proximinal in X. Thus X is a $R(1)$ space.*

We now present an alternate proof for the fact that c_0 is an $R(1)$ space. We recall that the earlier proof of Lemma 1.3.3 uses the fact that c_0 has a monotone basis. Though simple, this proof does not lend itself to derive the result in more general cases. In contrast, the alternate proof given below, for c_0 being an $R(1)$ space, using the notion of quasi polyhedral points, would prove useful in showing that any subspace of c_0 is also a $R(1)$ space.

We begin with the alternate proof of c_0 being a $R(1)$ space and use similar path, to prove the result that any subspace of c_0 is a $R(1)$ space. The lemma below shows that any norm attaining functional, of norm of one in l_1 is a quasi polyhedral point of l_1.

Lemma 1.5.3 *If $f \in l_1 \cong (c_0)^*$ with $f \in NA_1(c_0)$, then f is a quasi polyhedral point of l_1.*

Proof: We recall that if $f \in NA(c_0)$ then f has only finite number of non zero entries. Let $2\alpha = \min\{|f(n)| : f(n) \neq 0\}$. Then $\alpha > 0$. Taking $X = l_1$ we have

$$J_{X^*}(f) = \{\phi \in l_\infty : \|\phi\| = 1 \text{ and } \phi(n) = \operatorname{sgn}(f(n)),\ \forall\, n \in \wedge_f\}, \qquad (1.1)$$

where $\wedge_f = \{n \in \mathbb{N} : f(n) \neq 0\}$. Consider any $g \in l_1$ with $\|g\| = 1$ and $\|f - g\| < \alpha$. Then clearly

$$g(n) \neq 0 \text{ and } \operatorname{sgn}(g(n)) = \operatorname{sgn}(f(n)),\ \forall\, n \in \wedge_f.$$

Thus $\wedge_f \subseteq \wedge_g$ and using (1.1) we have

$J_{X^*}(g) = \{\phi \in l_\infty : \|\phi\| = 1 \text{ and } \phi(n) = \text{ sgn}(g(n)),\ \forall\ n \in \wedge_g\} \subseteq J_{X^*}(f).$

This implies f is quasi polyhedral point of l_1. $\square$

It now follows from the above lemma and Corollary 1.5.2 that c_0 is a $R(1)$ space.

We now show that a similar conclusion holds for a subspace of c_0. We begin with a remark.

Remark 1.5.1 *Let Y be a subspace of Banach space Z, then $Y^* \cong Z^*/Y^\perp$ and $Y^{**} \cong Y^{\perp\perp} \subseteq Z^{**}$. Let $Q : Z^* \to Z^*/Y^\perp \cong Y^*$ denote the quotient map. If $\phi \in Y^{**} \cong Y^{\perp\perp}$, $H \in Z^*$ and $h = Q(H)$ then it is easily verified that*

$$\phi(H) = \phi(h).$$

Further, $Y^\perp$ is proximinal in X^*, being weak closed. If $h \in Y^*$ and $\|h\| = 1$, by Remark 1.2.1, there exists $H \in Z^*$ with $\|H\| = 1$ and $Q(H) = h$. Note that $H\ |_Y = h$.*

Lemma 1.5.4 *[20]Let X be a subspace of c_0. Then every functional in $NA_1(X)$ is a quasi polyhedral point of X^*.*

Proof: We will prove that if $g \in NA_1(X)$ then, there exists $\epsilon(g) > 0$ such that $h \in S(X^*)$ and $\|h - g\| < \epsilon(g)$ would imply $J_{X^{**}}(h) \subseteq J_{X^{**}}(g)$. This, in turn, would imply every functional in $NA_1(X)$ is a quasi polyhedral point of X^* and by Corollary 1.5.2, Y would be proximinal in X.

Clearly, we only need to show that if $g \in NA_1(X)$ and $(h_n) \subseteq S(X^*)$ converges to g then $J_{X^{**}}(h_n) \subseteq J_{X^{**}}(g)$ eventually. Recall that

$$c_0{}^* \cong l_1, \text{ and } \ l_1{}^* \cong l_\infty.$$

We denote by Q the canonical quotient map from l^1 onto $X^* \cong l_\infty/X^\perp$. The bidual of X is identified with the subspace $X^{\perp\perp}$ of l_∞. For each n, pick H_n in the unit sphere of l^1 such that $Q(H_n) = h_n$. Since $B(l_1)$ is weak* sequentially compact, taking a subsequence if necessary, we may and do assume that the sequence H_n converges *weak** to $G \in l_1$. We have $H_n \mid_X = h_n \; \forall \; n$ and further, (H_n) converges weak* to G implies that $G(x) = \lim_{n\to\infty} H_n(x) = \lim_{n\to\infty} h_n(x)$. But $\lim_{n\to\infty} h_n(x) = g(x), \; \forall \; x \in X$ and so $G \mid_X = g$. That is, $Q(G) = g$. Further $\|G\| \leq \lim \inf_n \|H_n\| \leq 1$. Since $\|g\| = 1$, we have $\|G\| = \|g\| = 1$.

For any H in l^1, we denote by

$$\mathrm{supp}(H) = \{k \geq 1 : H(e_k) \neq 0\}$$

where (e_k) is the natural basis of c_0. Note that for any $H \in l_1$ we have

$$J_{l_{1^*}}(H) = \{t = (t_k) \in S_{l_\infty} : t_k = \mathrm{sgn}(H(e_k)) \; \forall \; k \in \; \mathrm{supp}(H)\}. \tag{1.2}$$

Using Remark 1.5.1, we clearly have

$$\begin{aligned} J_{X^{**}}(h_n) &= \{\phi \in S(X^{**}) : \phi(h_n) = 1\} \\ &= \{\phi \in X^{\perp\perp} : \|\phi\| = 1, \; \phi(H_n) = 1\} \\ &= J_{l_{1^*}}(H_n) \cap X^{\perp\perp} \end{aligned} \tag{1.3}$$

and similarly

$$J_{X^{**}}(g) = J_{l_{1^*}}(G) \cap X^{\perp\perp}. \tag{1.4}$$

We observe now that $G \in NA_1(c_0)$. In fact, since $g \in NA_1(X)$, G attains its norm at some point of the unit sphere of X. It follows easily that supp(G) is a finite set.

Let $G = (G_k)_{k=1}^{\infty}$ and $H_n = (H_{n,k})_{k=1}^{\infty}$. Since the sequence (H_n) converges weak* to G in l_1, we have

$$G_k = \lim_{n\to\infty} H_{n,k}, \; \forall \; k \geq 1.$$

Thus $G_k \neq 0$ for some k implies $H_{n,k} \neq 0$ for all large enough n. Since G_k is zero except for finite number of k, there exists N such that for all $n \geq N$, we have supp$(G) \subset$ supp(H_n) and

$$\mathrm{sgn}(H_n(e_k)) = \mathrm{sgn}(G(e_k)),$$

for all $k \in$ supp(G). Thus by (1.2)

$$J_{l_1^*}(H_n) \subseteq J_{(l_1)^*}(G), \; \forall \; n \geq N.$$

It now follows from (1.3) and (1.4) that

$$J_{X^{**}}(h_n) \subseteq J_{X^{**}}(g) \ \forall\ n \geq N.$$

This concludes the proof of lemma. □

We can now present the main result of this section which follows immediately from Corollary 1.5.2 and the above Lemma 1.5.4.

Theorem 1.5.5 *[20] Every subspace of* c_0 *is a* $R(1)$ *space.*

2 Proximinal subspaces of finite codimension

In the following two sections, we derive another "more applicable" characterization of proximinal subspaces of finite codimension from Theorem 1.2.2 of Garkavi. For this, we require some definitions and facts related to convex sets and a characterization of extreme points of a finite dimensional closed convex set. These preliminaries are done in the following section before we give the main characterization theorem.

