

Algebraic vector bundles on spheres

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Abstract

We complete the determination of the first non-stable \mathbb{A}^1 -homotopy sheaf of SL_n by treating the case where n is even. Using techniques of obstruction theory involving the \mathbb{A}^1 -Postnikov tower, supported by some ideas from the theory of unimodular rows, we classify vector bundles of rank $\geq \lfloor \frac{d}{2} \rfloor$ on split smooth affine quadrics of dimension d . These computations allow us to answer a question posed by Nori, which gives a criterion for completability of certain unimodular rows. Furthermore, we study compatibility of our computations of \mathbb{A}^1 -homotopy sheaves with real and complex realization.

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1 Introduction

In [AF12], we began a study of the first non-stable \mathbb{A}^1 -homotopy sheaf of the special linear group. In that paper, the computations of \mathbb{A}^1 -homotopy sheaves were used in conjunction with techniques of obstruction theory to give a cohomological classification of vector bundles on smooth affine threefolds (over algebraically closed fields having characteristic unequal to 2). This paper, which is a continuation of some of the themes of [AF12], answers some questions that were implicitly raised before.

Henceforth, fix a field k that is assumed to be infinite, perfect, and to have characteristic unequal to 2. We consider here the smooth affine variety Q_{2n-1} defined, for any integer $n \geq 1$, by the hypersurface $\sum_{i=1}^n x_i y_i = 1$ in \mathbb{A}^{2n} . Projecting onto x_1, \dots, x_n , the quadric Q_{2n-1} admits a

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morphism to $\mathbb{A}^n \setminus 0$ that is Zariski locally trivial and has affine space fibers; as a consequence this morphism is an isomorphism in the Morel-Voevodsky \mathbb{A}^1 -homotopy category $\mathcal{H}(k)$ [MV99]. This isomorphism can be used to identify the \mathbb{A}^1 -homotopy type of Q_{2n-1} as $\Sigma_s^{n-1} \mathbf{G}_m^{\wedge n}$, i.e., Q_{2n-1} is a smooth affine model of an \mathbb{A}^1 -homotopy sphere (see, e.g., [MV99, §3 Example 2.20]).

If we write $Gr_{r,\infty}$ for the infinite Grassmannian, Morel's \mathbb{A}^1 -homotopy classification of vector bundles [Mor12, Theorem 7.1] identifies the set of isomorphism classes of rank r vector bundles on Q_{2n-1} as the set of \mathbb{A}^1 -homotopy classes of maps $[Q_{2n-1}, BGL_r]_{\mathbb{A}^1}$. For $n \geq 2$, the space Q_{2n-1} has trivial Picard group, and therefore to classify vector bundles of rank r , it suffices to classify vector bundles with trivial determinant, which can be identified with the set $[Q_{2n-1}, BSL_r]_{\mathbb{A}^1}$. The benefit of this identification is that BSL_r is \mathbb{A}^1 -1-connected, and therefore the canonical map from pointed to unpointed \mathbb{A}^1 -homotopy classes of maps is a bijection. If we write $\mathcal{V}_r(X)$ for the set of isomorphism classes of rank r vector bundles on a smooth affine variety X , then for any integers $n, r \geq 2$ there are canonical isomorphisms

$$\mathcal{V}_r(Q_{2n-1}) \cong [\Sigma_s^{n-1} \mathbf{G}_m^{\wedge n}, BSL_r]_{\mathbb{A}^1},$$

Moreover, the (abelian) group $[\Sigma_s^{n-1} \mathbf{G}_m^{\wedge n}, BSL_r]_{\mathbb{A}^1}$ is the set of sections over k of the \mathbb{A}^1 -homotopy sheaf $\pi_{n-1,n}^{\mathbb{A}^1}(BSL_r)$.

Since the space BSL_r is \mathbb{A}^1 -connected, results of F. Morel identify the sheaf $\pi_{n-1,n}^{\mathbb{A}^1}(BSL_r)$ as the n -fold contraction of the sheaf $\pi_{n-1}^{\mathbb{A}^1}(BSL_r)$. When $n-1 \leq r-1$, the sheaf $\pi_{n-1}^{\mathbb{A}^1}(BSL_r)$ is already “stable” in the sense that it coincides with the sheaf \mathbf{K}_n^Q , i.e., the sheafification for the Nisnevich topology on smooth varieties of the Quillen K-theory presheaf (see [AF12, §2] for a more detailed discussion of the stable range in this context). In [AF12, Theorem 3.9], under the additional hypothesis that r was odd, we described $\pi_r^{\mathbb{A}^1}(BSL_r)$. Here, we finish the computation of $\pi_r^{\mathbb{A}^1}(BSL_r)$ by treating the case where r is even.

Theorem 1 (See Theorem 2.3 and Remark 2.4). *For any integer $n \geq 1$, there is a short exact sequence of strictly \mathbb{A}^1 -invariant sheaves of the form*

$$0 \longrightarrow \mathbf{T}_{2n+1} \longrightarrow \pi_{2n}^{\mathbb{A}^1}(BSL_{2n}) \longrightarrow \mathbf{K}_{2n}^Q \longrightarrow 0,$$

where \mathbf{T}_{2n+1} admits a description as the fiber product of strictly \mathbb{A}^1 -invariant sheaves

$$\begin{array}{ccc} \mathbf{T}_{2n+1} & \longrightarrow & \mathbf{I}^{2n+1} \\ \downarrow & & \downarrow \\ \mathbf{S}_{2n+1} & \longrightarrow & \mathbf{K}_{2n+1}^M/2, \end{array}$$

\mathbf{I}^{2n+1} is the unramified sheaf corresponding to the $2n+1$ -st power of the fundamental ideal in the Witt ring, $\mathbf{K}_{2n+1}^M/2$ is the unramified mod 2 Milnor K-theory sheaf, and \mathbf{S}_{2n+1} is the cokernel of a homomorphism $\mathbf{K}_{2n+1}^Q \rightarrow \mathbf{K}_{2n+1}^M$ that coincides with Suslin's homomorphism upon taking sections over fields.

Remark 2. When $n = 1$ in the above statement, the sheaf \mathbf{T}_3 is simply \mathbf{I}^3 . Indeed, in that case, Suslin proved [Sus84, Proposition 4.5] that the image of \mathbf{K}_3^Q in \mathbf{K}_3^M is precisely $2\mathbf{K}_3^M$ (strictly

speaking, Suslin establishes this contingent upon a portion of Milnor's conjecture on quadratic forms, but that is now known to hold by [OVV07, Theorem 4.1]). Therefore, the morphism $\mathbf{S}_3 \rightarrow \mathbf{K}_3^M/2$ is an isomorphism. In that case, using the identification $\mathbf{K}_2^Q \xrightarrow{\sim} \mathbf{K}_2^M$ that follows from Matsumoto's theorem, one sees that $\pi_2^{\mathbb{A}^1}(BSL_2)$ is an extension of \mathbf{K}_2^M by \mathbf{I}^3 . This description is consistent with Morel's identification $\pi_1^{\mathbb{A}^1}(SL_2) \cong \pi_2^{\mathbb{A}^1}(BSL_2) \xrightarrow{\sim} \mathbf{K}_2^{MW}$ [Mor12, Theorem 5.40]: our description corresponds precisely to the fact that \mathbf{K}_2^{MW} can be written as an extension of \mathbf{K}_2^M by \mathbf{I}^3 (see, e.g., Proposition 2.1).

Note that the sheaf \mathbf{S}_{2n+1} (resp. the sheaf \mathbf{S}_{2n} appearing in [AF12, Theorem 3.9]) admits an epimorphism from $\mathbf{K}_{2n+2}^M/(2n+1)!$ (resp. $\mathbf{K}_{2n+1}^M/(2n)!$). The question of whether this epimorphism is an isomorphism is, as was discussed in [AF12, Remark 5], equivalent to a question posed by Suslin. Unfortunately, the only case where a positive answer to Suslin's question is known is the case $n = 1$, as discussed in the previous remark. While Theorem 1 does not immediately provide enough information to completely describe the set of isomorphism classes of rank $n - 1$ vector bundles on Q_{2n-1} , it does reduce the problem to understanding contractions of \mathbf{S}_n . Moreover, upon n -fold contraction, the problem of providing an explicit description of \mathbf{S}_n becomes in a sense geometric and, with some input from the theory of unimodular rows, we can then give a rather explicit classification of rank $(n - 1)$ -vector bundles on Q_{2n-1} (the vector bundles of rank $\geq n$ are easy to describe as well). The next result can thus be viewed as a tiny piece of evidence that Suslin's question admits a positive answer; the dichotomy between the odd and even cases persists through all our results.

Theorem 3 (See Theorems 3.4 and 3.5). *If n is an integer ≥ 1 , and $W(k)$ denotes the Witt group of k , there are canonical isomorphisms*

$$\mathcal{V}_{n-1}(Q_{2n-1}) \xrightarrow{\sim} \begin{cases} \mathbb{Z}/(n-1)! & \text{if } n = 2m \\ \mathbb{Z}/(n-1)! \times_{\mathbb{Z}/2} W(k) & \text{if } n = 2m + 1; \end{cases}$$

where the maps in the fiber product are the rank homomorphism and the reduction modulo 2 map.

In [Sus77], Suslin gave a condition that was sufficient to ensure that a unimodular row (see Section 3 for some recollections about unimodular rows) over any ring R can be completed to an invertible matrix over R . In [Kum97], Nori inquired about a possible generalization of Suslin's theorem. In [Fas12], the second author constructed a counterexample to Nori's original question and proposed a refined version. The computations of Theorem 3 can be used to answer this refined version of Nori's question.

Theorem 4 (See Theorems 4.2 and 4.4). *Suppose k is a field, $R = k[x_1, \dots, x_n]$ be a polynomial ring in n variables, $\phi : R \rightarrow A$ is a k -algebra homomorphism such that $\sum \phi(x_i)A = A$, and f_1, \dots, f_n are elements of R such that reduced subscheme of \mathbb{A}^n defined by the ideal $I(f_1, \dots, f_n)$ coincides with $0 \in \mathbb{A}^n$; write $\mathbf{f} : \mathbb{A}^n \setminus 0 \rightarrow \mathbb{A}^n \setminus 0$ for the morphism induced by (f_1, \dots, f_n) . Assume that $\text{length}(R/(f_1, \dots, f_n))$ is divisible by $(n - 1)!$.*

- *If n is odd, then $(\phi(f_1), \dots, \phi(f_n))$ is completable.*
- *If n is even, then one can attach an element $\deg(\mathbf{f}) \in W(k)$ to \mathbf{f} , and if $\deg(\mathbf{f}) = 0$, then $(\phi(f_1), \dots, \phi(f_n))$ is completable.*

If Q_{2n} is the smooth affine quadric defined by the hypersurface $\sum_i x_i y_i = z(z+1)$ in \mathbb{A}^{2n} , then it is expected that Q_{2n} is also a motivic sphere (this is true for $n = 1, 2$). We give a description of the set of isomorphism classes of rank n vector bundles on Q_{2n} in Theorem 3.15 as well. Combining this description of isomorphism classes of vector bundles with Theorem 3, allows us to deduce Theorem 4.5, which discusses compatibility with complex realization of the computations of Theorem 2.3 and [AF12, Theorem 3.9]. In a sense, this compatibility explains that the factors of $n!$ that appear in the homotopy sheaves arise from complex Bott periodicity, while the factors of \mathbb{I}^n that appear arise because of real Bott periodicity. Finally, Theorem 4.7 discusses compatibility of the computation of the second non-stable homotopy sheaf of SL_2 (from [AF12, Theorem 3.20]) with complex realization, but since this is a low-dimensional result, the techniques are somewhat more explicit.

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Some notational preliminaries

We use the following notation. Assume k is a field. Write $\mathcal{S}m_k$ for the category of schemes that are smooth, separated and have finite type over $\text{Spec } k$ and $\mathcal{S}pc_k := \Delta^\circ \mathcal{S}hv_{\text{Nis}}(\mathcal{S}m_k)$ (resp. $\mathcal{S}pc_{k,\bullet}$) for the category of (pointed) simplicial sheaves on the site of smooth schemes equipped with the Nisnevich topology; objects of $\mathcal{S}pc_k$ (resp. $\mathcal{S}pc_{k,\bullet}$) will be referred to as (pointed) k -spaces, or simply as (pointed) spaces if k is clear from context. Write $\mathcal{H}(k)$ (resp. $\mathcal{H}_\bullet(k)$) for the Morel-Voevodsky \mathbb{A}^1 -homotopy category.

Given two (pointed) spaces X and \mathcal{Y} , we set $[X, \mathcal{Y}]_{\mathbb{A}^1} := \text{Hom}_{\mathcal{H}(k)}(X, \mathcal{Y})$; morphisms in pointed homotopy categories will be denoted similarly with base-points explicitly written if it is not clear from context. We write S_s^i for the constant sheaf on $\mathcal{S}m_k$ associated with the simplicial i -sphere, and \mathbf{G}_m will always be pointed by 1. The \mathbb{A}^1 -homotopy sheaves of a pointed space (X, x) , denoted $\pi_i^{\mathbb{A}^1}(X, x)$ are defined as the Nisnevich sheaves associated with the presheaves $U \mapsto [S_s^i \wedge U_+, (X, x)]_{\mathbb{A}^1}$. We also write $\pi_{i,j}^{\mathbb{A}^1}(X, x)$ for the Nisnevich sheafification of the presheaf $U \mapsto [S_s^i \wedge \mathbf{G}_m^{\wedge j} \wedge U_+, (X, x)]_{\mathbb{A}^1}$.

