A cohomological classification of vector bundles on smooth affine threefolds

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Abstract

We give a cohomological classification of vector bundles on smooth affine threefolds over algebraically closed fields having characteristic unequal to 2. As a consequence we deduce that cancellation holds for arbitrary rank projective modules over the corresponding algebras. The proofs of these results involve three main ingredients. First, we give a description of the first non-stable $\mathbb{A}^1$-homotopy sheaf of the general linear group. Second, these computations can be used in concert with F. Morel's $\mathbb{A}^1$-homotopy classification of vector bundles on smooth affine schemes and obstruction theoretic techniques (stemming from a version of the Postnikov tower in $\mathbb{A}^1$-homotopy theory) to reduce the classification results to cohomology vanishing statements. Third, we prove the required vanishing statements.

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1 Introduction

Assume k is an algebraically closed field. If X is a smooth affine k-variety of dimension 3, classical results of N. Mohan Kumar and M.P. Murthy [KM82] prove the existence of vector bundles on X with given Chern classes. Among other things, they prove that there is a unique rank 3 vector bundle with given Chern classes. Recent work of the second author [Fas11] further showed that stably free rank 2 bundles over such X are in fact free.

This paper, which is the first in a series including [AF12a, AF12b], studies problems regarding projective modules using the Morel-Voevodsky $\mathbb{A}^1$-homotopy theory. The main outcome of our approach, as we hope becomes clear in this introduction, can be summarized in the following slogan: there is a framework in which intuition and results about classical homotopy groups of spheres and special linear groups can be suitably "algebraized" to control the theory of projective modules on smooth affine algebras. As a byproduct of our approach, we can recover, refine and extend the statements mentioned above. For example, if we write $\mathcal{V}_n(X)$ for the set of isomorphism classes of rank n vector bundles on X, we can establish the following results.

Theorem 1 (see Theorems 6.10 and 6.11)**.** *Suppose X is a smooth affine 3-fold over an algebraically closed field k having characteristic unequal to 2. The map assigning to a vector bundle of rank $r \leq 3$ the sequence $(c_1, \ldots, c_r)$ of its Chern classes gives isomorphisms of pointed sets:*

$$\mathcal{V}_2(X) \xrightarrow{\sim} Pic(X) \times CH^2(X), \text{ and}$$
$$\mathcal{V}_3(X) \xrightarrow{\sim} Pic(X) \times CH^2(X) \times CH^3(X).$$

One says that cancellation holds for projective modules of rank r over smooth affine algebras of dimension d if stably isomorphic projective modules of rank r are in fact isomorphic. The Bass-Schanuel cancellation theorem (see [BS62, Theorem 2] or [Bas64, Theorem 9.3]) shows that cancellation always holds for projective modules of rank $r > d$. Suslin's famous cancellation theorem [Sus77, Theorem 1] states that cancellation holds for projective modules of rank r if $r \geq d$. In [Sus79] (see the discussion after Theorem 6), Suslin asked whether cancellation holds for projective modules of rank $r \geq \frac{d+1}{2}$. However, Mohan Kumar [MK85] constructed examples showing that cancellation sometimes fails for projective modules of rank $r = d - 2$. Whether cancellation holds for rank $d - 1$ projective modules over affine algebras of dimension d is, in general, an open problem. From the above theorem, we deduce the following result, which provides an answer to the first non-trivial case of Suslin's cancellation question for general algebraically closed fields.

Corollary 2 (see Corollary 6.13)**.** *If X is a smooth affine 3-fold over an algebraically closed field k having characteristic unequal to 2, then cancellation holds for projective modules of any rank over $k[X]$.*

If k is a field having cohomological dimension 1, and X is a smooth affine variety of dimension d over k, Suslin proved that stably free rank d bundles are free [Sus82]. If k is furthermore C_1, Bhatwadekar showed how to deduce cancellation for rank d bundles over a smooth affine d-fold from Suslin's result [Bha03]. Based on these results, it was hoped that cancellation for rank $d - 1$ vector bundles over smooth affine d-folds over algebraically closed fields could be reduced to the assertion that stably free rank $d-1$ bundles on such varieties are in fact free (we will say that "cancellation can be reduced to the stably free case"); the fact that stably free modules rank $d-1$ bundles over smooth affine d-folds over an algebraically closed base are free was recently established by the second author in collaboration with R. Rao and R. Swan [FRS12]. Dhorajia and Keshari showed that if k is the algebraic closure of a finite field, then cancellation could be reduced to the stably free case [DK12]. However, it is apparently much harder to reduce cancellation to the stably free case over larger fields (even when $d=3$), and it was therefore desirable to find a general framework in which to approach the cancellation problem.