2.1 Finite dimensional convex sets

Let X be a (real) vector space and $C \subseteq X$ be convex. Pick any $x \in C$ and let $Y = sp(C - x)$. Then the set $Y + x$ is called *the affinehull of* C and consists of union of all lines, passing through any two distinct points of C. If $z \in C$ and $Z = sp(C - z)$ then $Y = Z$. To see this, note that if $y \in Y$, there exists $(\alpha_1 \dots \alpha_n) \in \mathbb{R}^n$ and $\{x_1 \dots x_n\} \subseteq C$ for some $n \in \mathbb{N}$ such that $y = \sum_{i=1}^{n} \alpha_i(x_i - x)$. Now

$$y = \sum_{i=1}^{n} \alpha_i(x_i - x) = \sum_{i=1}^{n} \alpha_i(x_i - z) - \sum_{i=1}^{n} \alpha_i(x - z).$$

Thus $Y \subseteq Z$. Interchanging Y and Z, we conclude $Y = Z$. The $d$imension of the convex set C is defined as the dimension of the subspace Y, which is the same as the dimension of the affinehull of C. We refer the reader to [36] and [16], for further details related to the following definitions and results, about convex sets.

Definition 2.1.1 Let X be a topological vector space and $C \subseteq X$ be convex. Then the *relative interior of* C (rel. int C) is the interior of C, considered as a subset of if affinehull and the *relative boundary of* C is the boundary of C, considered as a subset of its affinehull.

We will make frequent use of the following well known fact.

Theorem 2.1.2 *Let X be a finite dimensional normed linear space and $C \subseteq X$ be (non-empty) convex. Then*

(i) rel. int. C is non-empty

(ii) interior of C is non-empty $\Leftrightarrow \dim C = \dim X$.

Let X be a vector space, C a convex subset of X and E an extremal, convex subset of C. If $F \subseteq E$, then F is extremal in E if and only if F is extremal in C. Thus

$$\mathrm{ext}(E) = E \cap \ \mathrm{ext}(C). \tag{2.5}$$

Note that intersection of extremal subsets of a convex set is extremal, if non-empty.

Definition 2.1.3 *[13]* Let X be a vector space and $C \subseteq X$ convex. A convex extremal subset of C is called *a face of* C.

For more details of faces of convex sets and related concepts, we refer the reader to [13].

Example 2.1.1 *Let X be a normed linear space and $x \in S_X$. Then $J_{X^*}(x)$ is a face of B_{X^*}. To see this note that $J_{X^*}(x)$ is a convex, nonempty subset of B_{X^*}. If f and g are in B_{X^*} and $h \in J_{X^*}(x) \cap (f, g)$ then $h = \lambda f + (1-\lambda)g$ for some λ, $0 < \lambda < 1$. Further $\|h\| = 1$ and $h(x) = 1$. Now $1 = h(x) = \lambda f(x) + (1-\lambda)g(x)$ and this in the $f(x) \leq 1$ and $g(x) \leq 1$ implies $f(x) = g(x) = 1$. Hence $[f, g] \subseteq J_{X^*}(x)$.*

Note that intersection of faces of a convex set is a face, if non-empty.

Definition 2.1.4 Let X be a normed linear space and $C \subseteq X$. convex. Let $x \in C$ and $f \in X^*$. We say f *supports* C at x if $f(x) = \sup_C f = \sup\{f(y) : y \in C\}$.

If $x \in C$ and f supports C at x then $H \cap C$ is a face of C where $H = \{y \in X : f(y) = f(x)\}$. The following well known separation theorem is needed in the sequel.

Theorem 2.1.5 *[35](Separation Theorem)Let X be a non- zero normed linear space and $C \subseteq X$ be a convex , bounded set with non-empty interior. If x is in the boundary of C, then there exists a f in X^* such that $f(x) = \sup\{f(z) : z \in C\}$ and $f(w) < f(x)$ if w is in the interior of C. Equivalently, f supports C at x and $H \cap C$ is a proper face of C ($H \cap C \subsetneq C$), where $H = \{y \in X : f(y) = f(x)\}$.*

The following proposition is a useful corollary of the above theorem.

Proposition 2.1.6 *Let X be a normed linear space and $C \subseteq X$ be convex, closed and not a singleton set. If rel. int C is nonempty then for any $x \in$ rel. bd. C, there exists $f \in X^* \backslash \{0\}$ such that f supports C at x and $H \cap C$ is a proper face of C where $H = \{y \in X : f(y) = f(x) = \sup_C f\}$.*

Proof: Let $Y = sp(C - x)$ and consider $C - x = D$ as a subset of Y. Then $Y \neq \{0\}$, int D is nonempty and 0 is in the boundary of D. If $z \in$ rel. int C, then $z - x \in$ int D. By Theorem 2.1.5, there exists $g \in Y^* \backslash \{0\}$ such that $0 = g(0) = \sup\{g(y) : y \in D\}$ and $g(z - x) < 0$. Let $f \in X^*$ be a Hahn-Banach extension of g. Then $f \in X^*$ and

$$f(x) = \sup\{f(y) : y \in C\} \text{ and } f(z) < f(x).$$

Thus f supports C at x and if $H = \{y \in X : f(y) = f(x)\}$ then $H \cap C$ is a face of C. Further $z \in C \backslash H$ and $H \cap C$ is a proper face of C.

Corollary 2.1.7 *Let X be a finite dimensional normed linear space X and $C \subseteq X$ closed, convex and not a singleton set. If $x \in$ rel. bd C then $\exists\ f \in X^* \backslash \{0\}$ such that f supports C at x and $H \cap C$ is a proper face of C, where $H = \{y \in X : f(y) = f(x)\}$.*

Proof: Note that rel int C is non-empty by Theorem 2.1.2 and apply Proposition 2.1.6.

We now fix some notation, used hereafter. Let X be a normed linear space and $\{f_1 \dots f_n\} \subseteq X^*$. We define subsets $J_X(f_1, \dots f_i)$ for $1 \le i \le n$ inductively as follows.

$$J_X(f_1) = \{x \in B_X : f_1(x) = \|f_1\|\}. \tag{2.6}$$

Having define $J_X(f_1 \dots f_{i-1})$ we define

$$J_X(f_1, \dots f_i) = \{x \in J_X(f_1 \dots f_{i-1}) : f_i(x) = \sup\{f_i(y) : y \in J_X(f_1 \dots f_{i-1})\}\} \tag{2.7}$$

for $2 \le i \le n$. Note that $J_X(f_1) \neq \emptyset \Leftrightarrow f_1 \in NA(x)$. The sets $J_X(f_1 \dots f_i)$ can be empty and if non-empty, they are faces of B_X.

However, if X is finite dimensional then the sets $J_X(f_1 \dots f_i)$ are non-empty for $1 \le i \le n$. Further if $\dim X = n$ and $(f_1 \dots f_n)$ is a basis of X^*, then $J_X(f_1 \dots f_n)$ is a singleton set. We set

$$\alpha_i = \sup\{f_i(x) : x \in J_X(f_1, \dots, f_{i-1})\} \tag{2.8}$$

for $2 \le i \le n$. Clearly

$$\begin{aligned} J_X(f_1 \dots f_i) &= \{x \in J_X(f_1 \dots f_{i-1}) : f_i(x) = \alpha_i\} \\ &= \bigcap_{j=1}^{i} \{x \in B_X : f_j(x) = \alpha_j\} \end{aligned} \tag{2.9}$$

for $2 \le i \le n$. Further if $x_0 \in J_X(f_1 \dots f_i)$ then

$$J_X(f_1 \dots f_i) = \{x \in B_X : f_j(x) = f_j(x_0) \text{ for } 1 \le j \le i\} \tag{2.10}$$

for $1 \le i \le n$. We are now in a position to characterize extreme points of closed unit balls of finite dimensional normed linear spaces.