A presheaf of sets \mathcal{F} on $\mathcal{S}m_k$ is called \mathbb{A}^1 -invariant if for any smooth k -scheme U the morphism $\mathcal{F}(U) \rightarrow \mathcal{F}(U \times \mathbb{A}^1)$ induced by pullback along the projection $U \times \mathbb{A}^1 \rightarrow U$ is a bijection. A Nisnevich sheaf of groups \mathcal{G} is called *strongly* \mathbb{A}^1 -invariant if the cohomology presheaves $H_{\text{Nis}}^i(\cdot, \mathcal{G})$ are \mathbb{A}^1 -invariant for $i = 0, 1$. A Nisnevich sheaf of abelian groups \mathbf{A} is called *strictly* \mathbb{A}^1 -invariant if the cohomology presheaves $H_{\text{Nis}}^i(\cdot, \mathbf{A})$ are \mathbb{A}^1 -invariant for every $i \geq 0$. Henceforth, unless otherwise indicated, the word sheaf will mean Nisnevich sheaf on $\mathcal{S}m_k$, and the undecorated symbol H^i will mean “ i -th cohomology (of a sheaf) with respect to the Nisnevich topology.”

If $n \geq 0$ is an integer, a space \mathcal{X} is called \mathbb{A}^1 - n -connected if $\pi_0^{\mathbb{A}^1}(\mathcal{X}) = *$, and, for any choice of base-point $x \in \mathcal{X}(k)$ and any integer $i \leq n$, $\pi_i^{\mathbb{A}^1}(\mathcal{X}, x) = 0$. If G is an algebraic group and we view G as a pointed space, the base-point is always the identity section $\text{Spec } k \rightarrow G$ and for this reason will usually be suppressed. Likewise, the space BG , defined by means of the simplicial bar construction, has a canonical base-point corresponding to the unique 0-simplex, and this will

usually be suppressed from notation as well (just as in [AF12], we abuse notation and write BG for any space that has the \mathbb{A}^1 -homotopy type of the simplicial bar construction just mentioned).

2 The first non-stable homotopy sheaf of SL_{2n}

The goal of this section is to compute the group $\pi_{2n-1}^{\mathbb{A}^1}(SL_{2n})$ for $n \geq 1$. We begin by reviewing some notation and results from [Mor04] regarding Milnor-Witt K-theory. After that, we review some details regarding fibration sequences; a more detailed presentation of this material is given in [AF12, §3], and we will use a number of results from that work.

Some exact sequences

Write $K_*^{MW}(k)$ for the graded Milnor-Witt K-theory ring. Recall that $K_*^{MW}(k)$ is generated by symbols $[a] \in k^\times$ of degree $+1$ and a symbol η of degree -1 satisfying various relations [Mor04, Definition 5.1]. Write $I^*(k)$ for the graded ring corresponding to the powers of the fundamental ideal in the Witt ring; recall that $I^m(k)$ is additively generated by the classes of m -fold Pfister forms. Assigning to a symbol $a \in k^\times$ the class of the Pfister form $\langle\langle a \rangle\rangle$ defines a group homomorphism $K_1^{MW}(k) \rightarrow I^1(k)$; this homomorphism extends to a graded ring homomorphism $K_*^{MW}(k) \rightarrow I^*(k)$. Likewise, if $K_*^M(k)$ denotes the graded Milnor K-theory ring, there is also a homomorphism of graded rings $K_*^{MW}(k) \rightarrow K_*^M(k)$ that sends η to 0.

Let $k_*(k) = K_*^M(k)/2K_*^M(k)$ (we beg the reader's indulgence for this unfortunate choice of notation, which will persist only through this paragraph). There is a canonical homomorphism of graded rings $K_*^M(k) \rightarrow k_*(k)$. The Milnor conjecture on quadratic forms [OVV07] defines an isomorphism of graded rings $I^*(k)/I^{*+1}(k) \xrightarrow{\sim} k_*(k)$. Morel [Mor04, Theorem 5.3] shows that these various homomorphisms fit into a cartesian square of graded rings of the form

$$\begin{array}{ccc} K_*^{MW}(k) & \longrightarrow & K_*^M(k) \\ \downarrow & & \downarrow \\ I^*(k) & \longrightarrow & k_*(k). \end{array}$$

The above square can be sheafified in an appropriate sense: the objects and morphisms in the fiber square are compatible with residue maps and yield a cartesian square of unramified sheaves of graded rings

$$\begin{array}{ccc} \mathbf{K}_*^{MW} & \longrightarrow & \mathbf{K}_*^M \\ \downarrow & & \downarrow \\ \mathbf{I}^* & \longrightarrow & \mathbf{K}_*^M/2. \end{array}$$

We refer the reader to [Mor05, §2.2-4] for a detailed discussion of the unramified Milnor K-theory sheaf \mathbf{K}_n^M , the unramified sheaf \mathbf{I}^n and the homomorphism $\mathbf{I}^* \rightarrow \mathbf{K}_*^M/2$, which Morel calls a sheafification of Milnor's homomorphism. We refer the reader to [Mor12, §2] for the construction of the sheaf \mathbf{K}_n^{MW} and the homomorphism in the left hand column and the top row. Because the above diagram is cartesian, one deduces immediately the existence of the following exact sequences.

Proposition 2.1. *For every integer n , there are short exact sequences of the form*

$$0 \longrightarrow \mathbf{I}^{n+1} \longrightarrow \mathbf{K}_n^{MW} \longrightarrow \mathbf{K}_n^M \longrightarrow 0,$$

and for every integer $n \geq 0$, there are short exact sequences of the form

$$0 \longrightarrow 2\mathbf{K}_n^M \longrightarrow \mathbf{K}_n^{MW} \longrightarrow \mathbf{I}^n \longrightarrow 0.$$

Moreover, the map $\mathbf{K}_n^{MW} \rightarrow \mathbf{K}_{n-1}^{MW}$ induced by multiplication by η factors as a composite $\mathbf{K}_n^{MW} \rightarrow \mathbf{I}^n \rightarrow \mathbf{K}_{n-1}^{MW}$, where the two constituent maps are those in the above exact sequences.

Proof. The only thing that remains to be checked is the final statement. To that end, the map $K_n^{MW}(k) \rightarrow I^n(k)$ is defined by sending a symbol $[a_1] \cdots [a_n]$ to $\langle \langle a_1, \dots, a_n \rangle \rangle$, and the map $I^n(k) \rightarrow K_{n-1}^{MW}(k)$ is defined by sending a Pfister form $\langle \langle a_1, \dots, a_n \rangle \rangle$ to $\eta[a_1] \cdots [a_n]$. \square

Recollections on fiber sequences

Recall from [AF12] that the fiber sequence

$$SL_{2n} \longrightarrow SL_{2n+1} \longrightarrow SL_{2n+1}/SL_{2n}$$

yields an exact sequence of sheaves

$$\pi_{2n}^{\mathbb{A}^1}(SL_{2n}) \longrightarrow \pi_{2n}^{\mathbb{A}^1}(SL_{2n+1}) \xrightarrow{q_{2n}} \mathbf{K}_{2n+1}^{MW} \longrightarrow \pi_{2n-1}^{\mathbb{A}^1}(SL_{2n}) \longrightarrow \mathbf{K}_{2n}^Q \longrightarrow 0,$$

while the fiber sequence

$$SL_{2n+1} \longrightarrow SL_{2n+2} \longrightarrow SL_{2n+2}/SL_{2n+1}$$

gives an exact sequence

$$\mathbf{K}_{2n+2}^{MW} \xrightarrow{\delta_{2n+1}} \pi_{2n}^{\mathbb{A}^1}(SL_{2n+1}) \longrightarrow \mathbf{K}_{2n+1}^Q \longrightarrow 0.$$

The composition $q_{2n} \circ \delta_{2n+1}$ is trivial by [AF12, Lemma 3.1], and thus the morphism $\pi_{2n}(SL_{2n+1}) \rightarrow \mathbf{K}_{2n}^{MW}$ factors through a map $\pi_{2n}^{\mathbb{A}^1}(SL_{2n+2}) = \mathbf{K}_{2n+1}^Q \rightarrow \mathbf{K}_{2n+1}^{MW}$, and we obtain an exact sequence of the form:

$$\mathbf{K}_{2n+1}^Q \xrightarrow{\psi_{2n+1}} \mathbf{K}_{2n+1}^{MW} \longrightarrow \pi_{2n-1}^{\mathbb{A}^1}(SL_{2n}) \longrightarrow \mathbf{K}_{2n}^Q \longrightarrow 0$$

with a morphism ψ_{2n+1} that we want to identify.

The image of ψ_{2n+1}

Lemma 2.2. *The morphism ψ_{2n+1} has image contained in $2\mathbf{K}_{2n+1}^M$.*

Proof. Consider the diagram

$$\begin{array}{ccccccc}
 & & \pi_{2n}^{\mathbb{A}^1}(SL_{2n+1}) & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & 2\mathbf{K}_{2n+1}^M & \longrightarrow & \mathbf{K}_{2n+1}^{MW} & \longrightarrow & \mathbf{I}^{2n+1} \longrightarrow 0 \\
 & & \downarrow & \searrow \eta & \downarrow & & \\
 & & \pi_{2n-1}^{\mathbb{A}^1}(SL_{2n-1}) & \longrightarrow & \pi_{2n-1}^{\mathbb{A}^1}(SL_{2n}) & \longrightarrow & \mathbf{K}_{2n}^{MW} \longrightarrow \pi_{2n-2}^{\mathbb{A}^1}(SL_{2n-1}) \\
 & & \downarrow & & \downarrow & & \\
 & & \mathbf{K}_{2n}^Q & \longrightarrow & \mathbf{K}_{2n}^M & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

The short exact sequence in the second row and the vertical short exact sequence involving \mathbf{I}^{2n+1} are those from Proposition 2.1. Moreover, the commutativity of the triangle with the arrow labeled η as its bottom edge is also a consequence of Proposition 2.1. The commutativity of the lower triangle with η on the diagonal was established in the discussion preceding [AF12, Lemma 3.1] (the composite map is the connecting homomorphism in a long exact sequence in a fiber sequence involving a Stiefel variety).

Now, any element in $\pi_{2n}^{\mathbb{A}^1}(SL_{2n+1})$ goes to zero in $\pi_{2n-1}^{\mathbb{A}^1}(SL_{2n})$, and therefore the composite into \mathbf{K}_{2n}^{MW} is also zero. By commutativity of the diagram, the image of an element in $\pi_{2n}^{\mathbb{A}^1}(SL_{2n+1})$ in \mathbf{I}^{2n+1} is also zero. Therefore, the map $\pi_{2n}^{\mathbb{A}^1}(SL_{2n+1}) \rightarrow \mathbf{K}_{2n+1}^{MW}$ has image in $2\mathbf{K}_{2n+1}^M$.

On the other hand, consider the diagram

$$\begin{array}{ccc}
 \pi_{2n+1}^{\mathbb{A}^1}(SL_{2n+2}/SL_{2n+1}) & & \\
 \downarrow & \searrow & \\
 \pi_{2n}^{\mathbb{A}^1}(SL_{2n+1}) & \longrightarrow & \mathbf{K}_{2n+1}^{MW} \\
 \downarrow & & \\
 \pi_{2n}^{\mathbb{A}^1}(SL_{2n+2}) & & \\
 \downarrow & & \\
 0 & &
 \end{array}$$

The diagonal map is the zero map by [AF12, Lemma 3.1], and therefore the map $\pi_{2n}^{\mathbb{A}^1}(SL_{2n+1}) \rightarrow \mathbf{K}_{2n+1}^{MW}$ factors through the map $\pi_{2n}^{\mathbb{A}^1}(SL_{2n+1}) \rightarrow \pi_{2n}^{\mathbb{A}^1}(SL_{2n+2}) = \mathbf{K}_{2n+1}^Q$. Combining these two observations, the image of $\pi_{2n}^{\mathbb{A}^1}(SL_{2n+1}) \rightarrow \mathbf{K}_{2n+1}^{MW}$ is contained in the image of a map $\mathbf{K}_{2n+1}^Q \rightarrow 2\mathbf{K}_{2n+1}^M \subset \mathbf{K}_{2n+1}^{MW}$. \square

Let \mathbf{T}_{2n+1} be the kernel of the morphism $\pi_{2n-1}^{\mathbb{A}^1}(SL_{2n}) \rightarrow \mathbf{K}_{2n}^Q$, so that we have an exact sequence of sheaves

$$0 \longrightarrow \mathbf{T}_{2n+1} \longrightarrow \pi_{2n-1}^{\mathbb{A}^1}(SL_{2n}) \longrightarrow \mathbf{K}_{2n}^Q \longrightarrow 0.$$

Theorem 2.3. *The cartesian square (see above Proposition 2.1)*

$$\begin{array}{ccc} \mathbf{K}_{2n+1}^{MW} & \longrightarrow & \mathbf{I}^{2n+1} \\ \downarrow & & \downarrow \\ \mathbf{K}_{2n+1}^M & \longrightarrow & \mathbf{K}_{2n+1}^M/2 \end{array}$$

induces a cartesian square of the form

$$\begin{array}{ccc} \mathbf{T}_{2n+1} & \longrightarrow & \mathbf{I}^{2n+1} \\ \downarrow & & \downarrow \\ \mathbf{S}_{2n+1} & \longrightarrow & \mathbf{K}_{2n+1}^M/2. \end{array}$$

Proof. By definition of \mathbf{T}_{2n+1} , we have an exact sequence

$$\mathbf{K}_{2n+1}^Q \xrightarrow{\psi_{2n+1}} \mathbf{K}_{2n+1}^{MW} \longrightarrow \mathbf{T}_{2n+1} \longrightarrow 0.$$

Lemma 2.2 shows that the image of ψ_{2n+1} is completely determined by the composite

$$\mathbf{K}_{2n+1}^Q \xrightarrow{\psi_{2n+1}} \mathbf{K}_{2n+1}^{MW} \longrightarrow \mathbf{K}_{2n+1}^M,$$

which is precisely the morphism considered in [AF12, Lemma 3.8]. \square

Remark 2.4. Theorem 1 follows by combining Theorem 2.3 with the isomorphism $\pi_i^{\mathbb{A}^1}(SL_n) \cong \pi_{i+1}^{\mathbb{A}^1}(BSL_n)$ arising from the \mathbb{A}^1 -fiber sequence $SL_n \rightarrow ESL_n \rightarrow BSL_n$. In the sequel, we will often use the computation in this form. Recall also from [AF12, Theorem 3.9] that there is an epimorphism of sheaves

$$\mathbf{K}_{2n+1}^M/(2n)! \rightarrow \mathbf{S}_{2n+1}$$

that is conjecturally an isomorphism.