Henceforth, assume k is an arbitrary field. Morel and Voevodsky [MV99] introduced $\mathcal{H}(k)$, the $\mathbb{A}^1$-homotopy category of smooth schemes over k. For spaces X and Y, set $[X,Y]_{\mathbb{A}^1} := \mathrm{Hom}_{\mathcal{H}(k)}(X,Y)$ (we will clarify the word space later, but, for the time being, it suffices to believe that smooth schemes and BGL_n are both spaces). One of the main results of [MV99] is that stable isomorphism classes of vector bundles on *arbitrary* smooth k-schemes could be understood in terms of $\mathbb{A}^1$-homotopy theory. More precisely, they introduced spaces BGL_n and a space BGL_∞ such that the set $[X, \mathbb{Z} \times BGL_\infty]$ is $K_0(X)$, i.e., the functor K_0 (restricted to smooth k-schemes) is representable in the $\mathbb{A}^1$-homotopy category.

From the beginning, homotopy theoretic ideas have served as an important source of inspiration in the study of projective modules (see the introduction to [Bas64]). Thus, by analogy, extending the representability result of Morel-Voevodsky, one might expect that $\mathcal{V}_n(\cdot)$ admits a description in terms of maps to the classifying space BGL_n. Such representability statements hold true for $n=1$, i.e., the Picard group is representable by BGL_1. Unfortunately, for $n \geq 2$, the functor $\mathcal{V}_n(\cdot)$ fails to be $\mathbb{A}^1$-invariant for smooth schemes in general, i.e., the map $\mathcal{V}_n(X) \to \mathcal{V}_n(X \times \mathbb{A}^1)$ fails to be a bijection in general. Indeed, already for $X = \mathbb{P}^1$, the failure of $\mathbb{A}^1$-homotopy invariance is well-known. Even worse, for arbitrary smooth schemes, the failure of homotopy invariance is, in a certain sense, "as bad as can be" [AD08].

Nevertheless, classical results of Lindel establishing the Bass-Quillen conjecture [Lin82] showed that the functor $\mathcal{V}_n(\cdot)$ *is* $\mathbb{A}^1$-invariant when restricted to the category of smooth affine k-schemes. Using Lindel's results, together with results of Suslin and Vorst on the so-called K_1-analogue of the Serre problem, Morel showed [Mor12] that if X is a smooth affine k-scheme (at least over a perfect field k), then $[X, BGL_n]_{\mathbb{A}^1} = \mathcal{V}_n(X)$, at least for $n \geq 3$, i.e., a partial representability result remains true. Combined with recent results of L.-F. Moser [Mos11], the above result can be extended to the case $n=2$ as well.

The above results can be viewed as an algebro-geometric analog of Steenrod's homotopy classification of topological vector bundles on CW complexes [Ste99, §19.3]. However, Steenrod also opened the door to enumeration of vector bundles on manifolds using techniques of obstruction theory. Notably, given results known at the time about homotopy groups of (classifying spaces of) special orthogonal groups, Dold and Whitney [DW59] provided explicit cohomological descriptions of sets of isomorphism classes of (real, oriented) vector bundles having a given rank on complexes of dimension ≤ 4 in terms of characteristic classes. Our approach is to transpose these ideas into algebraic geometry.

One of the main impediments to applying techniques of obstruction theory, say via the Postnikov tower, in $\mathbb{A}^1$-homotopy theory arises from our limited knowledge of $\mathbb{A}^1$-homotopy sheaves. Since classical homotopy groups are notoriously difficult to compute directly from the definition, and since $\mathbb{A}^1$-homotopy sheaves are defined abstractly in terms of maps in a certain category, performing computations might seem even more hopeless. Nevertheless, we first devote some attention to providing some new computations of $\mathbb{A}^1$-homotopy sheaves. Before stating our results, we briefly recall some known computations.