Theorem 2.1.8 *Let X be an n- dimensional normed linear space and $x \in B_X$. Then $x \in$ ext(B_X) if and only if there exists a basis $(f_1 \dots f_n)$ of X^* such that $\{x\} = J_X(f_1 \dots f_n)$.*

Proof: If $\{x\} = J_X(f_1 \dots f_n)$ then $\{x\}$ is extremal and $x \in$ ext(B_X).

Conversely assume that $x \in \text{ext}(B_X)$. Observe that if $\{x\} = J_X(f_1 \dots f_i)$ for some linearly independent subset $\{f_1 \dots f_i\}$ of X^*, then $(f_1 \dots f_i)$ can be extended to a basis $(f_1 \dots f_n)$ and $\{x\} = J_X(f_1 \dots f_n)$.

Since $x \in \text{ext}(B_X)$, x belongs to the boundary of B_X. By Corollary 2.1.7, $\exists f_1 \in X^* \backslash \{0\}$ such that $J_X(f_1)$ is a proper face of B_X and $x \in J_X(f_1)$. If $\{x\} = J_X(f_1)$, we are done. Otherwise, inductively assume that $x \in J_X(f_1, \dots f_{i-1})$ for some linearly independent subset $\{f_1 \dots f_{i-1}\}$ of X^*. If $\{x\} = J_X(f_1 \dots f_{i-1})$, nothing needs to be proved. So assume that the convex set $J_X(f_1 \dots f_{i-1})$ is not a singleton set. Note that x is an extreme point of the extremal set $J_X(f_1, \dots, f_{i-1})$. By Corollary 2.1.7, $\exists f_i \in X^* \backslash \{0\}$ such that $x \in J_X(f_1 \dots f_i)$ and $J_X(f_1 \dots f_i)$ is a proper face of $J_X(f_1 \dots f_{i-1})$. Note, using (2.9), that f_j is a constant on the set $J_X(f_1 \dots f_{i-1})$ for $1 \leq j \leq i-1$, while f_i is not. This is so since $J_X(f_1 \dots f_i) \subsetneqq J_X(f_1 \dots f_{i-1})$. It now follows that $(f_1 \dots f_i)$ is a linearly independent set and the induction is complete. If $\{x\} = J_X(f_1 \dots f_i)$, we are done. Otherwise proceed inductively to set a basis $(f_1 \dots f_n)$ of X^* such that $\{x\} = J_X(f_1 \dots f_n)$. $\square$

Let X be a normed linear space and Y be a subspace of codimension n in X. Then $\dim Y^\perp = n$. We use Theorem 2.1.8 to characterize extreme points of $B_{(Y^\perp)^*}$. Let E be a basis of $(Y^\perp)^{**}$. Then $E = (\hat{f}_1 \dots \hat{f}_n)$ for some basis $(f_1 \dots f_n)$ of $Y^\perp$. We identify $(\hat{f}_1 \dots \hat{f}_n)$ with $(f_1 \dots f_n)$ and write

$$J_{(Y^\perp)^*}(f_1) = \{\phi \in B_{(^\perp)^*} : \phi(f_1) = \|f_1\|\},$$

$$\beta_i = \sup\{\psi(f_i) : \psi \in J_{(Y\perp)^*}(f_1, \dots, f_{i-1})\} \tag{2.11}$$

and

$$J_{(Y^\perp)^*}(f_1 \dots f_i) = \{\phi \in J_{(Y^\perp)^*}(f_1, \dots, f_{i-1}) : \phi(f_i) = \beta_i\} \tag{2.12}$$

for $2 \leq i \leq n$. By Theorem 2.1.8, $\phi \in \text{ext}\ (B_{(Y^\perp)^*})$ if and only if there is a basis $(f_1 \dots f_n)$ of $Y^\perp$ such that $\{\phi\} = J_{(Y^\perp)^*}(f_1, \dots, f_n)$. With notation as above, we are in a position to state and prove the characterization theorem for proximinal subspaces of finite codimension, mentioned earlier.

Theorem 2.1.9 *[24] Let Y be a subspace of codimension n in a normed linear space X. Then Y is proximinal if and only if for every basis $(f_1 \dots f_n)$ of $Y^\perp$*

(i) $J_X(f_1 \dots f_i) \neq \phi$ *for* $1 \leq i \leq n$ *and*

(ii) $\alpha_i = \beta_i$ *for* $1 \leq i \leq n$

where α_i *and* β_i *are given by (2.8) and (2.11) respectively.*

Proof: Assume Y is proximinal and let $(f_1 \dots f_n)$ be a basis of $Y^\perp$. Then $J_{(Y^\perp)^*}(f_1 \dots f_n)$ is a singleton set, say $\{\phi\}$, and by Theorem 2.1.8, $\phi \in$ ext $(B_{(Y^\perp)^*})$. Now by Garkavi's characterization Theorem 1.2.2 there is x in B_X such that $\phi(f) = f(x) \ \forall \ f \in Y^\perp$. So $\phi(f_1) = f_1(x) = \|f_1\|$ and $x \in J_X(f_1)$. Further $\alpha_1 = \beta_1 = \|f_1\|$. Assume inductively that $J_X(f_1, \dots, f_{i-1}) \neq \phi$ and $\alpha_j = \beta_j$ for $1 \leq j \leq i-1$ for some $2 \leq i \leq n$.

For any $z \in B_X$, let $\psi_z = \hat{z} \mid_{Y^\perp}$. Then ψ_z is in $B_{(Y^\perp)^*}$. If $z \in J_X(f_1 \dots f_{i-1})$, $f_j(z) = f_j(x) = \alpha_j = \beta_j$ for $1 \leq j \leq i-1$. So $\psi_z \in J_{(Y^\perp)^*}(f_1 \dots f_{i-1})$. Thus

$$\{\psi_z : z \in J_X(f_1 \dots f_{i-1})\} \subseteq J_{(Y^\perp)^*}(f_1 \dots f_{i-1}).$$

Therefore

$$\begin{aligned} \alpha_i &= \sup\{f_i(z) : z \in J_X(f_1 \dots f_{i-1})\} \\ &= \sup\{\psi_z(f_i) : z \in J_X(f_1 \dots f_{i-1})\} \\ &\leq \sup\{\psi(f_i) : \psi \in J_{(Y^\perp)^*}(f_1 \dots f_{i-1})\} \\ &= \beta_i. \end{aligned}$$

Now $x \in B_X$ and $f_j(x) = \phi(f_j) = \beta_j = \alpha_j$ for $1 \leq j \leq i-1$. By (2.9), $x \in J_X(f_1 \dots f_{i-1})$. Further

$$\beta_i = \phi(f_i) = f_i(x) \leq \alpha_i.$$

Hence $\alpha_i = \beta_i$ and clearly $x \in J_X(f_i \dots f_i)$. This completes the induction and the proof for the necessity part of the theorem.