Remark 2.5. Assume for this remark that k is a field having characteristic zero. The inclusions $SL_m(k) \rightarrow SL_{m+1}(k)$ induce homomorphisms

$$f_{m,n} : H_n(SL_m(k), \mathbb{Z}) \longrightarrow H_n(SL_{m+1}(k), \mathbb{Z}),$$

which are isomorphisms if $m \geq n + 1$, and if $m = n$ is odd [HT10, Theorem 1.1]. Moreover, there is a sequence of the form

$$H_{m-1}(SL_m(k), \mathbb{Z}) \longrightarrow H_m(SL_m(k), \mathbb{Z}) \xrightarrow{\epsilon_m} K_m^{MW}(k)$$

for any $m \geq 1$ [BM99]. By [HT10, Theorem 1.1], this sequence is exact in the middle and ϵ_m is surjective if m is even, while its image is $2K_m^M(F)$ if m is odd. Given these results one defines a map $f_{2n+1} : K_{2n+1}^Q(k) \rightarrow K_{2n+1}^{MW}(k)$ as the following composition:

$$\begin{array}{ccc}
 K_{2n+1}^Q(k) & \xlongequal{\quad} & \pi_{2n+1}(BSL_\infty(k)^+) \rightarrow H_{2n+1}(BSL_\infty(k)^+, \mathbb{Z}) \xlongequal{\quad} H_{2n+1}(BSL_\infty(k), \mathbb{Z}) \\
 & \searrow f_{2n+1} & \downarrow \\
 & & H_{2n+1}(BSL_{2n+1}(k), \mathbb{Z}) \\
 & & \downarrow \epsilon_{2n+1} \\
 & & K_{2n+1}^{MW}(k).
 \end{array}$$

Observe that since $2n+1$ is odd, the image of f_{2n+1} is included in $2K_{2n+1}^M(k)$. While it is not necessary for our purposes, we expect that one can show that the morphism induced by ψ_{2n+1} upon taking sections over fields coincides with f_{2n+1} .

Contracted homotopy sheaves

Recall that if \mathbf{A} is a strictly \mathbb{A}^1 -invariant sheaf, one defines the contracted sheaf \mathbf{A}_{-1} by means of the formula $\mathbf{A}_{-1}(U) = \ker(s^* : \mathbf{A}(\mathbf{G}_m \times U) \rightarrow \mathbf{A}(U))$, where $s : U \rightarrow \mathbf{G}_m \times U$ is the map coming from the identity section of \mathbf{G}_m . One then defines the i -fold contracted sheaf \mathbf{A}_{-i} inductively by $\mathbf{A}_{-i} = (\mathbf{A}_{-i+1})_{-1}$. A convenient summary of calculations of contractions used here, and other basic properties of the contraction construction is presented in [AF12, §5].

Lemma 2.6. *If $j \geq 0$, $i \geq 1$, and $n \geq 2$ are integers, then there are canonical isomorphisms $\pi_{i,j}^{\mathbb{A}^1}(GL_n) \cong \pi_i^{\mathbb{A}^1}(GL_n)_{-j}$.*

Proof. The fibration sequence $GL_n \rightarrow EGL_n \rightarrow BGL_n$ gives isomorphisms $\pi_{i,j}^{\mathbb{A}^1}(GL_n) \cong \pi_{i+1,j}^{\mathbb{A}^1}(BGL_n)$, and BGL_n is \mathbb{A}^1 -connected. The result then follows from [Mor12, Theorem 5.13] (note: we cannot apply the aforementioned result directly to GL_n since it fails to be \mathbb{A}^1 -connected). \square

Since the contraction construction is exact, we deduce the following results from Lemma 2.6, [AF12, Theorem 3.9] and Theorem 2.3 (resp. [AF12, Theorem 3.20]).

Proposition 2.7. *Suppose $n \geq 1$ and $j \geq 0$ are integers. There are short exact sequences of the form*

$$\begin{aligned}
 0 &\longrightarrow (\mathbf{T}_{2n+1})_{-j} \longrightarrow \pi_{2n-1,j}^{\mathbb{A}^1}(GL_{2n}) \longrightarrow \mathbf{K}_{2n-j}^Q \longrightarrow 0, \text{ and} \\
 0 &\longrightarrow (\mathbf{S}_{2(n+1)})_{-j} \longrightarrow \pi_{2n,j}^{\mathbb{A}^1}(GL_{2n+1}) \longrightarrow \mathbf{K}_{2n+1-j}^Q \longrightarrow 0,
 \end{aligned}$$

where there is an epimorphism $\mathbf{K}_{n+1-j}^M/n! \rightarrow (\mathbf{S}_{n+1})_{-j}$.

Proposition 2.8. *For any integer $j \geq 0$, there is a short exact sequence of the form*

$$0 \longrightarrow (\mathbf{S}_4'')_{-j} \longrightarrow \pi_{2,j}^{\mathbb{A}^1}(GL_2) \longrightarrow (\mathbf{K}_3^{Sp})_{-j} \longrightarrow 0,$$

where $(\mathbf{S}_4'')_{-4}$ sits in a short exact sequence of the form

$$(\mathbf{I}^5)_{-4} \longrightarrow (\mathbf{S}_4'')_{-4} \longrightarrow (\mathbf{S}_4')_{-4} \longrightarrow 0$$

and there is an epimorphism $\mathbf{K}_{4-j}^M/12 \rightarrow (\mathbf{S}_4')_{-j}$.

As discussed in the introduction, the epimorphism $\mathbf{K}_{n+1}^M/n! \rightarrow \mathbf{S}_{n+1}$ is an isomorphism provided a question posed by Suslin has a positive answer. In the sequel, we will several times need to use the fact that the map on cohomology induced by the epimorphism just mentioned is an isomorphism. This conclusion can be deduced independently of a positive answer to Suslin's question by establishing that appropriate contractions of the sheaves in question are isomorphic. The next two lemmas summarize the results in the form we need.

Lemma 2.9. *The epimorphism $\mathbf{K}_{n+1}^M/n! \rightarrow \mathbf{S}_{n+1}$ induces isomorphisms $\mathbf{K}_{n+1-j}^M/n! \rightarrow (\mathbf{S}_{n+1})_{-j}$ for any $j \geq n-1$.*

Proof. Recall first that \mathbf{S}_{n+1} is defined by the exact sequence

$$\mathbf{K}_{n+1}^Q \longrightarrow \mathbf{K}_{n+1}^M \longrightarrow \mathbf{S}_{n+1} \longrightarrow 0$$

where the morphism $\mathbf{K}_{n+1}^Q \rightarrow \mathbf{K}_{n+1}^M$ coincides with Suslin's homomorphism when evaluated on fields (which are infinite and finitely generated over the base field).

On the other hand, using Rost's theory of cycle modules [Ros96], one can construct a morphism of sheaves $\alpha_{n+1} : \mathbf{K}_{n+1}^M \rightarrow \mathbf{K}_{n+1}^Q$ whose sections over fields (again, assumed to be infinite and finitely generated over the base field) coincides with the natural homomorphism from Milnor K-theory to Quillen K-theory. Indeed, this follows from [Ros96, Remark 5.4], which we quickly summarize. The natural transformation of functors $K_*^M(\cdot) \rightarrow K_*^Q(\cdot)$ on the category of fields that are finitely generated over the base field (which is assumed infinite throughout) is compatible with residue maps and transfer maps and yields a morphism of cycle modules. Any morphism of cycle modules yields a corresponding morphism of the associated “unramified” sheaves (see [Ros96, Remark 5.2]). Moreover, since a morphism of cycle modules is by definition compatible with the action of $K_*^M(\cdot)$ by left multiplication, it follows that the contracted morphism $(\alpha_{n+1})_{-j} : \mathbf{K}_{n+1-j}^M \rightarrow \mathbf{K}_{n+1-j}^Q$ coincides with α_{n+1-j} for any $j \in \mathbb{N}$.

Next, by [Sus84, Corollary 4.4], the composite

$$\mathbf{K}_{n+1}^M \xrightarrow{\alpha_{n+1}} \mathbf{K}_{n+1}^Q \longrightarrow \mathbf{K}_{n+1}^M$$

is multiplication by $(-1)^n \cdot n!$ and we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} \mathbf{K}_{n+1}^M & \xrightarrow{(-1)^n \cdot n!} & \mathbf{K}_{n+1}^M & \longrightarrow & \mathbf{K}_{n+1}^M/n! & \longrightarrow & 0 \\ \alpha_{n+1} \downarrow & & \parallel & & \downarrow & & \\ \mathbf{K}_{n+1}^Q & \longrightarrow & \mathbf{K}_{n+1}^M & \longrightarrow & \mathbf{S}_{n+1} & \longrightarrow & 0. \end{array}$$

Since contraction is an exact functor, by contracting j times we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc}
 \mathbf{K}_{n+1-j}^M & \xrightarrow{(-1)^n \cdot n!} & \mathbf{K}_{n+1-j}^M & \longrightarrow & \mathbf{K}_{n+1-j}^M/n! & \longrightarrow & 0 \\
 \alpha_{n+1-j} \downarrow & & \parallel & & \downarrow & & \\
 \mathbf{K}_{n+1-j}^Q & \longrightarrow & \mathbf{K}_{n+1-j}^M & \longrightarrow & (\mathbf{S}_{n+1})_{-j} & \longrightarrow & 0.
 \end{array}$$

If $j \geq n - 1$, then α_{n+1-j} is an isomorphism and the result follows. \square

The next result uses the notation of [AF12, Lemmas 3.18-19 and Theorem 3.20].

Lemma 2.10. *The epimorphism $\mathbf{K}_4^M/12 \rightarrow \mathbf{S}'_4$ induces an isomorphism $\mathbf{K}_{4-j}^M/12 \rightarrow (\mathbf{S}'_4)_{-j}$ for any $j \geq 3$, and there is a short exact sequence of the form*

$$0 \longrightarrow \mathbf{I} \longrightarrow \mathbf{S}''_4 \longrightarrow \mathbb{Z}/12 \longrightarrow 0.$$

Proof. By definition, the sheaf \mathbf{S}'_4 admits the following presentation:

$$\mathbf{K}_4^{Sp} \xrightarrow{\varphi_4 \circ f_{4,2}} \mathbf{K}_4^M \longrightarrow \mathbf{S}'_4 \longrightarrow 0,$$

where $f_{4,2} : \mathbf{K}_4^{Sp} \rightarrow \mathbf{K}_4^Q$ is the forgetful morphism and $\varphi_4 : \mathbf{K}_4^Q \rightarrow \mathbf{K}_4^M$ is Suslin's homomorphism. Using [AF12, Proposition 4.2, Proposition 5.4], we see that $(\mathbf{K}_4^{Sp})_{-j} = \mathbf{GW}_{4-j}^{2-j}$, where the latter denotes the Nisnevich sheaf associated to the Grothendieck-Witt group \mathbf{GW}_{4-j}^{2-j} (see [AF12, §4] for more details regarding this group). Since the forgetful functor preserves Gersten resolutions, we get $(f_{4,2})_{-j} = f_{4-j,2-j}$. In view of the above lemma, it suffices to prove that $f_{4-j,2-j}(\mathbf{GW}_{4-j}^{2-j}) = 2\mathbf{K}_{4-j}^Q$ for $j \geq 3$. Consider the hyperbolic homomorphism $H_{4-j,2-j} : \mathbf{K}_{4-j}^Q \rightarrow \mathbf{GW}_{4-j}^{2-j}$. The composition $f_{4-j,2-j} \circ H_{4-j,2-j}$ is multiplication by 2 by [AF12, Lemma 4.3] and it suffices to show that $H_{4-j,2-j}$ is an epimorphism provided $j \geq 3$; this follows from [FRS11, Lemma 2.3].

By definition, \mathbf{S}''_4 is the cokernel of a morphism $\mathbf{K}_4^{Sp} \rightarrow \mathbf{K}_4^{MW}$ and fits into a short exact sequence of the form $\mathbf{I}^5 \rightarrow \mathbf{S}''_4 \rightarrow \mathbf{S}'_4 \rightarrow 0$. Contracting the morphism defining \mathbf{S}''_4 four times, and using the identifications of the previous paragraph, together with the identification $(\mathbf{K}_4^{MW})_{-4} \cong \mathbf{K}_0^{MW}$, we see that $(\mathbf{S}''_4)_{-4}$ is the cokernel of a morphism

$$\mathbf{GW}_0^{-2} \longrightarrow \mathbf{K}_0^{MW}.$$

Observe that $\mathbf{GW}_0^{-2} = \mathbf{GW}_0^2$ and [FRS11, Lemma 2.3] tells us that the hyperbolic homomorphism determines an isomorphism $\mathbb{Z} \xrightarrow{\sim} \mathbf{GW}_0^2$. On the other hand, we know that \mathbf{K}_0^{MW} sits in a short exact sequence of the form $0 \longrightarrow \mathbf{I} \rightarrow \mathbf{K}_0^{MW} \rightarrow \mathbf{K}_0^M \rightarrow 0$. Moreover, the proof of [AF12, Lemma 3.17] shows that the composite map $\mathbb{Z} \rightarrow \mathbf{K}_0^{MW} \rightarrow \mathbb{Z}$ is the map studied in the previous paragraph; the result follows from this observation. \square

3 Unimodular rows and vector bundles on split quadrics

In this section, we begin by reviewing some ideas from the theory of unimodular rows. We then use the computations of Section 2 and [AF12, §3] together with techniques of obstruction theory using the Postnikov tower in \mathbb{A}^1 -homotopy theory (we refer the reader to [AF12, §6] for a digest of all the results that will be used) to give a general procedure to describe sets of isomorphism classes of vector bundles. We refer to vector bundles on the split smooth affine quadric Q_d having rank $\lfloor \frac{d}{2} \rfloor$ as those of critical rank: above this rank the classification of vector bundles is a stable problem; at or below this rank, the problem is unstable.