Morel showed that SL_n is connected from the standpoint of $\mathbb{A}^1$-homotopy theory. Geometrically, this corresponds to the fact that, over any extension field K/k, any two elements of $SL_n(K)$ can be connected by the image of a morphism from $\mathbb{A}^1_K$; that this can be done is a consequence of the classical fact that $SL_n(K)$ is generated by elementary matrices. Morel also computed [Mor12] the sheaf $\boldsymbol{\pi}_1^{\mathbb{A}^1}(SL_n)$:

$$\boldsymbol{\pi}_1^{\mathbb{A}^1}(SL_n) = \begin{cases} \mathbf{K}_2^{\mathrm{MW}} & \text{if } n = 2 \\ \mathbf{K}_2^{M} & \text{if } n > 2. \end{cases}$$

Roughly speaking, the sheaf $\boldsymbol{\pi}_1^{\mathbb{A}^1}(SL_n)$ encodes information about the non-trivial relations between elementary matrices, and the above result can be viewed as an incarnation of a classical theorem of Steinberg/Matsumoto [Mat69], though it is independent of that statement. Here, $\mathbf{K}_n^M$ is the n-th unramified Milnor K-theory sheaf (the abelian group of sections of this sheaf over a field extension K/k is precisely the n-th Milnor K-theory group of K), and $\mathbf{K}_n^{\mathrm{MW}}$ is the Milnor-Witt K-theory sheaf introduced in [Mor06, Mor12]. Furthermore, for $n \geq 3$, the sheaf $\boldsymbol{\pi}_1^{\mathbb{A}^1}(SL_n)$ is "in the stable range".

Remark 3. As written above, $\boldsymbol{\pi}_1^{\mathbb{A}^1}(SL_2)$ appears not to fit the regular pattern that appears for $n \geq 3$, i.e., it appears to be an unstable group: this is simply a feature of the presentation. The low-dimensional isomorphism $SL_2 \cong Sp_2$ can be used to give an alternate computation: $\boldsymbol{\pi}_1^{\mathbb{A}^1}(SL_2) \cong \mathbf{K}_2^{Sp}$, where $\mathbf{K}_2^{Sp}$ is the sheafification of the second symplectic K-theory for the Nisnevich topology. In that case, the identification $\mathbf{K}_2^{MW} \cong \mathbf{K}_2^{Sp}$ can be viewed as a manifestation of Suslin's description of the second symplectic K-theory of a field [Sus87b, §6]. After making the notion of stable range precise (see Theorems 2.9 and 2.10), we we see that all the homotopy sheaves of SL_n described so far are already "stable" groups.

On the other hand, Wendt [Wen10] provides a rather general description of sections of $\mathbb{A}^1$-homotopy sheaves of SL_n (and, more generally, Chevalley groups) as "unstable Karoubi-Villamayor K-theory." While having such a description is appealing, it is practically speaking intractable since there are essentially no techniques available to study such unstable K-theory groups. Our first main computation can be viewed as providing a computationally tractable description of the first "honestly" unstable $\mathbb{A}^1$-homotopy sheaf of SL_n; moreover, while the constituents of the description might appear involved, we will see below there are a number of techniques available to facilitate their study.

Theorem 4 (See Theorems 3.9 and 3.20)**.** *If k is an infinite perfect field having characteristic unequal to 2, there are canonical short exact sequences of strictly $\mathbb{A}^1$-invariant sheaves*

$$0 \longrightarrow \mathbf{S}_4'' \longrightarrow \boldsymbol{\pi}_2^{\mathbb{A}^1}(SL_2) \longrightarrow \mathbf{K}_3^{Sp} \longrightarrow 0$$
$$0 \longrightarrow \mathbf{S}_4 \longrightarrow \boldsymbol{\pi}_2^{\mathbb{A}^1}(SL_3) \longrightarrow \mathbf{K}_3^{Q} \longrightarrow 0,$$

and for $n \geq 4$ there are isomorphisms $\boldsymbol{\pi}_2^{\mathbb{A}^1}(SL_n) \cong \mathbf{K}_3^Q$. Here, $\mathbf{K}_i^Q$ is the sheafification of the i-th Quillen K-theory functor for the Nisnevich topology, $\mathbf{K}_3^{Sp}$ is the sheafification of the third symplectic

K-theory group for the Nisnevich topology, and there is a canonical epimorphism $\mathbf{K}^M_4/6 \to \mathbf{S}_4$*. The sheaf* $\mathbf{S}''_4$ *sits in an exact sequence of the form*

$$\mathbf{I}^5 \longrightarrow \mathbf{S}''_4 \longrightarrow \mathbf{S}'_4 \longrightarrow 0,$$

where $\mathbf{I}^5$ *is the unramified sheaf associated with the* 5*th power of the fundamental ideal in the Witt ring, and there is a canonical epimorphism* $\mathbf{K}^M_4/12 \to \mathbf{S}'_4$.