To prove sufficiently, assume (i) and (ii) hold for every basis $(f_1 \dots f_n)$ of $Y^\perp$. Let $\phi \in B_{(Y^\perp)^*}$. Then by Theorem 1.4.2, $\phi = \sum\limits_{i=1}^{k} \lambda_i \ \phi_i$, where $\phi_i \in$ ext$(B_{(Y^\perp)^*})$, $\lambda_i \geq 0$ for $1 \leq i \leq k$ and $\sum\limits_{i=1}^{k} \lambda_i = 1$, for some positive integer k. Assume that Garkavi's criteria holds for all extreme points of $B_{(Y^\perp)^*}$. That is, if $\psi \in$ ext$(B_{(Y^\perp)^*})$ there exists $z \in B_X$ such that $\psi(f) =$

$f(z)\ \forall\ f \in Y^{\perp}$. Then we can get a finite subset $\{x_i \ldots x_k\}$ of B_X such that $\phi_i(f) = f(x_i)\ \forall\ f \in Y^{\perp}$ and $1 \le i \le k$. Let $x = \sum\limits_{i=1}^{k} \lambda_i\ x_i$. Then $x \in B_X$ since $\lambda_i \ge 0$ and $\sum\limits_{i=1}^{k} \lambda_i = 1$. Further,

$$\phi(f) = \sum_{i=1}^{k} \lambda_i\ \phi_i(f) = \sum_{i=1}^{k} \lambda_i\ f(x_i) = f\left(\sum_{i=1}^{k} \lambda_i x_i\right) = f(x)$$

for all $f \in Y^{\perp}$. Thus Garkavi's criteria holds for ϕ. Hence it suffices to show that Garkavi's criteria holds for every extreme point of $B_{(Y^{\perp})^*}$.

Let $\phi \in \text{ ext}(B_{(Y^{\perp})^*})$. Then by Theorem 2.1.8 there is a basis $(f_1 \ldots f_n)$ of $Y^{\perp}$ such that

$$\{\phi\} = J_{(Y^{\perp})^*}(f_1 \ldots f_n) = \{\psi \in B_{(Y^{\perp})^*} : \psi(f_i) = \beta_i \text{ for } 1 \le i \le n\}.$$

By (i), $J_X(f_1 \ldots f_n) \ne \emptyset$. Pick any $x \in J_X(f_1 \ldots f_n)$. Then $\|x\| \le 1$ and by (ii),

$$f_i(x) = \alpha_i = \beta_i = \phi(f_i),\ \text{for } 1 \le i \le n$$

Since $(f_1 \ldots f_n)$ is a basis of $Y^{\perp}$, it now follows that $\phi(f) = f(x)\ \forall\ f \in Y^{\perp}$, and this completes the proof.

2.2 Strongly Proximinal Subspaces

Let X be a normed linear space and $\{f_1, f_2 \ldots, f_i\} \subseteq X^*$. For $\epsilon > 0$, we set

$$J_X(f_1, \epsilon) = \{x \in B_X : f_1(x) > \|f_1\| - \epsilon\}.$$

Having defined $J_X(f_1, \ldots, f_j, \epsilon)$ for $1 \le j \le i-1$ inductively, we define

$$J_X(f_1, \ldots, f_i, \epsilon) = \{x \in J_X(f_1 \ldots f_{i-1}, \epsilon) : f_i(x) > \alpha_i - \epsilon\}.$$

Note that

$$J_X(f_1, \ldots, f_i, \epsilon) = \cap_{j=1}^{i}\{x \in B_X : f_j(x) > \alpha_j - \epsilon\},\ \text{for } 1 \le i \le n.$$

We now define a notion stronger than proximinality.

Definition 2.2.1 Let X be a normed linear space and $A \subseteq X$ be a proximinal set. Let $x \in X$. We say A is *strongly proximinal at* x if given $\epsilon > 0$ there exists $\delta > 0$ such that

$$a \in A, \ \|x - a\| < d(x, A) + \delta \Rightarrow d(a, P_A(x)) < \epsilon.$$

We say A is *strongly proximinal in* X if A is strongly proximinal at each $x \in X$. As in proximinality, strong proximinality at x is a nontrivial condition only when $x \in X \backslash A$.

Usual compactness arguments show that if A is a proximinal subset of a finite dimensional normed linear space then A is strongly proximinal.

The following characterization of strongly proximinal subspaces of finite codimension can be derived from Theorem 2.1.9. We refer the reader to [21] for details.

Theorem 2.2.2 *[21] Let Y be a proximinal subspace of finite codimension n in a normed linear space X. Then Y is strongly proximinal in X if and only if for every basis $(f_1 \dots f_n)$ of $Y^\perp$,*

$$\lim_{\epsilon \longrightarrow 0} [\sup\{d(x, J_X(f_1 \dots f_i)) : x \in J_X(f_1 \dots f_i, \epsilon)\}] = 0$$

for $1 \leq i \leq n$.

Corollary 2.2.3 *Let Y be a strongly proximinal subspace of finite codimension in a normed linear space X. If Z is a subspace of X that contains Y then Z is strongly proximinal in X.*

Proof: By Corollary 1.2.3, Z is proximinal in X. The conclusion now follows easily from the above theorem.

Corollary 2.2.4 *[21] Let X be a normed linear space, $f \in X^*$ and $H = \ker f$ be a proximinal hyperplane in X. Then H is strongly proximinal in X if and only if*

$$\lim_{\epsilon \longrightarrow 0} [\sup\{d(x, J_X(f)) : x \in J_X(f, \epsilon)\}] = 0.$$

We now proceed to characterize strongly proximinal hyperplanes, in terms of a differential property of the dual norm on X^*. For this, we need the following definition and facts. Let F be a real valued convex function defined on a Banach space X. For fixed x and y in X, $\frac{F(x+ty)-F(x)}{t}$ is an increasing function of t and therefore $\lim\limits_{t\to 0^+} \frac{F(x+ty)-F(x)}{t}$ exists. Further, if $t > 0$,

$$\frac{F(x+ty)-F(x)}{t} \geq \lim_{t\to 0^+} \frac{F(x+ty)-F(x)}{t}.$$

The set of subdifferentials of F at x, denoted by $\partial F(x)$, is defined by

$$\partial F(x) = \{\phi \in X^* : \phi(y) \leq F(x+y) - F(x),\ \forall\ y \in X\}.$$

For $\phi \in \partial F(x)$ and $y \in X$, we have

$$\phi(y) \leq \lim_{t\to 0^+} \frac{F(x+ty)-F(x)}{t}.$$

If F is continuous at x then $\partial F(x)$ is non empty and moreover there exists (see Proposition 2.24 of [33]) a $\phi \in \partial F(x)$ such that

$$\phi(y) = \lim_{t\to 0^+} \frac{F(x+ty)-F(x)}{t}.$$

We refer the reader to [33],[34] and [16], for more details on differentiality of convex functions.

Definition 2.2.5 *Let X be a Banach space and $F : X \to \mathbb{R}$ be a convex function. We say F is strongly subdifferentiable (SSD) at $x \in X$, if the one sided limit $\lim\limits_{t\to 0^+} \frac{F(x+ty)-F(x)}{t}$ exists uniformly for $y \in S_X$.*

In the special case where F is the norm functional on X, the set $\partial F(x)$ is easily characterized.

Fact 2.2.6 *Let X be a Banach space, $x \in S_X$ and F be the norm functional on X. then $\partial F(x) = J_{X^*}(x)$*

Proof: Let $f \in \partial F(x)$. Then $f(y) - f(x) \leq \|y\| - \|x\|$ for all $y \in X$. Taking $y = x + z$, we have $f(z) \leq \|z + x\| - \|x\| \leq \|z\|$ for all $z \in X$. So $\|f\| \leq 1$. Taking $y = 0$, we have $f(x) = \|x\|$ and $f \in J_{X^*}(x)$.

Conversely if $f \in J_{X^*}(x)$, then for any $y \in X$, $f(y) - f(x) = f(y) - \|x\| \leq \|y\| - \|x\|$ for all $y \in X$.