Unimodular rows

Let R be a ring and let $n \geq 3$ be an integer. Recall that a row (a_1, \dots, a_n) of elements of R is called unimodular if there exists (b_1, \dots, b_n) such that $\sum a_i b_i = 1$. We denote by $Um_n(R)$ the set of unimodular rows of length n over R . We consider this set as a pointed set, the base point being the row $e_1 := (1, 0, \dots, 0)$. The group $GL_n(R)$ acts on $Um_n(R)$ by multiplication on the right and so do all subgroups of $GL_n(R)$. In this paper, we will be mostly interested in the groups $SL_n(R)$ and $E_n(R)$, the subgroup generated by elementary matrices.

Let X and Y be two schemes over a field k . Recall that two morphisms of schemes (over k) $f, g : X \rightarrow Y$ are said to be naively \mathbb{A}^1 -homotopic if there exists a morphism $F : X \times \mathbb{A}^1 \rightarrow Y$ such that $F(0) = f$ and $F(1) = g$. We can consider the equivalence relation generated by naive \mathbb{A}^1 -homotopies and we write $\text{Hom}_{\mathbb{A}^1}(X, Y)$ for the set of naive \mathbb{A}^1 -homotopy classes of morphisms from X to Y .

Suppose k is a field, and R is a (commutative unital) k -algebra. A unimodular row (a_1, \dots, a_n) can be seen as a morphism $\text{Spec } R \rightarrow \mathbb{A}^n \setminus 0$ and therefore $Um_n(R) = \text{Hom}(\text{Spec } R, \mathbb{A}^n \setminus 0)$. In this context, we have $Um_n(R)/E_n(R) = \text{Hom}_{\mathbb{A}^1}(\text{Spec } R, \mathbb{A}^n \setminus 0)$ provided that R is smooth [Fas11b, Theorem 2.1].

For any pair of smooth k -schemes X and Y , the map $\text{Hom}_{\mathcal{S}m_k}(X, Y) \rightarrow [X, Y]_{\mathbb{A}^1}$ factors through a map

$$\text{Hom}_{\mathbb{A}^1}(X, Y) \rightarrow [X, Y]_{\mathbb{A}^1}$$

since naively \mathbb{A}^1 -homotopic morphism become equal in $\mathcal{H}(k)$. In the special case where X is smooth affine and $Y = \mathbb{A}^n \setminus 0$, the map $\text{Hom}_{\mathbb{A}^1}(X, \mathbb{A}^n \setminus 0) \rightarrow [X, \mathbb{A}^n \setminus 0]_{\mathbb{A}^1}$ is in fact a bijection [Mor12, Remark 7.10]. It follows that the right-hand side is generated by morphisms of schemes $X \rightarrow \mathbb{A}^n \setminus 0$, i.e., unimodular rows of length n over $\mathcal{O}_X(X)$.

Since $\mathbb{A}^n \setminus 0$ is $\mathbb{A}^1 - (n - 2)$ -connected, we can further identify the set $[X, \mathbb{A}^n \setminus 0]_{\mathbb{A}^1}$ with the cohomology group $H^{n-1}(X, \mathbf{K}_n^{MW})$ provided X is isomorphic (in $\mathcal{H}(k)$) to a smooth scheme of dimension $\leq n - 1$ [Mor12, Theorem 7.16, footnote 11]. More precisely, we can write

$$H^{n-1}(\mathbb{A}^n \setminus 0, \mathbf{K}_n^{MW}) = GW(k) \cdot \xi,$$

where ξ is an explicit element of $H^{n-1}(\mathbb{A}^n \setminus 0, \mathbf{K}_n^{MW})$ that we call the orientation class [Fas11b, §3.3]. The bijection $[X, \mathbb{A}^n \setminus 0]_{\mathbb{A}^1} \rightarrow H^{n-1}(X, \mathbf{K}_n^{MW})$ is then given by pulling-back the class ξ .

Vector bundles on Q_{2n-1}

Let $n \geq 2$ and $A_{2n-1} = k[x_1, \dots, x_n, y_1, \dots, y_n] / \langle \sum x_i y_i - 1 \rangle$. We denote by Q_{2n-1} the scheme $\text{Spec } A_{2n-1}$. The goal of this section is to describe, up to isomorphism, all vector bundles of sufficiently large rank over Q_{2n-1} . As observed above, projection onto x_1, \dots, x_n yields a morphism of schemes $p_{2n-1} : Q_{2n-1} \rightarrow \mathbb{A}^n \setminus 0$ that is a Zariski locally trivial smooth morphism with fibers isomorphic to \mathbb{A}^{n-1} . In particular, p_{2n-1} is an isomorphism in $\mathcal{H}(k)$.

A refined vanishing statement

Lemma 3.1. *If \mathbf{A} is a strictly \mathbb{A}^1 -invariant sheaf and if $n \geq 2$ is an integer, then*

$$H^i(Q_{2n-1}, \mathbf{A}) \simeq \begin{cases} \mathbf{A}(k) & \text{if } i = 0, \\ \mathbf{A}_{-n}(k) & \text{if } i = n - 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since p_{2n-1} is an isomorphism in $\mathcal{H}(k)$ and since \mathbf{A} is strictly \mathbb{A}^1 -invariant (equivalently, the Eilenberg-Mac Lane space $K(\mathbf{A}, i)$ is \mathbb{A}^1 -local for every integer $i \geq 0$), it follows that the pullback morphism $H^i(\mathbb{A}^n, \mathbf{A}) \rightarrow H^i(Q_{2n-1}, \mathbf{A})$ is an isomorphism. In $\mathcal{H}(k)$, we have an identification $\mathbb{A}^n \setminus 0 \cong \Sigma_s^{n-1} \mathbf{G}_m^{\wedge n}$.

The statement for $i = 0$ is clear, since $\mathbb{A}^n \setminus 0$ has a k -point and the pullback morphism induced by the structure map is a split injection. By the suspension isomorphism in cohomology and the definition of contraction, it follows that there are isomorphisms

$$\tilde{H}^i(\mathbb{A}^n \setminus 0, \mathbf{A}) \cong \tilde{H}^{i-(n-1)}(\text{Spec } k, \mathbf{A}_{-n}).$$

The remaining statements follow immediately. □

Remark 3.2. Observe that the isomorphism $H^{n-1}(Q_{2n-1}, \mathbf{A}) \simeq \mathbf{A}_{-n}(k)$ is non-canonical.

Vector bundles of large rank

Corollary 3.3. *If $n \geq 1$ is any integer, any vector bundle E of rank $m \geq n$ over Q_{2n-1} is free.*

Proof. If $n = 1$, then $Q_{2n-1} \cong \mathbf{G}_m \subset \mathbb{A}^1$ and the result is clear. If $n \geq 2$, the \mathbb{A}^1 -weak equivalence $Q_{2n-1} \rightarrow \mathbb{A}^n \setminus 0$ shows that $\text{Pic}(Q_{2n-1})$ is trivial. Therefore, any vector bundle on Q_{2n-1} has trivial determinant. Now, since BSL_m is \mathbb{A}^1 -1-connected for any integer n , if we arbitrarily pick a base-point $*$ in Q_{2n-1} , it follows that the canonical map

$$[(Q_{2n-1}, *), BSL_m]_{\mathbb{A}^1} \longrightarrow [Q_{2n-1}, BSL_m]_{\mathbb{A}^1}$$

is a bijection. Therefore, to describe the set of isomorphism classes of rank m vector bundles on Q_{2n-1} , it suffices to describe the set on the left.

We can describe the set of pointed \mathbb{A}^1 -homotopy classes of maps $Q_{2n-1} \rightarrow BSL_m$ as follows. By means of the \mathbb{A}^1 -weak equivalence $Q_{2n-1} \rightarrow \mathbb{A}^n \setminus 0 \cong \Sigma_s^{n-1} \mathbf{G}_m^{\wedge n}$, we have

$$[(Q_{2n-1}, *), BSL_m]_{\mathbb{A}^1} \cong [\Sigma_s^{n-1} \mathbf{G}_m^{\wedge n}, BSL_m]_{\mathbb{A}^1}.$$

By [Mor12, Theorem 5.13], we have identifications

$$[\Sigma_s^{n-1} \mathbf{G}_m^{\wedge n}, BSL_m]_{\mathbb{A}^1} \cong \pi_{n-1}^{\mathbb{A}^1}(BSL_m)_{-n}(k).$$

If $m \geq n$, then $\pi_{n-1}^{\mathbb{A}^1}(BSL_m)_{-n} = (\mathbf{K}_{n-1}^Q)_{-n}$ by the results of Morel (see, e.g., [AF12, Theorem 2.9] for a convenient summary). Since $(\mathbf{K}_{n-1}^Q)_{-n} = 0$, the result follows. \square

Vector bundles of critical rank I: the case n even

We now study vector bundles of rank $n - 1$ on Q_{2n-1} under the additional assumption that n is even. In that case, we have $\pi_i^{\mathbb{A}^1}(BSL_{n-1}) = \mathbf{K}_i^Q$ if $2 \leq i \leq n - 2$ and an exact sequence (from [AF12, Theorem 3.9])

$$0 \longrightarrow \mathbf{S}_n \longrightarrow \pi_{n-1}^{\mathbb{A}^1}(BSL_{n-1}) \longrightarrow \mathbf{K}_{n-1}^Q \longrightarrow 0.$$

We recall that the sheaf \mathbf{S}_n admits the following explicit description. The \mathbb{A}^1 -fiber sequence

$$\mathbb{A}^n \setminus 0 \longrightarrow BSL_{n-1} \longrightarrow BSL_n$$

yields an exact sequence

$$\pi_n^{\mathbb{A}^1}(BSL_n) \longrightarrow \pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0) \longrightarrow \pi_{n-1}^{\mathbb{A}^1}(BSL_{n-1}) \rightarrow \pi_{n-1}^{\mathbb{A}^1}(BSL_n) \longrightarrow 0.$$

The sheaf \mathbf{S}_n is defined to be the image of $\pi_{2n-1}^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0) = \mathbf{K}_n^{MW}$ in $\pi_{n-1}^{\mathbb{A}^1}(BSL_{n-1})$. In [AF12, Theorem 3.9] we furthermore prove that the epimorphism $\mathbf{K}_n^{MW} \rightarrow \mathbf{S}_n$ factors as a sequence of epimorphisms of sheaves

$$\mathbf{K}_n^{MW} \longrightarrow \mathbf{K}_n^M \longrightarrow \mathbf{K}_n^M / (n-1)! \longrightarrow \mathbf{S}_n;$$

here the left hand map is the natural map that sends η to 0, the middle map is the quotient by $(n-1)!$ and the right hand map is a map induced by the fact that the image of \mathbf{K}_n^Q in \mathbf{K}_n^M is contained in $(n-1)!\mathbf{K}_n^M$ (see [AF12, Lemma 3.8]).

Theorem 3.4. *If $n \geq 2$ is an even integer, then there is an isomorphism between the group of isomorphism classes of rank $n - 1$ vector bundles on Q_{2n-1} and the group $\mathbb{Z}/(n-1)!$. Moreover, each isomorphism class admits a representative given by the unimodular row (x_1^m, x_2, \dots, x_n) for $1 \leq m \leq (n-1)!$.*

Proof. The proof of Corollary 3.3 yields a bijection

$$\mathcal{V}_{n-1}(Q_{2n-1}) \xrightarrow{\sim} \pi_{n-1}^{\mathbb{A}^1}(BSL_{n-1})_{-n}(k).$$

We described the relevant contraction in Proposition 2.7: since n is even, $n - 1$ is odd, and we have a short exact sequence of the form

$$0 \longrightarrow (\mathbf{S}_n)_{-n}(k) \longrightarrow \pi_{n-1}^{\mathbb{A}^1}(BSL_{n-1})_{-n}(k) \longrightarrow (\mathbf{K}_{n-1}^Q)_{-n}(k) \longrightarrow 0.$$

Since $(\mathbf{K}_{n-1}^Q)_{-n} = 0$, it follows that there is a bijection $\mathcal{V}_{n-1}(Q_{2n-1}) \xrightarrow{\sim} (\mathbf{S}_n)_{-n}(k)$. Lemma 2.9 yields $(\mathbf{S}_n)_{-n}(k) = \mathbb{Z}/(n-1)!$ thus proving the first assertion.