Using the earlier remarks on convex functionals, we have for any $t > 0$

$$\frac{\|x + ty\| - \|x\|}{t} \geq \lim_{t \to 0+} \frac{\|x + ty\| - \|x\|}{t}$$

and for $y \in X$, there is a $f \in J_{X^*}(x)$ such that $f(y) = \lim_{t \to 0^+} \frac{\|x+ty\| - \|x\|}{t}$.

We now give a characterization of SSD points on the unit sphere of a Banach space.

Proposition 2.2.7 *[10] Let X be a Banach space and $x \in S_X$. Then the following conditions are equivalent.*

(a) The norm of X is SSD at x.

(b) Given $\epsilon > 0$, there exists $\delta > 0$ such that for f is in B_{X^*} satisfying $f(x) > 1 - \delta$, we have $d(f, J_{X^*}(x)) < \epsilon$.

Proof: a) $\Rightarrow$ b). Assume a). Suppose b) does not hold. Then there exists $\epsilon > 0$ and a sequence $(f_n) \subseteq S_{X^*}$ such that $\lim_{n \to \infty} f_n(x) = 1$ but $d(f_n, J_{X^*}(x)) > \epsilon$ for all n. Now $J_{X^*}(x)$ is a weak* compact, convex subset of B_{X^*} and $B[f_n, \epsilon] \cap J_{X^*}(x)$ is empty for all n. Using the separation theorem for the locally convex space X^* with the weak* topology, we can get a sequence $(z_n) \subseteq S_X$ such that

$$f_n(z_n) - \epsilon \geq \sup\{h(z_n) : h \in J_{X^*}(x)\}.$$

For each n, select $h_n \in J_{X^*}(x)$ such that

$$h_n(z_n) = \lim_{t \to 0^+} \frac{\|x + tz_n\| - \|x\|}{t}.$$

Now,

$$\begin{aligned} \frac{\|x + tz_n\| - \|x\|}{t} &\geq \frac{f_n(x + tz_n) - h_n(x)}{t} \\ &= t^{-1}(f_n - h_n)(x) + (f_n - h_n)(z_n) + h_n(z_n) \end{aligned}$$

and so

$$\frac{\|x+tz_n\|-\|x\|}{t} - \lim_{t\to 0^+} \frac{\|x+tz_n\|-\|x\|}{t} \geq t^{-1}(f_n-h_n)(x) = \epsilon$$

Taking $(t_n)^{-1} = \frac{\epsilon}{2(h_n-f_n)(x)}$, we have $t_n > 0$ for all n and $\lim_{n\to\infty} t_n = 0$. Hence

$$\begin{aligned}\frac{\|x+t_nz_n\|-\|x\|}{t_n} - \lim_{t\to 0^+} \frac{\|x+tz_n\|-\|x\|}{t} &\geq t_n^{-1}(f_n-h_n)(x)+\epsilon \\ &\geq \epsilon/2\end{aligned}$$

for all n. This contradicts a). So b) holds.

b) $\Rightarrow$ a). Let $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that $g \in B_{X^*}$ and $g(x) > 1-\delta$ implies $d(g, J_{X^*}(x)) < \epsilon$. Select any $y \in X$ such that $\|x-y\| < \delta$. Then for any $g \in J_{X^*}(y)$, we have

$$g(x) = g(y+x-y) = 1 - g(x-y) > 1-\delta$$

and so $d(g, J_{X^*}(x)) < \epsilon$. Thus there is $f \in J_{X^*}(x)$ such that $\|f-g\| < \epsilon$. Now

$$g(y-x) \geq \|y\| - \|x\| \geq f(y-x) \;\; \forall\, y \in X.$$

Taking $y = x+tz$ where $z \in S_X$ and $0 < t < \delta$, we have

$$g(z) \geq \frac{\|x+tz\|-\|x\|}{t} \geq f(z).$$

Thus

$$\left|\frac{\|x+tz\|-\|x\|}{t} - f(z)\right| \leq |(f-g)(z)| < \epsilon$$

for all $z \in S_X$ and $0 < t < \delta$. Now $f \in J_{X^*}(x)$ and so, for any $t > 0$,

$$f(z) \leq \lim_{t\to 0^+} \frac{\|x+tz\|-\|x\|}{t} \leq \frac{\|x+tz\|-\|x\|}{t}$$

It is now clear that

$$\begin{aligned}\left|\frac{\|x+tz\|-\|x\|}{t} - \lim_{t\to 0^+} \frac{\|x+tz\|-\|x\|}{t}\right| &\leq \left|\frac{\|x+tz\|-\|x\|}{t} - f(z)\right| \\ &< \epsilon\end{aligned}$$

for all $z \in S_X$ and for all $0 < t < \delta$. This implies a).

The result below shows that if X^* is SSD at f, then $f \in NA(X)$. We quote the Principle of local reflexivity, that is needed in the sequel.

Principle of local reflexivity:[7] Let X be a Banach space, $E \subseteq X^{**}$ and F, a finite dimensional subspace of X^*. Then given $\epsilon > 0$, there is an ϵ-isometry $T : X^{**} \to X$ such that $T_{|E \cap X}$ is identity and $f(T\phi) = \phi(f)$ for all $f \in F$. (We say $T : X \to Y$ is an ϵ-isometry if $1 - \epsilon \le \|T(x)\| \le 1 + \epsilon \;\; \forall\, x \in S_X$).

Theorem 2.2.8 *[21] Let X be a Banach space and $f \in S_{X^*}$. Then the norm of X^* is SSD at f if and only if $f \in NA_1(X)$ and given $\epsilon > 0$ there exists $\delta_\epsilon > 0$, such that for $x \in B_X$ satisfying $f(x) > 1 - \delta_\epsilon$, we have $d(x, J_X(f)) < \epsilon$.*

Proof: We first show that $f \in NA(X)$ and for $x \in X$, $d(x, J_{X^{**}}(f)) = d(x, J_X(f))$.

By Proposition 2.2.7, a) $\Rightarrow$ b), given $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that $\phi \in B_{X^{**}}$, $\phi(f) > 1 - \delta_\epsilon$ implies $d(\phi, J_{X^{**}}(f)) < \epsilon$. Pick any $x \in X$ and let $d = d(\hat{x}, J_{X^{**}}(f)) = d(x, J_{X^{**}}(f))$. Since $J_{X^{**}}(f)$ is a weak* compact subset of X^{**}, by Proposition 1.1.6, $J_{X^{**}}(f)$ is a proximinal set. So we can get ϕ in $J_{X^{**}}(f)$ such that $\|x - \phi\| = d$. Now by the Principle of local reflexivity, there exists $x_1 \in B_X$ such that $\|x - x_1\| < d + \epsilon/2$ and $f(x_1) > 1 - \delta_{\epsilon/2^2}$. Get $\phi_1 \in J_{X^{\lambda-1}}(f)$ such that $\|x_1 - \phi_1\| < \epsilon/2^2$. Using the Principle of local reflexivity, there exists $x_2 \in B_X$ such that $\|x_1 - x_2\| < \epsilon/2^2$ and $f(x_2) > 1 - \delta_{\epsilon/2^3}$. Proceeding thus inductively we construct a sequence $(x_n)_{n=1}^\infty$ in B_X such that $\|x_{n-1} - x_n\| < \epsilon/2^n$ and $f(x_n) > 1 - \delta_{\epsilon/2^{n+1}}$, for all n. Clearly $(x_n)_{n=1}^\infty$ is Cauchy and we let $z = \lim_{n\to\infty} x_n$. Now $z \in B_X$ and $f(z) = 1$ and so $z \in J_X(f)$. Thus $f \in NA_1(X)$. Now for any n,

$$\|x - x_n\| \le d + \epsilon/2 + \epsilon/2^2 + \ldots + \epsilon/2^N.$$

Taking limit as $n \to \infty$ we have, $\|x - z\| \le d + \epsilon$. Since $\epsilon > 0$ is arbitrary and $z \in J_X(f)$, we have $d(x, J_X(f)) \le d = d(x, J_{X^{**}}(f))$. Clearly $J_X(f) \subseteq J_{X^{**}}(f)$ and $d(x, J_X(f)) \ge d(x, J_{X^{**}}(f))$. Thus $d(x, J_X(f)) = d(x, J_{X^{**}}(f))$.

If now $x \in B_X$ and $f(x) > 1 - \delta_\epsilon$, we have

$$d(x, J_X(f)) = d(x, J_{X^{**}}(f)) < \epsilon.$$

Conversely assume that $f \in NA_1(X)$ and given $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that $x \in B_X$, $f(x) > 1 - \delta_\epsilon$ implies $d(x, J_X(f)) < \epsilon$. We will show that the norm of X^* is SSD at f. Select any $\phi \in B_{X^{**}}$ satisfying $\phi(f) > 1 - \delta_{\epsilon/2}$. By Proposition 2.2.7, it is enough to show that $d(\phi, J_{X^{**}}(f)) < \epsilon$. By Goldslein's theorem, there is a net $(x_\alpha) \subseteq B_X$ that converges weak* to ϕ. Since $\lim_\alpha f(x_\alpha) = \phi(f) = 1$, w.l.o.g. we can assume $f(x_\alpha) > 1 - \delta_{\epsilon/2}\ \forall\ \alpha$. So

$$d(x_\alpha, J_{X^{**}}(f)) \leq d(x_\alpha, J_X(f)) < \epsilon/2\ \forall\ \alpha. \tag{2.13}$$

The closed unit ball, $B_{X^{**}}$, and $J_{X^{**}}(f)$ are weak * compact sets. So, $A = J_{X^{**}}(f) + \epsilon/2 B_{X^{**}}$ is weak* compact. If $d(\phi, J_{X^{**}}(f)) > \epsilon/2$, then ϕ does not belong to the weak* compact set A and ϕ is in the weak* open set A^c, the complement of A. Since (x_α) converges weak* to ϕ, $x_\alpha \in A^c$ eventually. This contradicts (2.13) and $d(\phi, J_{X^{**}}(f)) \leq \epsilon/2 < \epsilon$.

The result below, characterizing strongly proximinal hyperplanes, follows immediately from the above Theorem and Corollary 2.2.4.

Corollary 2.2.9 *[21] Let X be a Banach space and $f \in X^*$. Then $H = \ker f$ is strongly proximinal in X if and only if the norm of X^* is SSD at f.*

Corollary 2.2.10 *[21] Let X be a Banach space and $f \in S_{X^*}$. If the norm of X^* is SSD at f then $J_X(f)$ is weak* dense in $J_{X^{**}}(f)$.*

Proof: Pick any $\phi \in J_{X^{**}}(f)$. Then there exists a net $(x_\alpha) \subseteq B_X$ that converges weak* to ϕ. Hence $\lim_\alpha f(x_\alpha) = 1$ and using Theorem 2.2.8, we can get a net $(y_\alpha) \subseteq J_X(f)$ such that $\lim_\alpha \|x_\alpha - y_\alpha\| = 0$. Clearly (y_α) converges weak* to ϕ and ϕ is in the weak* closure of $J_X(f)$.

Remark 2.2.1 *[21]Let X be a Banach space and $f \in X^*$. Note that $\overline{J_X(f)}^{\omega^*} \subseteq J_X^{**}(f)$. The above Corollary indicates that equality holds if f is a SSD point of X^*. In general, the above inclusion can be strict. Consider, for example, the Banach space $X = l_1$. Then $(l_1)^* \cong l_\infty$. If $f = (1, (1 - \frac{1}{n})_{n \geq 2})$, then $J_X(f) = \{e_i\}$ where $l_1 = (1, 0, 0, 0 \ldots)$. It can be easily verified that $J_{X^{**}}(f)$ is not a singleton set. Hence $\overline{J_X(f)}^{\omega^*} \subsetneqq J_{X^{**}}(f)$.*

We now return to the notion of quasi polyhedral point (QP-point) given by Definition 1.4.6. We first show that a QP-point is a SSD point.

Lemma 2.2.11 *[21] Let X be a Banach space and x in S_X be a QP-point. Then the norm of X is SSD at x.*

Proof: Since x is a QP-point, there exists $\delta > 0$ such that if $\omega \in B(x, 2\delta) \cap S_X$ then $J_{X^*}(\omega) \subseteq J_{X^*}(x)$. Select any $y \in S_X$ and fix $0 < t < \delta$. If $\omega = \frac{x+ty}{\|x+ty\|}$ then $\omega \in S_X$ and $\|x - \omega\| < 2\delta$. Thus $J_{X^*}(x + ty) = J_{X^*}(\omega) \subseteq J_{X^*}(x)$.

Pick any s such that $0 < s < t$ and let $\lambda = s/t$. Then $x + sy = \lambda(x + ty) + (1 - \lambda)x$. Now, for any f in $J_{X^*}(x + ty)$,

$$\|x + sy\| \geq f(x + sy) = \lambda\|x + ty\| + (1 - \lambda)\|x\| \geq \|x + sy\|.$$

Hence $f(x + sy) = \|x + sy\|$ and

$$\frac{\|x + sy\| - \|x\|}{s} = \frac{f(x + sy) - f(x)}{s} = f(y).$$

It is now clear that for all y in S_X, we have

$$\lim_{s \to 0^+} \frac{\|x + sy\| - \|x\|}{s} = \frac{\|x + ty\| - \|x\|}{t} = \frac{f(x + ty) - f(x)}{t} = f(y)$$

and the norm of X is SSD at x.

We now give a characterization of QP-points, that helps to visualize QP-points on the unit sphere.

Fact 2.2.12 *Let X be a Banach space and $x \in S_X$. Then x is a QP-point of X if and only if $\exists\ \epsilon > 0$ such that if $y \in S_X$ and $\|x - y\| < \epsilon$, then the line segment $[x, y]$ lies on the sphere S_X.*

Proof: Since x is a QP-point, $\exists\ \epsilon > 0$ such that $J_{X^*}(y) \subseteq J_{X^*}(x)$ for all $y \in S_X \cap B(x, \epsilon)$. Select any $y \in S_X \cap B(x, \epsilon)$ and $f \in J_{X^*}(y)$. Then $f \in J_{X^*}(x)$ and if $\omega \in [x, y]$ then $f(x) = f(y) = f(\omega) = 1$. Since $\|\omega\| \leq 1$, this implies $\|\omega\| = 1$ and $[x, y] \subseteq S_X$.

For the converse, let $x \in S_X$ and select $\epsilon > 0$ so that the given condition holds. Let $\alpha = \epsilon/2$ and $z \in S_X \cap B(x, \alpha)$. Considering the 2-dimensional subspace generated by x and z and using the given condition, we can easily get $y \in B(x, \epsilon) \cap S_X$ such that $z \in (x, y)$. So there exists $\lambda, 0 < \lambda < 1$, such that $z = \lambda x + (1 - \lambda)y$. If $f \in J_{X^*}(z)$ then

$$f(z) = 1, \ f(x) \leq 1 \text{ and } f(y) \leq 1$$

and so $1 = f(z) = \lambda f(x) + (1 - \lambda) \ f(y) \leq 1$. This implies $f(x) = f(y) = 1$ and $f \in J_{X^*}(x)$. Thus $J_{X^*}(z) \subseteq J_{X^*}(x)$ for all $z \in S_X \cap B(x, \alpha)$ and x is a QP-point of X. □

For a norm attaining functional f in X^*, the functional f being a QP-point can be characterized in terms of the sets $J_X(.)$, instead of the sets $J_{X^{**}}(.)$, as the following result shows.