To identify the vector bundles of rank $n-1$ more explicitly, begin by observing that the \mathbb{A}^1 -fiber sequence

$$\mathbb{A}^n \setminus 0 \longrightarrow BSL_{n-1} \rightarrow BSL_n$$

yields an exact sequence of (groups and) pointed sets

$$[Q_{2n-1}, SL_n]_{\mathbb{A}^1} \longrightarrow [Q_{2n-1}, \mathbb{A}^n \setminus 0]_{\mathbb{A}^1} \longrightarrow [Q_{2n-1}, BSL_{n-1}]_{\mathbb{A}^1} \longrightarrow [Q_{2n-1}, BSL_n]_{\mathbb{A}^1}$$

where the first map on the left is induced by the projection $SL_n \rightarrow \mathbb{A}^n \setminus 0$. By Corollary 3.3, we have $[Q_{2n-1}, BSL_n]_{\mathbb{A}^1} = *$ while our computation above yields $[Q_{2n-1}, BSL_{n-1}]_{\mathbb{A}^1} = (\mathbf{S}_n)_{-n}(k) = H^{n-1}(Q_{2n-1}, \mathbf{S}_n)$. Since SL_n satisfies the affine BG property and is \mathbb{A}^1 -invariant, we get an equality $[Q_{2n-1}, SL_n]_{\mathbb{A}^1} = SL_n(Q_{2n-1})/E_n(Q_{2n-1})$ by Morel's results (see for instance [Fas11b, Corollary 4.6]). Thus, the above sequence of pointed sets reduces to an exact sequence of (groups and) pointed sets

$$SL_n(Q_{2n-1})/E_n(Q_{2n-1}) \longrightarrow [Q_{2n-1}, \mathbb{A}^n \setminus 0]_{\mathbb{A}^1} \longrightarrow H^{n-1}(Q_{2n-1}, \mathbf{S}_n) \longrightarrow *$$

Now we have a bijection $[Q_{2n-1}, \mathbb{A}^n \setminus 0] = H^{n-1}(Q_{2n-1}, \mathbf{K}_n^{MW})$ as explained in Section 3 and the map $[Q_{2n-1}, \mathbb{A}^n \setminus 0] \rightarrow H^{n-1}(Q_{2n-1}, \mathbf{S}_n)$ is exactly the morphism induced by the morphism of sheaves $\mathbf{K}_n^{MW} \rightarrow \mathbf{S}_n$. In particular, this is a group homomorphism and it identifies $H^{n-1}(Q_{2n-1}, \mathbf{S}_n)$ with the orbits of $H^{n-1}(Q_{2n-1}, \mathbf{K}_n^{MW})$ under the action of $SL_n(Q_{2n-1})$.

We now use the sequence of epimorphisms of sheaves

$$\mathbf{K}_n^{MW} \longrightarrow \mathbf{K}_n^M \longrightarrow \mathbf{K}_n^M/(n-1)! \longrightarrow \mathbf{S}_n$$

to obtain a sequence of surjective homomorphisms (use Lemma 3.1 once again!)

$$H^{n-1}(Q_{2n-1}, \mathbf{K}_n^{MW}) \twoheadrightarrow H^{n-1}(Q_{2n-1}, \mathbf{K}_n^M) \twoheadrightarrow H^{n-1}(Q_{2n-1}, \mathbf{K}_n^M/(n-1)!) \twoheadrightarrow H^{n-1}(Q_{2n-1}, \mathbf{S}_n)$$

Since $H^{n-1}(Q_{2n-1}, \mathbf{K}_n^M) = \mathbb{Z}$ and the homomorphism on the right-hand side is an isomorphism by Lemma 2.9, this proves that any unimodular row is equivalent, under the action of $SL_n(Q_{2n-1})$, to a row of the form (x_1^m, x_2, \dots, x_n) for $1 \leq m \leq (n-1)!$. \square

Vector bundles of critical rank II: the case n odd

We now study isomorphism classes of rank $n-1$ vector bundles on Q_{2n-1} when n is odd. In that case, recall from section 2 that there is an exact sequence of the form

$$0 \longrightarrow \mathbf{T}_n \longrightarrow \pi_{n-1}^{\mathbb{A}^1}(BSL_{n-1}) \longrightarrow \mathbf{K}_{n-1}^Q \longrightarrow 0$$

where \mathbf{T}_n is the image of the sheaf $\mathbf{K}_n^{MW} = \pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0)$ in $\pi_{n-1}^{\mathbb{A}^1}(BSL_{n-1})$ under the morphism of sheaves induced by the morphism of spaces $\mathbb{A}^n \setminus 0 \rightarrow BSL_{n-1}$.

Theorem 3.5. *If $n \geq 3$ is an odd integer, then there is an isomorphism between the group of isomorphism classes of rank $n-1$ vector bundles on Q_{2n-1} and $\mathbb{Z}/(n-1)! \times_{\mathbb{Z}/2} W(k)$.*

Proof. The proof begins in the same fashion as the proof of Theorem 3.4. Following the same steps there, we obtain a bijection

$$\mathcal{V}_{n-1}(Q_{2n-1}) \xrightarrow{\sim} \pi_{n-1}^{\mathbb{A}^1}(BSL_{n-1})_{-n}(k).$$

Again, applying Proposition 2.7 and now using the fact that n odd implies $n - 1$ is even, we obtain an isomorphism

$$(\mathbf{T}_n)_{-n}(k) \xrightarrow{\sim} \pi_{n-1}^{\mathbb{A}^1}(BSL_{n-1})_{-n}(k).$$

Similarly, one concludes that $(\mathbf{T}_n)_{-n}(k)$ is the group of orbits of $H^{n-1}(Q_{2n-1}, \mathbf{K}_n^{MW})$ under the action of $SL_n(Q_{2n-1})$.

By Theorem 2.3, we have a fiber product diagram of strictly \mathbb{A}^1 -invariant sheaves:

$$\begin{array}{ccc} \mathbf{T}_n & \longrightarrow & \mathbf{I}^n \\ \downarrow & & \downarrow \\ \mathbf{S}_n & \longrightarrow & \mathbf{K}_n^M/2 \end{array}$$

The projection $\mathbf{T}_n \rightarrow \mathbf{S}_n$ fits into a commutative diagram

$$\begin{array}{ccc} \mathbf{K}_n^{MW} & \longrightarrow & \mathbf{T}_n \\ \downarrow & & \downarrow \\ \mathbf{K}_n^M & \longrightarrow & \mathbf{S}_n \end{array}$$

where $\mathbf{K}_n^{MW} \rightarrow \mathbf{K}_n^M$ is the map sending η to 0. As in the proof of Theorem 3.4, we deduce from this diagram that $(\mathbf{S}_n)_{-n}(k) = \mathbb{Z}/(n-1)!$.

Combining the above observations, we conclude that there is a fiber product diagram of the form

$$\begin{array}{ccc} (\mathbf{T}_n)_{-n}(k) & \longrightarrow & (\mathbf{I}^n)_{-n}(k) \\ \downarrow & & \downarrow \\ \mathbb{Z}/(n-1)! & \longrightarrow & \mathbb{Z}/2 \end{array}$$

and the result follows from the straightforward computation that $(\mathbf{I}^n)_{-n} = \mathbf{W}$. \square

Remark 3.6. As in the situation when n is even, we can give an explicit collection of unimodular rows which give the stably free modules of rank $n - 1$. By definition of the fiber product, we have an exact sequence

$$0 \longrightarrow 2\mathbb{Z}/(n-1)! \longrightarrow \mathbb{Z}/(n-1)! \times_{\mathbb{Z}/2} W(k) \longrightarrow W(k) \longrightarrow 0.$$

Now $\mathbb{Z}/(n-1)! \times_{\mathbb{Z}/2} W(k)$ can be seen as a quotient of the group $H^{n-1}(Q_{2n-1}, \mathbf{K}_n^{MW}) = Um_n(Q_{2n-1})/E_n(Q_{2n-1})$. It can be deduced from [Fas12, Remark 2.6] that the unimodular rows $(\alpha x_1, x_2, \dots, x_n)$ with $\alpha \in k^\times$ generate the factor $W(k)$ in the exact sequence above. It follows that these unimodular rows, together with the rows $(x_1^{2^m}, x_2, \dots, x_n)$ for $1 \leq m \leq (n-1)!/2$, generate the group $\mathbb{Z}/(n-1)! \times_{\mathbb{Z}/2} W(k)$.

Remark 3.7. Let $f_1, \dots, f_n \in k[x_1, \dots, x_n]$ be functions such that $V(f_1, \dots, f_n)$ is a point in \mathbb{A}^n . The variety $\sum_i x_i f_i = 1$ is a smooth affine variety that is \mathbb{A}^1 -weakly equivalent to $\mathbb{A}^n \setminus 0$. By Morel's theorem, the set of isomorphism classes of vector bundles on such a variety is canonically in bijection with the set of isomorphism classes of vector bundles on Q_{2n-1} . However, the varieties so defined are *not* in general isomorphic to Q_{2n-1} . These varieties are torsors under vector bundles over $\mathbb{A}^n \setminus 0$. For example, when $n = 2$, there are pairwise non-isomorphic varieties of this form [DF11, Theorem 2.5]. Theorems 3.4 and 3.5 also provide a description of the set of isomorphism classes of rank $n - 1$ bundles on any such variety.

Vector bundles below critical rank

Since the Picard group of Q_7 is trivial, and we understand vector bundles of rank ≥ 3 on Q_7 by the results already proven, the next result completes the description of vector bundles on Q_7 .

Proposition 3.8. *There is a canonical bijection $\mathcal{V}_2(Q_7) \xrightarrow{\sim} \pi_2^{\mathbb{A}^1}(SL_2)_{-4}(k)$ and a short exact sequence of the form*

$$0 \longrightarrow I(k) \longrightarrow \pi_2^{\mathbb{A}^1}(SL_2)_{-4} \longrightarrow \mathbb{Z}/12 \longrightarrow 0.$$

Proof. As above, we identify $\mathcal{V}_2(Q_7)$ with $\pi_3^{\mathbb{A}^1}(BSL_2)_{-4}(k)$. By Proposition 2.8 we have a short exact sequence of the form

$$0 \longrightarrow (S''_4)_{-4}(k) \longrightarrow \pi_3^{\mathbb{A}^1}(BSL_2)_{-4} \longrightarrow (\mathbf{K}_3^{Sp})_{-4}(k) \longrightarrow 0.$$

We observed above that $\mathbf{K}_3^{Sp} = \mathbf{GW}_3^2$ and since $(\mathbf{GW}_3^2)_{-4} = \mathbf{GW}_{-1}^{-2} = 0$ by [AF12, Proposition 5.4], the result follows immediately from Lemma 2.10. \square

Remark 3.9. As with the case of rank 3 bundles on Q_7 , the rank 2 vector bundles on Q_7 are all given by stably free modules. It is possible to give explicit representatives for each of these stably free vector bundles: see [Fas11a, §3] for more information on how to associate a symplectic bundle of rank 2 to an unimodular row of length 4. For example, the unimodular rows of the form (x_1^m, x_2, x_3, x_4) with $1 \leq m \leq 12$ give rise to non-isomorphic rank 2 vector bundles.

Vector bundles on Q_{2n}

For $n \geq 1$, let

$$A_{2n} = \operatorname{Spec} k[x_1, \dots, x_n, y_1, \dots, y_n, z] / \langle \sum x_i y_i - z(1 + z) \rangle$$

and set $Q_{2n} := \operatorname{Spec} A_{2n}$. By convention Q_0 is the disjoint union of two copies of $\operatorname{Spec} k$. When $n = 1$, one identifies Q_{2n} as the quotient of SL_2 by its maximal torus \mathbf{G}_m acting by, say, right multiplication. The inclusion of \mathbf{G}_m into the Borel subgroup of upper triangular matrices determines a Zariski locally trivial smooth morphism with fibers isomorphic to \mathbb{A}^1 of the form

$$Q_2 \longrightarrow \mathbb{P}^1;$$

in particular, this morphism is an isomorphism in $\mathcal{H}(k)$.

Ideally, one would like to show that Q_{2n} is itself a motivic sphere for arbitrary n (in which case the proofs of the results below would be essentially identical to those given for the quadrics Q_{2n-1} above). The techniques of [AD07] show that Q_4 has the \mathbb{A}^1 -homotopy type of $\Sigma_s^2 \mathbf{G}_m^{\wedge 2}$. Indeed, in that case, one knows that Q_4 can be covered by two quasi-affine (but not affine) subschemes that are \mathbb{A}^1 -contractible (see [AD07, Remark 5.2]) and whose intersection is \mathbb{A}^1 -weakly equivalent to $\mathbb{A}^2 \setminus 0$. Unfortunately, we do not know if this is true for $n > 2$. Nevertheless, after a single suspension Q_{2n} has the \mathbb{A}^1 -homotopy type of a sphere; this observation has been made independently by a number of people including F. Morel and D. Dugger-D. Isaksen, but has not been written down.

Lemma 3.10. *There is an isomorphism $\Sigma_s^1 Q_{2n} \cong \Sigma_s^1 \mathbb{P}^{1 \wedge n}$ in $\mathcal{H}(k)$.*

Proof. Consider the closed immersion $Q_{2n-2} \hookrightarrow Q_{2n}$ defined by the equations $x_n = y_n = 0$. Let $Z \subset Q_{2n}$ be the closed subscheme defined by $x_n = 0$. Projection defines a morphism $Z \rightarrow Q_{2n-2}$ that makes Z into a trivial line bundle over Q_{2n-2} . The complement of Z in Q_{2n} is an open subscheme isomorphic to $\mathbb{A}^{2n-1} \times \mathbf{G}_m$.

The normal bundle of $Z \hookrightarrow Q_{2n}$ is a line bundle over the total space of a line bundle on Q_{2n-2} . If $n \geq 3$, this bundle is trivial since $\text{Pic}(Z) = \text{Pic}(Q_{2n-2})$ is trivial. When $n = 2$, this line bundle is still trivial by explicit computation. As a consequence, by choosing a trivialization, the homotopy purity theorem [MV99, §4 Theorem 2.23] combined with [MV99, §4 Proposition 2.17.2] then gives a cofibration sequence

$$\mathbb{A}^{2n-1} \times \mathbf{G}_m \longrightarrow Q_{2n} \rightarrow \mathbb{P}^1 \wedge (Q_{2n-2})_+ \longrightarrow \Sigma_s^1(\mathbb{A}^{2n-1} \times \mathbf{G}_m) \longrightarrow \cdots$$

Since Q_{2n-2} has a k -point, fixing such a point, we can identify $(Q_{2n-2})_+ = Q_{2n-2} \vee S_s^0$. In that case, $\mathbb{P}^1 \wedge (Q_{2n-2})_+ = \mathbb{P}^1 \wedge (Q_{2n-2} \vee S_s^0) = (\mathbb{P}^1 \wedge Q_{2n-2}) \vee \mathbb{P}^1$.