Fact 2.2.13 *Let X be a Banach space and $f \in NA_1(X)$. Then f is a QP-point of X^* if there exists $\alpha > 0$ such that $J_X(g) \subseteq J_X(f)$, for all $g \in B(f, \alpha) \cap NA_1(X)$.*

Proof: The necessity follows the above Fact. To prove sufficiency, let $f \in NA_1(X)$ satisfy the condition of the lemma. If $g \in NA_1(X)$ and $\|f - g\| < \epsilon$, then by assumption $J_X(g) \subseteq J_X(f)$. Pick any z in $J_X(g)$. Then $f(z) = g(z) = 1$ and so $(f + g)(z) = 2$. Since $\|z\| = 1$, this implies $\|f + g\| = 2$. Hence

$$\|f + g\| = 2 \text{ for all } g \in B(f, \epsilon) \cap NA_1(X).$$

By the Bishop-Phelps theorem, the set $B(f, \epsilon) \cap NA_1(X)$, is dense in $B(f, \epsilon) \cap S_{X^*}$ and this with the continuity of the norm function yields

$$\|f + g\| = 2 \text{ for all } g \in B(f, \epsilon) \cap S_{X^*}.$$

It is now easy to verify the above equality implies $[f, g] \subseteq S_{X^*}$ if $g \in B(f, \epsilon/2) \cap S_{X^*}$. By Fact 2.2.12, f is a QP-point of X^*. □

The theorem below gives a useful sufficient condition for a proximinal subspace of finite codimension to be strongly proximinal.

Theorem 2.2.14 *[21] Let X be a Banach space with every $f \in NA_1(X)$ being a QP-point of the norm of X^*. If Y is a subspace of finite codimension in X such that $Y^\perp \subseteq NA(X)$ then Y is strongly proximinal in X.*

Proof: By Corollary 1.5.2, Y is proximinal in X. By Lemma 2.2.11, every $f \in NA_1(X)$ is a SSD point of X^*.

Let $(f_1 \dots f_n)$ be a basis of $Y^{\perp}$. We now show that we can select positive scalars $\lambda_i,\ 1 \le i \le n$, such that

$$J_X(f_1, \dots, f_i) = J_X\left(\sum_{j=1}^{i} \lambda_j\ f_j\right),\ \text{for } 1 \le i \le n. \tag{2.14}$$

We use induction on n. We take $\lambda_1 = 1$ and note that the case $n = 1$ is trivial. Inductively assume that $\lambda_j > 0$ for $1 \le j \le i-1$ have been chosen so that if $g_{i-1} = \sum_{j=1}^{i-1} \lambda_j\ f_j$ then $J_X(g_{i-1}) = J_X(f_1, f_2, \dots, f_{i-1})$. Now $g_{i-1} \in Y^{\perp}$ and so is a QP-point of X^*, by assumption. Choose $\lambda_i > 0$ small enough so that

$$J_X(g_{i-1} + \lambda_i f_i) \subseteq J_X(g_{i-1}).$$

By induction assumption,

$$J_X(g_{i-1}, f_i) = J_X(f_1, f_2, \dots, f_{i-1}, f_i).$$

We have $J_X(g_{i-1} + \lambda_i f_i) \subseteq J_X(g_{i-1})$ and $\lambda_i > 0$. It is now easy to verify that $J_X(g_{i-1}, f_i) = J_X(g_{i-1} + \lambda_i f_i)$ and we have

$$J_X(\sum_{j=1}^{i} \lambda_j f_j) = J_X(g_{i-1} + \lambda_i f_i) = J_X(g_{i-1}, f_i) = J_X(f_1, f_2, \dots, f_{i-1}, f_i).$$

This completes the induction and (2.14) holds. It now follows that

$$x \in J_X(\sum_{j=1}^{i} \lambda_j f_j) \ \Rightarrow\ f_i(x) = \alpha_i \text{ for } 1 \le i \le n, \tag{2.15}$$

where $\alpha_1 = \|f_1\|$ and $\alpha_i = \sup\{f_i(y) : y \in J_X(f_1, f_2, \dots, f_{i-1})\}$, for $2 \le i \le n$.

We now proceed to show that the condition of Theorem 2.2.2 holds for the basis $(f_1, f_2, \cdots, f_n)$. Recall that

$$J_X(f_1 \dots, f_i, \epsilon) = \cap_{j=1}^{i}\{x \in B_X : f_j(x) > \alpha_j - \epsilon\}.$$

Now $\sum_{j=1}^{i} \lambda_j f_j \in Y^{\perp} \subseteq NA(X) \subseteq$ QP-points of X^*, for $1 \leq i \leq n$. Thus the norm of X^* is SSD at $\sum_{j=1}^{i} \lambda_j f_j$ for $1 \leq i \leq n$. So by Theorem 2.2.8

$$\lim_{\epsilon \to 0} \sup \left\{ d \left(y, J_X \left(\sum_{j=1}^{i} \lambda_j f_j \right) \right) : y \in J_X \left(\sum_{j=1}^{i} \lambda_j f_j, \epsilon \right) \right\} = 0$$

for $1 \leq i \leq n$. It is easy to check that this with (2.14) and (2.15) implies

$$\lim_{\epsilon \to 0} \sup\{d(y, J_X(f_1 \ldots f_i)) : y \in J_X(f_1 \ldots, f_i, \epsilon)\} = 0$$

for $1 \leq i \leq n$. By Theorem 2.2.2, Y is strongly proximinal in X.

Note that, by Lemma 1.5.4, any subspace of c_0 satisfies the condition of the above theorem.

Let X be a Banach space and $B \subseteq S_{X^*}$. We say B is a *(James) boundary* of X if for each $x \in X$, $\exists\ f \in B$ such that $f(x) = \|x\|$. If X has a boundary B such that any weak* limit of B with norm one is not in $NA_1(X)$, then we say X has *Property* P [8]. Fonf and Lindenstrauss [8] showed that if a Banach space X has Property P then X satisfies the condition of Theorem 2.2.14. That is, every $f \in NA_1(X)$ is a QP-point of X^*. It follows from a result of Gleit and MacGuigan [18] that any separable, polyhedral Banach space has Property P. Thus we get a whole class of Banach spaces X which satisfy the condition of Theorem 2.2.14. In particular, these spaces are $R(1)$-spaces.

2.3 The convex functional S_C

Let X be a Banach space and $C \subseteq X$ be a closed convex and bounded set. Define a map $S_C : X^* \to \mathbb{R}$ by $S_C(f) = \sup_C f$, for $f \in X^*$. Then S_C is a continuous, convex functional on X^*. The following result characterizes the subdifferential of the convex functional S_C.

Fact 2.3.1 *[23]* $\partial S_C(f) = \{\phi \in \bar{C}^{\omega^*} : \phi(f) = S_C(f)\}$.

Proof: If $\phi \in \partial S_C(f)$, then

$$\phi(g) \leq S_C(f+g) - S_C(f) \leq S_C(g) \quad \forall \ g \in X^*$$

and $\phi \in \bar{C}^{\omega^*}$ by the bipolar theorem. Since $S_C(0) = 0$, we have $\phi(-f) \leq -S_C(f)$ and so $\phi(f) = S_C(f)$. Conversely, if $\phi \in \bar{C}^{\omega^*}$, then $\phi(g) \leq S_C(g)\ \forall\ g \in X^*$ and if $\phi(f) = S_C(f)$ we have

$$\phi(g - f) \leq S_C(g) - S_C(f) \quad \forall \ g \in X^*.$$

So $\phi \in \partial S_C(f)$.

We get

$$J_C(f) = \{x \in C : f(x) = \sup_C f\}. \tag{2.16}$$

Note that if non-empty, $J_C(f)$ is a closed convex subset of C. Further, $\bar{C}^{\omega^*}$ is a bounded, weak* closed subset of X^{**} and so is weak* compact. The weak* continuous functional f in X^* attains its supremum over $\bar{C}^{\omega^*}$. Hence ∂S_C is a non-empty, weak* compact convex subset of $\bar{C}^{\omega^*}$. Using arguments very similar to those in the proofs of Proposition 2.2.7 and Theorem 2.2.8, we can show that.

Proposition 2.3.2 *[23]: Let X be a Banach space and C be a closed, convex and bounded subset of X. Then the following are equivalent.*

(i) The convex functional S_C is SSD at $f \in X^*$.

(ii) Given $\epsilon > 0$ there exists $\delta > 0$ such that if $\phi \in \bar{C}^{\omega^*}$ and $\phi(f) > S_C(f) - \delta$, then $d(\phi, \partial S_C(f)) < \epsilon$.

(iii) The set $J_C(f)$ is non-empty and given $\epsilon > 0$, there exists $\delta > 0$ such that if $x \in C$ and $f(x) > S_C(f) - \delta$, then $d(x, J_C(f)) < \epsilon$.

Corollary 2.3.3 *[23]Let C be as in the above Proposition. If S_C is SSD at f then $J_C(f)$ is non-empty and $\overline{J_C(f)}^{\omega^*} = \partial S_C(f)$.*

The above Corollary is to be compared with Corollaries 2.2.9 and 2.2.10.

Let X be a Banach space and Y be a subspace of finite codimension n in X. Let $(f_1 \dots f_n)$ be a basis of $Y^{\perp}$. Set $C_0 = B_X$ and $C_i = J_{C_{i-1}}(f_i)$ for $1 \leq i \leq n$, where the sets $J_{C_{i-1}}(f_i)$ are as defined by (2.16). It is clear from (2.6) and (2.7) that

$$J_{C_{i-1}}(f_i) = J_X(f_1 \dots f_i), \quad \text{for } 1 \leq i \leq n.$$

Theorem 2.3.4 *Let X be a Banach space and Y be a subspace of codimension n in X. For any basis $(f_1 \dots f_n)$ of $Y^\perp$, define C_{i-1}, $1 \le i \le n+1$ and $S_{C_{i-1}}$, $1 \le i \le n$ as above. If for each basis $(f_1 \dots f_n)$ of $Y^\perp$, the convex functional $S_{C_{i-1}}$ is SSD at f_i for $1 \le i \le n$, then Y is strongly proximinal in X.*

Proof: We have

$$J_{C_{i-1}}(f_i) = J_X(f_1 \dots f_i), \quad \text{for } 1 \le i \le n.$$

Let

$$D_0 = B_{Y^{\perp *}} \text{ and } D_i = \{\phi \in D_{i-1} : \phi(f_i) = \sup_{D_{i-1}} f_i\}, \text{ for } 1 \le i \le n.$$

Then

$$D_i = J_{Y^{\perp *}}(f_1, f_2, \cdots f_i) \text{ for } 1 \le i \le n,$$

where $J_{Y^{\perp *}}(f_1, f_2, \cdots f_i)$ is given by (2.12).

We have

$$\overline{C_0}^{\omega^*} |_{Y^\perp} = \overline{B_X}^{\omega^*} |_{Y^\perp} = B_{X^{**}} |_{Y^\perp} = B_{Y^{\perp *}} = D_0.$$

Inductively assume that

$$\overline{C_{i-1}}^{\omega^*} |_{Y^\perp} = D_{i-1}.$$

We will show that $\overline{C_i}^{\omega^*} |_{Y^\perp} = D_i$. Note that the induction assumption implies

$$S_{C_{i-1}}(f_i) = \sup_{D_{i-1}} f_i.$$

The convex functional $S_{C_{i-1}}$ is SSD at f_i. By Proposition 2.3.2 and Corollary 2.3.3, the set $J_{C_{i-1}}(f_i)$ is non empty and

$$\overline{J_{C_{i-1}(f_i)}}^{\omega^*} = \overline{C_i}^{\omega^*} = \partial S_{C_{i-1}}(f_i) \quad = \quad \{\phi \in \overline{C_{i-1}}^{\omega^*} : \phi(f_i) = S_{C_{i-1}}(f_i)\}$$

Hence using the induction assumption, we have

$$\overline{C_i}^{\omega^*} |_{Y^\perp} = \{\phi \in \overline{C_{i-1}}^{\omega^*} |_{Y^\perp} : \phi(f_i) = S_{C_{i-1}}(f_i)\} = \{\phi \in D_{i-1} : \phi(f_i) = \sup_{D_{i-1}} f_i\} = D_i.$$

The induction is complete and we have

$$\overline{C_i}^{\omega^*} |_{Y^\perp} = D_i, \text{ for } 1 \le i \le n.$$

This implies

$$\alpha_i = \sup\{f_i(x) : x \in J_X(f_1, \ldots, f_{i-1})\} = \beta_i = \sup\{\psi(f_i) : \psi \in J_{(Y^\perp)^*}(f_1, \ldots, f_{i-1})\}$$

for $1 \leq i \leq n$. Since $C_i = J_X(f_1 \ldots f_i)$ is non empty for $1 \leq i \leq n$, it now follows from Theorem 2.1.9 that Y is proximinal in X.

We now proceed to show that Y is strongly proximinal in X. By Theorem 2.2.2, it suffices to show that for every basis $(f_1, \cdots, f_n)$ and every $1 \leq j \leq n$, we have

$$\lim_{\epsilon \to 0} [\sup\{d(x, C_j) : x \in J_X(f_1 \ldots f_j, \epsilon)\}] = 0. \tag{2.17}$$

Note that for $1 \leq i \leq n-1$,

$$J_X(f_1 \ldots f_j, \epsilon) = \cap_{i=1}^{j} \{x \in B_X : f_i(x) > S_{C_{i-1}}(f_i) - \epsilon\}.$$

We now proceed to show (2.18). The case $j = 1$ follows from $i) \Rightarrow iii)$ of Proposition 2.3.2. Using the same conclusion we have

$$\lim_{\epsilon \to 0} [\sup\{d(x, C_i) : x \in J_{C_{i-1}}(f_i, \epsilon)\}] = 0, \tag{2.18}$$

where

$$J_{C_{i-1}}(f_i, \epsilon) = \{x \in C_{i-1} : f_i(x) > S_{C_{i-1}}(f_i) - \epsilon.\}$$

We now apply (2.18) for $i = 1, 2, \cdots, j$ to obtain (2.17) and this completes the proof.

It has been shown (Lemma 5.2 [23]) that if $X = K(l_2)$, the space of all compact operators on l_2 and Y is a subspace of codimension n in X with $Y^\perp \subseteq NA(X)$, then Y satisfies the conditions of Theorem 2.3.4. Hence we have

Theorem 2.3.5 *[23] Let $X = K(l_2)$ and Y be a subspace of finite codimension in X with $Y^\perp \subseteq NA(X)$. Then Y is strongly proximinal in X and thus $K(l_2)$ is a $R(1)$ space.*

There are more articles on proximinal subspaces of finite codimension, proximinality and strong proximinality in general and related topics, mostly in the recent literature, that we have not dealt with. Some of them are [2], [3], [4], [19], [26], [27], [28], [29], [31],[37], [38],[39]and [40].

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