The map $\mathbb{A}^{2n-1} \times \mathbf{G}_m \rightarrow \mathbf{G}_m$ given by projection onto the second factor is an \mathbb{A}^1 -weak equivalence, so the map $\Sigma_s^1(\mathbb{A}^{2n-1} \times \mathbf{G}_m) \rightarrow \Sigma_s^1 \mathbf{G}_m$ is also an \mathbb{A}^1 -weak equivalence. However, we know that $\mathbb{P}^1 \cong \Sigma_s^1 \mathbf{G}_m$ in $\mathcal{H}(k)$. With these identifications, the connecting homomorphism in the above cofibration sequence is a map

$$(\mathbb{P}^1 \wedge Q_{2n-2}) \vee \mathbb{P}^1 \longrightarrow \mathbb{P}^1,$$

and tracing through the definition of the connecting homomorphism, one identifies this map with the map collapsing the first wedge summand to a point. One deduces immediately that the map $\Sigma_s^1 Q_{2n} \rightarrow \Sigma_s^1(\mathbb{P}^1 \wedge Q_{2n-2})$ is an \mathbb{A}^1 -weak equivalence. The stated result follows immediately by induction since we already know Q_2 is unstably \mathbb{A}^1 -equivalent to \mathbb{P}^1 . \square

Lemma 3.11. *If \mathbf{A} is a strictly \mathbb{A}^1 -invariant sheaf, and n is an integer ≥ 1 , then*

$$H^i(Q_{2n}, \mathbf{A}) = \begin{cases} \mathbf{A}(k) & \text{if } i = 0. \\ \mathbf{A}_{-n}(k) & \text{if } i = n. \\ 0 & \text{else.} \end{cases}$$

Proof. As with the proof of Lemma 3.1, any k -rational point in Q_{2n} splits the cohomology of Q_{2n-1} as a sum of cohomology of $\text{Spec } k$ and reduced cohomology. Combing the suspension isomorphism and Lemma 3.10, we obtain isomorphisms of reduced cohomology groups

$$\tilde{H}^i(Q_{2n}, \mathbf{A}) \cong \tilde{H}^{i+1}(\Sigma_s^1 Q_{2n}, \mathbf{A}) \cong \tilde{H}^{i+1}(\Sigma_s^{n+1} \mathbf{G}_m^{\wedge n}, \mathbf{A}).$$

Again applying the suspension isomorphism and the definition of contraction, the result follows. \square

Example 3.12. Taking $\mathbf{A} = \mathbf{K}_{n+1}^M$ in Lemma 3.11, observe that one obtains isomorphisms $k^\times \rightarrow H^n(Q_{2n}, \mathbf{K}_{n+1}^M)$. Tracing through the proof of Lemma 3.10, one can realize the above isomorphism as follows. In the notation of that lemma, we have a map $H_n(Q_{2n}, \mathbf{K}_{n+1}^M) \rightarrow H_n(Th(\nu_{Z/Q_{2n}}), \mathbf{K}_{n+1}^M)$. Fixing a trivialization of the normal bundle to $Z \cong Q_{2n-2} \times \mathbb{A}^1 \subset Q_{2n}$, one obtains isomorphisms $Th(\nu_{Z/Q_{2n}}) \cong \mathbb{P}^1 \wedge (Q_{2n-2} \times \mathbb{A}^1)_+$. Since Z is defined by the equation $x_n = 0$, the differential gives a trivialization of the normal bundle. Proceeding iteratively, we eventually restrict to the subvariety Z_n defined by the equations $x_1 = \dots = x_n = 0$, which is isomorphic to $Q_0 \times \mathbb{A}^n$, i.e., the disjoint union of two copies of \mathbb{A}^n (the two components correspond to taking $z = 0$ or $z = -1$). Composing the maps obtained by making these choices, one identifies $\alpha \in k^\times$ with the element of $H^n(Q_{2n}, \mathbf{K}_{n+1}^M)$ corresponding to the invertible function $1 + (1 - \alpha)z$ on the component of $Q_0 \times \mathbb{A}^n$ with $z \neq 0$.

Lemma 3.13. *Two vector bundles E and E' of rank $m \geq n + 1$ on Q_{2n} are isomorphic if and only if $c_n(E) = c_n(E')$ in $CH^n(Q_{2n}) = \mathbb{Z}$.*

Proof. The case where $n = 1$ is clear since $Q_2 \cong \mathbb{P}^1$. Therefore, assume $n > 1$ so that $Pic(Q_{2n})$ is trivial. Lacking the homotopy theoretic description of Q_{2n} , we instead appeal to obstruction theoretic arguments. Since $Pic(Q_{2n})$ is trivial, we can still identify the set of isomorphism classes of rank m vector bundles on Q_{2n} with the set of pointed \mathbb{A}^1 -homotopy classes of maps $[(Q_{2n}, *), BSL_m]$. We can describe this set by obstruction theory using the \mathbb{A}^1 -Postnikov tower of BSL_m .

In view of Lemma 3.11, one sees inductively that there are no obstructions to lift a morphism $Q_{2n} \rightarrow BSL_m^{(i)}$ to a morphism $Q_{2n} \rightarrow BSL_m^{(i+1)}$, and irrespective of the choice of lift the subsequent obstruction vanishes since the group in which it lives is trivial. If $i + 1 \neq n$, there is only one possible lift, while in the case $i + 1 = n$, possible lifts are classified by $H^n(Q_{2n}, \pi_n^{\mathbb{A}^1}(BSL_m))$. Since $n < m$, the latter group is $CH^n(Q_{2n}) = \mathbb{Z}$. Moreover, one knows how to construct bundles corresponding to each element \mathbb{Z} . In view of the discussion before [AF12, Remark 6.7], the obstruction class is a (non-trivial) multiple of the Chern class c_n . Since $CH^n(Q_{2n})$ is torsion free, the result follows. \square

Remark 3.14. Vector bundles E of rank $m > n$ on Q_{2n} split as $E \simeq E' \oplus \mathcal{O}_{Q_{2n}}^{m-n}$, and therefore it is sufficient to understand the vector bundles of rank n . One way to see this is to observe that, if $i \geq n$, the obstructions to lifting an \mathbb{A}^1 -homotopy class of maps $X \rightarrow BSL_{i+i}$ to a map $X \rightarrow BSL_i$ vanish by Lemma 3.11.

Theorem 3.15. *Assume $n \geq 2$ is an integer. There are isomorphisms*

$$\mathcal{V}_n(Q_{2n}) \xrightarrow{\sim} \begin{cases} \mathbb{Z} \times (k^\times / (k^\times)^{n!}) & \text{if } n \text{ even, and} \\ \mathbb{Z} \times (k^\times / (k^\times)^{n!} \times_{k^\times / (k^\times)^2} I(k)) & \text{if } n \text{ odd.} \end{cases}$$

Proof. As above, we first describe the set of isomorphism classes of vector bundles of rank n on Q_{2n} by using the \mathbb{A}^1 -Postnikov tower of BSL_n . Using Lemma 3.11, we see that there is no obstruction to lifting a morphism $Q_{2n} \rightarrow BSL_n^{(i)}$ to a morphism $Q_{2n} \rightarrow BSL_n^{(i+1)}$ for any $i \in \mathbb{N}$. Moreover, each subsequent lift is uniquely determined except when we want to lift a morphism $Q_{2n} \rightarrow BSL_n^{(n-1)}$ to a morphism $Q_{2n} \rightarrow BSL_n^{(n)}$. The space of lifts is of the form

$[Q_{2n}, K(\pi_n^{\mathbb{A}^1}(BSL_n), n)]_{\mathbb{A}^1} = H^n(Q_{2n}, \pi_n^{\mathbb{A}^1}(BSL_n))$. By means of Lemma 3.11, we see that $H^n(Q_{2n}, \pi_n^{\mathbb{A}^1}(BSL_n)) = (\pi_n^{\mathbb{A}^1} BSL_n)_{-n}(k)$.

Now, the computations of Section 2 show that we have an exact sequence of sheaves

$$0 \longrightarrow \mathbf{A}_{n+1} \longrightarrow \pi_n^{\mathbb{A}^1}(BSL_n) \longrightarrow \mathbf{K}_n^Q \longrightarrow 0$$

where $\mathbf{A}_{n+1} = \mathbf{S}_{n+1}$ if n is even and $\mathbf{A}_{n+1} = \mathbf{T}_{n+1}$ if n is odd. Since the contraction construction is exact, contraction of the above exact sequence n -times yields

$$0 \longrightarrow (\mathbf{A}_{n+1})_{-n} \longrightarrow (\pi_n^{\mathbb{A}^1}(BSL_n))_{-n} \longrightarrow (\mathbf{K}_n^Q)_{-n} \longrightarrow 0.$$

Evaluating the result at $\text{Spec } k$, we see that $\pi_n^{\mathbb{A}^1}(BSL_n)_{-n}(\text{Spec } k)$ is an extension of $(\mathbf{K}_n^Q)_{-n} = \mathbb{Z}$ by $(\mathbf{A}_{n+1})_{-n}(k)$.

The group $(\mathbf{A}_{n+1})_{-n}(k)$ admits a description as the kernel of the homomorphism

$$H^n(Q_{2n}, \pi_n^{\mathbb{A}^1}(BSL_n)) \longrightarrow H^n(Q_{2n}, \mathbf{K}_n^Q)$$

given by the above morphism of sheaves. This homomorphism associates with a vector bundle E , classified by a map $Q_{2n} \rightarrow BSL_n$, the class in $H^n(Q_{2n}, \mathbf{K}_n^Q) = \mathbb{Z}$ pulled back from a certain universal lifting class on BSL_n ; the resulting class is a multiple of the Chern class $c_n(E)$ by the discussion just before [AF12, Remark 6.7]. It follows that $(\mathbf{A}_{n+1})_{-n}(k)$ parameterizes the vector bundles E of rank n whose n -th Chern class $c_n(E)$ is trivial. Lemma 3.13 thus implies that $(\mathbf{A}_{n+1})_{-n}(k)$ is exactly the set of isomorphism classes of projective modules E of rank n such that $E \oplus \mathcal{O}_{Q_{2n}} \simeq \mathcal{O}_{Q_{2n}}^{n+1}$. Therefore $Um_{n+1}(A_{2n})/SL_{n+1}(A_{2n}) = (\mathbf{A}_{n+1})_{-n}(k)$. Once again, Lemma 2.9 yields $(\mathbf{A}_{n+1})_{-n}(k) = k^\times / (k^\times)^{n!}$ if n is even and $(\mathbf{A}_{n+1})_{-n}(k) = k^\times / (k^\times)^{n!} \times_{k^\times / (k^\times)^2} I(k)$ if n is odd. \square

Remark 3.16. In case n is even, we can give a set of generators of the stably free modules of rank n on Q_{2n} as follows. Example 3.12 shows that $H^n(Q_{2n}, \mathbf{K}_{n+1}^M)$ is generated by the invertible functions $1 + (1 - \alpha)z$ on the component of $Q_0 \times \mathbb{A}^n$ with $z \neq 0$. Now the sequence of epimorphisms of sheaves

$$\mathbf{K}_{n+1}^{MW} \longrightarrow \mathbf{K}_{n+1}^M \longrightarrow \mathbf{K}_{n+1}^M/n! \longrightarrow \mathbf{S}_{n+1}.$$

shows that these generators correspond to unimodular rows $(x_1, \dots, x_n, 1 + (1 - \alpha)z)$ with $\alpha \in k^\times$. We thus obtain a set of generators by considering $\alpha \in k^\times / (k^\times)^{n!}$.

4 Applications

In this section, we discuss two applications of the description of the set of isomorphism classes of vector bundles on split quadrics from Section 3.

On a question of M. V. Nori

Our computation of the isomorphism classes of vector bundles of rank $n - 1$ on Q_{2n-1} allows us to address the following question of M. V. Nori on unimodular rows.

Question 4.1 (M. V. Nori). Suppose k is a field, $R = k[x_1, \dots, x_n]$ is a polynomial ring in n variables over k , $\phi : R \rightarrow A$ is a k -algebra homomorphism such that $\sum \phi(x_i)A = A$, and f_1, \dots, f_n are elements of R such that the reduced closed subscheme defined by the ideal $I(f_1, \dots, f_n)$ is $0 \in \mathbb{A}^n$. If $\text{length}(R/I(f_1, \dots, f_n))$ is divisible by $(n-1)!$, then is the unimodular row $(\phi(f_1), \dots, \phi(f_n))$ completable?

Nori's question admits the following reinterpretation. The homomorphism $\phi : R \rightarrow A$ such that $\sum \phi(x_i)A = A$ defines a unimodular row $v = (\phi(x_0), \dots, \phi(x_n))$ and a morphism of schemes $v : \text{Spec } R \rightarrow \mathbb{A}^n \setminus 0$. Now any polynomials f_1, \dots, f_n such that $\text{rad}(f_1, \dots, f_n) = (x_1, \dots, x_n)$ defines a morphism $\varphi : \mathbb{A}^n \setminus 0 \rightarrow \mathbb{A}^n \setminus 0$. If $\text{length}(R/I(f_1, \dots, f_n))$ is divisible by $(n-1)!$, then does the morphism $\varphi \circ v : \text{Spec } R \rightarrow \mathbb{A}^n \setminus 0$ lift to a morphism $\text{Spec } R \rightarrow SL_n$?

Since the question is about *all* k -algebras A and all unimodular rows of length n on A , it is reasonable to try to deal with the above question by looking at the universal algebra parameterizing unimodular rows of length n , namely the k -algebra A_{2n-1} . Indeed, let A be a k -algebra and v be a unimodular row of length n . Then the choice of $w \in \text{Um}_n(A)$ such that $v \cdot w^t = 1$ yields a lift of the morphism $v : \text{Spec } A \rightarrow \mathbb{A}^n \setminus 0$ to a morphism $v' : \text{Spec } A \rightarrow Q_{2n-1}$, i.e. we have a commutative diagram

$$\begin{array}{ccc} \text{Spec } R & \xrightarrow{v'} & Q_{2n-1} \\ & \searrow v & \downarrow p_{2n-1} \\ & & \mathbb{A}^n \setminus 0. \end{array}$$

Let now $\varphi : \mathbb{A}^n \setminus 0 \rightarrow \mathbb{A}^n \setminus 0$ be a morphism and $r : SL_n \rightarrow \mathbb{A}^n \setminus 0$ be the projection to the first row. The diagram

$$\begin{array}{ccccc} & & Q_{2n-1} & & \\ & \nearrow v' & \downarrow p_{2n-1} & & \\ \text{Spec } R & \xrightarrow{v} & \mathbb{A}^n \setminus 0 & & SL_n \\ & \searrow \varphi v & \downarrow \varphi & \swarrow r & \\ & & \mathbb{A}^n \setminus 0 & & \end{array}$$

thus proves that it suffices to show that $\varphi \circ p_{2n-1}$ factorizes through SL_n to prove that φv also factorizes through SL_n .

Theorem 4.2. *If n is an even integer, then Question 4.1 has an affirmative answer.*

Proof. The morphism $p_{2n-1} : Q_{2n-1} \rightarrow \mathbb{A}^n \setminus 0$ corresponds to the unimodular row (x_1, \dots, x_n) , whose class in $\text{Um}_n(Q_{2n-1})/SL_n(Q_{2n-1}) = \mathbb{Z}/(n-1)!$ is 1. The unimodular row corresponding to φp_{2n-1} is precisely (f_1, \dots, f_n) and we want to compute its class in $\mathbb{Z}/(n-1)!$. Now $\varphi : \mathbb{A}^n \setminus 0 \rightarrow \mathbb{A}^n \setminus 0$ induces a homomorphism $\varphi^* : \mathbb{Z} = H^{n-1}(Q_{2n-1}, \mathbf{K}_n^M) \rightarrow H^{n-1}(Q_{2n-1}, \mathbf{K}_n^M) = \mathbb{Z}$, which is precisely the multiplication by $\text{length}(R/I(f_1, \dots, f_n))$, and it follows therefore that the class of (f_1, \dots, f_n) in $\mathbb{Z}/(n-1)!$ is this length (modulo $(n-1)!$). The result follows. \square

When n is odd, the answer to Nori's question is known to be negative by [Fas12, Theorem 4.7]. In view of this counter-example, the second author proposed a stronger version of Nori's question,

which we now explain. If \mathbf{I}^n denotes the unramified sheaf corresponding to the n -th power of the fundamental ideal, then Lemma 3.1 shows that $H^{n-1}(Q_{2n-1}, \mathbf{I}^n) \cong H^{n-1}(\mathbb{A}^n \setminus 0, \mathbf{I}^n) \simeq (\mathbf{I}^n)_{-n}(k) = W(k)$. The isomorphism can be uniquely specified by choosing a trivialization of the normal sheaf of 0 in \mathbb{A}^n and thus an orientation class [Fas12, Remark 2.5]. Any morphism $\varphi : \mathbb{A}^n \setminus 0 \rightarrow \mathbb{A}^n \setminus 0$ yields a homomorphism $\varphi^* : W(k) \rightarrow W(k)$ that we call the degree of φ and write $\deg(\varphi)$. This degree is simply a concrete avatar of (the quadratic part of) F. Morel's Brouwer degree [Mor12, Corollary 24].

Remark 4.3. In [Fas12], a degree homomorphism is defined by considering the Grothendieck-Witt group $GW_{red}^{n-1}(\mathbb{A}^n \setminus 0)$ (here the subscript *red* means “reduced,” i.e., one has split off the summand corresponding to a base-point; see [Fas12, Lemma 2.4] for more details). This degree is exactly the same as the one defined above. Indeed, the Gersten-Grothendieck-Witt spectral sequence $E(n-1)^{p,q}$ shows that the edge homomorphism $E(n-1)_2^{n-1,0} = H^{n-1}(\mathbb{A}^n \setminus 0, \mathbf{I}^n) \rightarrow GW_{red}^{n-1}(\mathbb{A}^n \setminus 0)$ induces an isomorphism $H^{n-1}(\mathbb{A}^n \setminus 0, \mathbf{I}^n) \rightarrow GW_{red}^{n-1}(\mathbb{A}^n \setminus 0)$.

We now state and prove a result that constitutes a positive answer to a strengthening of Nori's original question; this provides an answer to [Fas12, Question 4.8].

Theorem 4.4. *Let A be a k -algebra, $n \in \mathbb{N}$ be an odd integer and let $v : \text{Spec } A \rightarrow \mathbb{A}^n \setminus 0$ be a unimodular row. If \mathfrak{m} is the maximal ideal corresponding to $0 \in \mathbb{A}^n$, assume we are given a homomorphism $f : k[x_1, \dots, x_n] \rightarrow k[x_1, \dots, x_n]$ such that $f(\mathfrak{m}) \subset \mathfrak{m}$ and such that $(n-1)!$ divides the length of $k[x_1, \dots, x_n]/f(\mathfrak{m})$. Let $\varphi : \mathbb{A}^n \setminus 0 \rightarrow \mathbb{A}^n \setminus 0$ be the morphism induced by f . If the degree $\deg(\varphi) = 0$, then the unimodular row $\varphi v : \text{Spec } R \rightarrow \mathbb{A}^n \setminus 0$ is completable.*

Proof. Theorem 3.5 shows that $Um_n(Q_{2n-1})/E_n(Q_{2n-1})$ is the fiber product of the groups $W(k)$ and $\mathbb{Z}/(n-1)!$ over $\mathbb{Z}/2$. The same arguments as in the proof of Theorem 4.2 show that we have to prove that the unimodular row $(f(x_1), \dots, f(x_n))$ is completable if $\deg(\varphi) = 0$ and $(n-1)!$ divides the length l of $k[x_1, \dots, x_n]/f(\mathfrak{m})$. However, the unimodular row $(f(x_1), \dots, f(x_n))$ corresponds to the pair $(\deg(\varphi), l)$ in the fiber product by definition of $\deg(\varphi)$ and Theorem 4.2. \square

Compatibility with realization

Assume $k = \mathbb{C}$. If (X, x) is a pointed space, and $X(\mathbb{C})$ is the associated topological space of complex points, complex realization [MV99, p. 120-121] gives a homomorphism

$$\bigoplus_{i+j=n} \pi_{i,j}^{\mathbb{A}^1}(X)(\mathbb{C}) \longrightarrow \pi_n(X(\mathbb{C}))$$

by summing the various component homomorphisms. Taking $X = SL_n$ or BSL_n , complex realization allows us to compare the computations of \mathbb{A}^1 -homotopy sheaves from [AF12] and Section 2 with those coming from classical homotopy theory. We will see that the above homomorphism is surjective in some situations. The precise description of first non-stable \mathbb{A}^1 -homotopy sheaves of SL_n was motivated by anticipation of results such as those established here.

Compatibility with complex realization

Bott periodicity (see also [Bot58, Theorem 5]) yields a computation of the homotopy groups of the unitary group in the stable range and the first non-stable homotopy group:

$$\pi_i(U(n)) = \begin{cases} 0 & \text{if } i < 2n, i \text{ even} \\ \mathbb{Z} & \text{if } i < 2n, i \text{ odd, and} \\ \mathbb{Z}/n! & \text{if } i = 2n. \end{cases}$$

Furthermore, it is classically known that $\pi_5(U(2)) = \mathbb{Z}/2$ (use [Whi50] together with the fact that $U(2)$ is an S^1 -bundle over $SU(2)$) and $\pi_6(U(2)) = \mathbb{Z}/12$ [BS53, Proposition 19.4].

Theorem 4.5. *For any integer $n \geq 3$, the homomorphisms*

$$\begin{aligned} \pi_{n-1,n}^{\mathbb{A}^1}(GL_n)(\mathbb{C}) &\longrightarrow \pi_{2n-1}(GL_n(\mathbb{C})) \cong \pi_{2n-1}(U(n)) = \mathbb{Z}, \text{ and} \\ \pi_{n-1,n+1}^{\mathbb{A}^1}(GL_n)(\mathbb{C}) &\longrightarrow \pi_{2n}(GL_n(\mathbb{C})) \cong \pi_{2n}(U(n)) = \mathbb{Z}/n!, \end{aligned}$$

induced by complex realization are isomorphisms.

Proof. We begin by establishing the second isomorphism of the theorem. For any integer $n \geq 3$, we first identify $\pi_{n-1,n+1}^{\mathbb{A}^1}(GL_n) \cong \pi_{n-1,n+1}^{\mathbb{A}^1}(SL_n) \cong \pi_{n,n+1}^{\mathbb{A}^1}(BSL_n)$. Because $\pi_1^{\mathbb{A}^1}(BSL_n)$ is trivial, the latter set can be canonically identified with the set of unpointed homotopy classes of maps $[Q_{2n+1}, BSL_n]_{\mathbb{A}^1}$. Similarly, we identify $\pi_{2n}(GL_n(\mathbb{C})) = \pi_{2n}(SU(n))$ with $[S^{2n+1}, BSU(n)]$ by means of the clutching construction.

Since $W(\mathbb{C}) = \mathbb{Z}/2$, Theorems 3.4 and 3.5 tell us that the set of isomorphism classes of rank n bundles on Q_{2n+1} has a natural group structure and is isomorphic to $\mathbb{Z}/n!\mathbb{Z}$ (irrespective of whether n is even or odd). Now, the map that sends a complex algebraic vector bundle to the underlying topological vector bundle defines a function

$$[Q_{2n+1}, BSL_{2n}]_{\mathbb{A}^1} \longrightarrow [S^{2n+1}, BSU(n)].$$

As mentioned above, topological vector bundles can be described by means of the clutching construction. Now, each of the vector bundles of rank n on Q_{2n+1} is given by a unimodular row. The homotopy class of the clutching function attached to the unimodular row is computed, e.g., in [ST75, Theorem 3.1] and this gives the required isomorphism.

To establish the first isomorphism of the statement we proceed as follows. Observe that we have canonical isomorphisms

$$\begin{aligned} [S_s^{n-1} \wedge \mathbf{G}_m^{\wedge n}, GL_n]_{\mathbb{A}^1} &\cong [S_s^{n-1} \wedge \mathbf{G}_m^{\wedge n}, SL_n]_{\mathbb{A}^1} \\ &\cong [S_s^{n-1} \wedge \mathbf{G}_m^{\wedge n}, \Omega_s^1 BSL_n]_{\mathbb{A}^1} \cong [S_s^n \wedge \mathbf{G}_m^{\wedge n}, BSL_n]_{\mathbb{A}^1}; \end{aligned}$$

the first isomorphism is a consequence of [AF12, Theorem 2.9], the second isomorphism follows from [Mor12, Theorem 5.46] and [MV99, §4 Proposition 1.15] once one observes that SL_n is \mathbb{A}^1 -connected, and the third isomorphism is simply the loop-suspension adjunction.

Again, since BSL_n is \mathbb{A}^1 -1-connected, the canonical map from pointed to unpointed \mathbb{A}^1 -homotopy classes is an isomorphism. Now, we know that the complex realization of $S_s^n \wedge \mathbf{G}_m^{\wedge n}$

is the sphere S^{2n} . On the other hand, we know that $Q_{2n}(\mathbb{C})$ is weakly equivalent to S^{2n} (more precisely, it is diffeomorphic to the tangent bundle of the standard $2n$ -sphere S^{2n}). The map sending a complex algebraic vector bundle to its associated topological vector bundle then determines a function

$$[S_s^n \wedge \mathbf{G}_m^{\wedge n}, BSL_n]_{\mathbb{A}^1} \longrightarrow [S^{2n}, BSU(n)],$$

which coincides with the homomorphism of the theorem statement under complex realization by means of the loops-suspension adjunction. It therefore suffices to show this map is an isomorphism.

Now, we have given an explicit identification of the set of isomorphism class of rank n vector bundles on Q_{2n} in Theorem 3.15. In particular, when $k = \mathbb{C}$, the set of isomorphism classes of rank n vector bundles on Q_{2n} is isomorphic to $CH^n(Q_{2n}) = \mathbb{Z}$. Likewise, $[S^{2n}, BSU(n)] = \pi_{2n}(BSU(n)) = \pi_{2n-1}(SU(n)) = \mathbb{Z}$ by Bott periodicity. Since the map in question is a homomorphism of free abelian groups, it suffices to observe that we can lift a generator.

The map $SU(n) \rightarrow S^{2n-1}$ induces a homomorphism $\pi_{2n-1}(SU(n)) \rightarrow \pi_{2n-1}(S^{2n-1}) = \mathbb{Z}$. Therefore, a rank n topological vector bundle is classified by the topological degree of the map of spheres induced by the clutching map. It is straightforward to check that the topological degree of the clutching function of the unimodular row defining a rank n vector bundle with n -th Chern 1 is 1.

Set $F_n := \text{hofib}(BSL_n \rightarrow BSL_n^{(n-1)})$. Since the space $S_s^n \wedge \mathbf{G}_m^{\wedge n}$ is \mathbb{A}^1 -($n-1$)-connected, the map

$$\pi_{n,n}^{\mathbb{A}^1}(F_n) \longrightarrow \pi_{n,n}^{\mathbb{A}^1}(BSL_n)$$

is a bijection. Under the assumption on n , the \mathbb{A}^1 -Freudenthal suspension theorem [Mor12, Theorem 5.61] then gives an isomorphism

$$\pi_{n,n}^{\mathbb{A}^1}(F_n) \longrightarrow \pi_{n+1,n}^{\mathbb{A}^1}(\Sigma_s^1 F_n).$$

By Lemma 3.10, we know that $\Sigma_s^1 Q_{2n} \cong \Sigma_s^{n+1} \mathbf{G}_m^{\wedge n}$. Therefore, the set on the right hand side is $[\Sigma_s^1 Q_{2n}, \Sigma_s^1 F_n]$. Note also that, since $\Sigma_s^1 Q_{2n}$ is \mathbb{A}^1 - n -connected, the map

$$[\Sigma_s^1 Q_{2n}, \Sigma_s^1 F_n]_{\mathbb{A}^1} \longrightarrow [\Sigma_s^1 Q_{2n}, \Sigma_s^1 BSL_n]$$

is an isomorphism.

Complex realization thus gives a map

$$[\Sigma_s^1 Q_{2n}, \Sigma_s^1 BSL_n] \longrightarrow [S^{2n+1}, \Sigma_s^1 BSU(n)],$$

and combining all of the results above it suffices to prove that this morphism is an isomorphism. To see this, it suffices to observe that $[Q_{2n}, BSL_n]_{\mathbb{A}^1} \rightarrow CH^n(X)$ and $[S^{2n}, BSU(n)] \rightarrow H^{2n}(S^{2n}, \mathbb{Z})$ given by the n -th Chern class are isomorphisms. Since both of these isomorphisms are stable in the sense that they are compatible with simplicial or ordinary suspension the result follows. \square

Remark 4.6. Consider the homomorphism $\pi_{2n-1-i,i}^{\mathbb{A}^1}(SL_n) \rightarrow \pi_{2n-1}(SU(n))$. If $i > n$, the sheaf $\pi_{2n-1-i,i}^{\mathbb{A}^1}(SL_n) = \mathbf{K}_{2n-i}^Q$. This sheaf becomes trivial after i -fold contraction, and therefore, the homomorphism in question is trivial. If $i < n$ it seems likely that the above homomorphism is trivial as well, even though in that range the sheaves $\pi_{2n-1-i,i}^{\mathbb{A}^1}(SL_n)$ are not expected to be trivial.

Theorem 4.7. *The homomorphisms*

$$\begin{aligned}\pi_{2,3}^{\mathbb{A}^1}(SL_2)(\mathbb{C}) &\longrightarrow \pi_5(SL_2(\mathbb{C})) \cong \pi_5(SU(2)) = \mathbb{Z}/2, \text{ and} \\ \pi_{2,4}^{\mathbb{A}^1}(SL_2)(\mathbb{C}) &\longrightarrow \pi_6(SL_2(\mathbb{C})) \cong \pi_6(SU(2)) = \mathbb{Z}/12\end{aligned}$$

are isomorphisms.

Proof. For the second isomorphism, begin by recalling from [AF12, Theorem 3.20] that there is an exact sequence of the form:

$$\mathbf{I}^5 \longrightarrow \mathbf{S}_4'' \longrightarrow \mathbf{S}_4' \longrightarrow 0.$$

Contracting this sequence 4 times, evaluating on \mathbb{C} (using the fact that $\mathbf{I}(\mathbb{C}) = 0$), and using Lemma 2.10 yields an isomorphism $\mathbb{Z}/12 \xrightarrow{\sim} (\mathbf{S}_4')_{-4}(\mathbb{C})$. Since $(\mathbf{K}_3^{Sp})_{-4} = (\mathbf{GW}_3^2)_{-4} = (\mathbf{GW}_1^0)_{-2} = 0$ by [AF12, Lemma 4.9 and Proposition 5.4], there is thus an isomorphism $\mathbb{Z}/12 \xrightarrow{\sim} \pi_{2,4}^{\mathbb{A}^1}(SL_2)(\mathbb{C})$.

By [BS53, Proposition 19.1] one knows that the classifying map of the Sp_2 -bundle $Sp_4/Sp_2 \rightarrow BSp_2$ provides a generator of $\pi_6(S^3) = \pi_7(BSp_2)$. Now the computation of $\pi_2^{\mathbb{A}^1}(SL_2)$ was achieved using the \mathbb{A}^1 -fiber sequence $Sp_2 \rightarrow Sp_4 \rightarrow Sp_4/Sp_2$, and the isomorphism $\mathbb{Z}/12 \rightarrow \pi_{2,4}^{\mathbb{A}^1}(SL_2)(\mathbb{C})$ is induced by the connecting homomorphism of the associated long exact sequence in \mathbb{A}^1 -homotopy sheaves. Since the complex realization of the fiber sequence $Sp_2 \rightarrow Sp_4 \rightarrow Sp_4/Sp_2$ is homotopy equivalent to the fiber sequence considered by Borel-Serre, the result follows.

For the first isomorphism, recall first that $\mathbf{K}_1^M/12(\mathbb{C}) = 0$. Then, using the fact that $\mathbf{K}_3^{Sp} = \mathbf{GW}_3^2$ observe that $(\mathbf{GW}_3^2)_{-3} = (\mathbf{GW}_1^0)_{-1} = \mathbb{Z}/2$ (again, use [AF12, Lemma 4.9 and Proposition 5.4]). Now, by Proposition 2.8, the fact that $\mathbf{K}_1^M/12(\mathbb{C}) = 0$, the fact that $\mathbf{I}^2(\mathbb{C}) = 0$, and the fact that $(\mathbf{K}_3^{Sp})_{-3} = \mathbb{Z}/2$, we see that $\pi_{2,3}^{\mathbb{A}^1}(SL_2)(\mathbb{C}) \cong \mathbb{Z}/2$. Thus, complex realization gives a map $\pi_{2,3}^{\mathbb{A}^1}(SL_2)(\mathbb{C}) = \mathbb{Z}/2 \rightarrow \mathbb{Z}/2$. The computation of [Whi50] shows that (see the proof of [Hu59, Theorem 15.2] for more details) the generator of $\pi_5(S^3)$ is obtained as follows: start with the Hopf map $\eta_{\mathbb{C}} : S^3 \rightarrow S^2$ and consider the composition $\Sigma \eta_{\mathbb{C}} \circ \Sigma^2 \eta_{\mathbb{C}}$. Now, there is the algebro-geometric Hopf map $\eta : \mathbb{A}^2 \setminus 0 \rightarrow \mathbb{P}^1$ (see [Mor12, §6.3 and Example 6.26]), and taking the \mathbf{G}_m and \mathbb{P}^1 -suspensions of this map we obtain:

$$\begin{aligned}\Sigma_{\mathbf{G}_m} \eta : \mathbb{P}^{1 \wedge 2} &\longrightarrow \mathbb{A}^2 \setminus 0 \\ \Sigma_{\mathbb{P}^1} \eta : \mathbb{A}^3 \setminus 0 &\longrightarrow \mathbb{P}^{1 \wedge 2}.\end{aligned}$$

The composite of these two maps has complex realization the generator of the $\pi_5(S^3)$ since the complex realization of η is the usual Hopf map $\eta_{\mathbb{C}}$. \square

Remark 4.8. The second statement of the above theorem can also be deduced from Proposition 3.8.

Comments on real realization

For $k = \mathbb{R}$, Morel and Voevodsky [MV99, p. 121-122] also show that sending a smooth k -scheme to $X(\mathbb{C})$ equipped with the $\mathbb{Z}/2$ -action by complex conjugation defines a “real realization” functor from $\mathcal{H}(\mathbb{R})$ to the $\mathbb{Z}/2$ -equivariant homotopy category. There is a homotopy equivalence $GL_n(\mathbb{R}) \cong O(n)$. Since $O(2)$ is an extension of $\mathbb{Z}/2$ by $SO(2)$, which has no homotopy groups in dimension > 1 . The groups $\pi_{n-1}(O(n))$ are determined by Bott periodicity. For

completeness, we quote the result from [Ker60]: the group $\pi_{r-1}(O(r))$ is equal to $0, \mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}, 0$ if $r = 3, 4, 5, 6$ or 7 and, more generally, $\mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}, \mathbb{Z}/2$ if $r \geq 8$, and $r \equiv 0, 1, 2, 3, 4, 5, 6$ or 7 modulo 8 . The situation involving compatibility with real realization is more subtle than that of complex realization.

Real realization gives rise to canonical homomorphisms

$$\pi_{i,j}^{\mathbb{A}^1}(GL_n)(\mathbb{R}) \longrightarrow \pi_i(GL_n(R)) \cong \pi_i(O(n));$$

in particular, \mathbb{A}^1 -homotopy groups of several different weights map to the *same* topological homotopy group. If $n \geq 2$ and $i \geq 2$, we can again use fiber sequences to study SL_n and $SO(n)$ instead of GL_n and $O(n)$. In that situation, the isomorphisms in question are compatible with the clutching construction (as above).

Similar to the situation involving complex realization, real realization is compatible with (simplicial) suspension, so the homomorphism above can also be identified as a morphism

$$\pi_{i+1,j}^{\mathbb{A}^1}(BGL_n)(\mathbb{R}) \longrightarrow \pi_{i+1}(BO(n)).$$

The computations of homotopy groups of $O(n)$ give rise to descriptions of the set of isomorphism classes of rank n topological vector bundles on S^n . Likewise, Theorems 3.5 and 3.4 give descriptions of the sets of isomorphism classes of real rank n vector bundles on Q_{2n+1} (which has real realization homotopy equivalent to S^n): these groups are equal to $\mathbb{Z}/(n-1)!\mathbb{Z}$ if n is odd and $\mathbb{Z}/(n-1)! \times_{\mathbb{Z}/2} W(k)$ if n is even (the indices have shifted). The descriptions of the set of isomorphism classes of real rank n vector bundles on Q_{2n} (which has real realization homotopy equivalent to S^n) is in bijection with $\mathbb{Z} \times \mathbb{Z}/2$ if n is even and $\mathbb{Z} \times \mathbb{Z}$ if n is odd. In particular, while neither realization map is (individually) surjective or injective, it is possible that the map $\bigoplus_j \pi_{n,j}^{\mathbb{A}^1}(BGL_n) \rightarrow \pi_n(BO(n))$ is surjective. Nevertheless, the factor of \mathbb{Z} that corresponds to $W(\mathbb{R})$ in Theorem 3.5 does admit an elementary explanation; we view the following remark as an explanation of the factors of \mathbf{I}^n that appear in Theorem 2.3.

Remark 4.9. A rank i vector bundle on S^n is classified by a map $S^n \rightarrow BSO(i)$. The obvious inclusion $SO(i) \hookrightarrow SO(i+1)$ induces a map $BSO(i) \rightarrow BSO(i+1)$. Those maps $S^n \rightarrow BSO(i)$ such that the composed maps $S^n \rightarrow BSO(i+1)$ are homotopically trivial (i.e., those rank i vector bundles that become trivial upon direct sum with a trivial line bundle) lift to a map $\tilde{f} : S^n \rightarrow SO(i+1)/SO(i) \cong S^i$. Taking $i = n$, the homotopy class of the map \tilde{f} is completely determined by its topological degree.

Now, given a rank $n-1$ vector bundle on Q_{2n-1} corresponding to a unimodular row, the classifying map $Q_{2n-1} \rightarrow BSL_{n-1}$ lifts to a map $Q_{2n-1} \rightarrow Q_{2n-1}$. Morel has associated with such a map a degree in $GW(k)$, and there is an associated degree in $W(k)$; as observed in the proof of Theorem 4.4, this degree can be identified with the degree of [Fas12]. Taking $k = \mathbb{R}$, one observes that the real points of a map $Q_{2n-1} \rightarrow BSL_{n-1}$ correspond to a rank $n-1$ vector bundle on S^{n-1} and the element of $W(\mathbb{R})$ constructed above is precisely the topological degree of this map.

Remark 4.10. The factor of \mathbf{I}^5 appearing in $\pi_2^{\mathbb{A}^1}(SL_2)$ exemplifies some of the complexities inherent in the discussion of real realization. Note that real realization gives a map $\pi_2^{\mathbb{A}^1}(SL_2)(\mathbb{R}) \rightarrow \pi_2(SL_2(\mathbb{R})) = \pi_2(S^1)$. However, $\mathbf{I}^5(\mathbb{R}) = \mathbb{Z}$, but $\pi_2(S^1) = 0$ so the factor of \mathbf{I}^5 is mapped to zero under real realization.

The explanation for the factor of \mathbf{I}^5 is different. One can show using the next Hopf fibration (this is an algebro-geometric version of $\nu : S^7 \rightarrow S^4$) that there is an epimorphism

$$\pi_3^{\mathbb{A}^1}(\mathbb{P}^{1 \wedge 2}) \longrightarrow \pi_2^{\mathbb{A}^1}(SL_2).$$

There is a canonical morphism $SL_2 \rightarrow \Omega_s^1 L_{\mathbb{A}^1} \Sigma_s^1 SL_2$ (here $L_{\mathbb{A}^1}$ is the \mathbb{A}^1 -localization functor), and this induces a morphism

$$\pi_2^{\mathbb{A}^1}(SL_2) \longrightarrow \pi_3^{\mathbb{A}^1}(\mathbb{P}^{1 \wedge 2})$$

that provides a splitting of this map. Now, under real realization, there is a map $\pi_3^{\mathbb{A}^1}(\mathbb{P}^{1 \wedge 2})(\mathbb{R}) \rightarrow \pi_3(S^2) = \mathbb{Z}$, and the factor of \mathbf{I}^5 encodes this factor of \mathbb{Z} . We will explain this construction in greater detail elsewhere.

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