In fact, the description of $\boldsymbol{\pi}^{\mathbb{A}^1}_2(SL_3)$ provided above is derived from a description of the first non-stable homotopy sheaf of SL_n when n is odd. We summarize this in the following result.

Theorem 5 (See Theorem 3.9)**.** *If* k *is an infinite perfect field, then for every odd integer* $n \geq 3$ *there are canonical short exact sequences of the form*

$$0 \longrightarrow \mathbf{S}_{n+1} \longrightarrow \boldsymbol{\pi}^{\mathbb{A}^1}_{n-1}(SL_n) \longrightarrow \mathbf{K}^Q_n \longrightarrow 0,$$

where there is an epimorphism $\mathbf{K}^M_{n+1}/n! \to \mathbf{S}_{n+1}$.

The proofs of Theorems 4 and 5 rely on the theory of $\mathbb{A}^1$-fiber sequences attached to Zariski locally trivial SL_n and Sp_n-bundles developed by Morel [Mor12] and Wendt [Wen11] and Morel's unstable connectivity results; these ideas are reviewed in Section 2. Moreover, these theorems immediately give identifications of homotopy sheaves of GL_n, SL_n and, with appropriate index shifts, the associated classifying spaces: it is for this reason that we may pass freely between discussion of (higher) homotopy sheaves of GL_n, its classifying space or SL_n. Combining the results of Morel and Wendt, one obtains that $\boldsymbol{\pi}^{\mathbb{A}^1}_2(GL_2)$ and $\boldsymbol{\pi}^{\mathbb{A}^1}_2(GL_3)$ are extensions of a "stable" sheaf by a certain quotient of $\mathbf{K}^{\mathrm{MW}}_4$. In a sense, the point of the theorems is to precisely identify this quotient, and this requires reinterpreting some classical results of Suslin [Sus84]; this is completed in Section 3, though some results of Section 4 are necessary as well. The first isomorphism reflects the fact that $SL_2 = Sp_2$ is just outside the stable range for symplectic K-theory, and the second $\mathbb{A}^1$-homotopy sheaf gets a contribution from symplectic K-theory. The second isomorphism reflects the fact that $\boldsymbol{\pi}^{\mathbb{A}^1}_2(SL_3)$ is just outside the stable range for the general linear group.

Remark 6. The question of whether the surjective map $\mathbf{K}^M_{n+1}/n! \to \mathbf{S}_{n+1}$ is an isomorphism is equivalent to a question posed by Suslin in [Sus84]. Indeed, if F is a field, Suslin constructs a homomorphism $K^Q_{n+1}(F) \to K^M_{n+1}(F)$ whose image contains $n!K^M_{n+1}(F)$. He observes that the question of whether the image of this map is precisely $n!K^M_{n+1}(F)$ is equivalent to a portion of Milnor's conjecture on quadratic forms for $n = 3$, and speculates about equality in general. Our choice of the letter $\mathbf{S}$ in the notation is intended to remind the reader of both surjectivity and Suslin.

Suslin's question is already non-trivial when $n = 3$. In that case, using the Voevodsky-Rost proof of the Bloch-Kato conjecture and the spectral sequence relating motivic cohomology to algebraic K-theory, one can give conditions involving motivic cohomology groups that imply $\mathbf{S}_4$ is isomorphic to $\mathbf{K}^M_4/6$; these points are discussed in detail in Appendix A.

The question of whether $\mathbf{K}^M_{n+1}/n! \to \mathbf{S}_{n+1}$ is an isomorphism is yet more difficult. Nevertheless, these issues only appear over non-algebraically closed fields. For example, if F is algebraically closed, since $\mathbf{K}^M_i(F)$ is divisible for arbitrary $i \geq 1$, it follows that $\mathbf{K}^M_{n+1}/n!(F)$ is trivial. Likewise, the Bloch-Kato conjecture implies vanishing of $\mathbf{K}^M_{n+1}/n!(F)$ for fields F of étale cohomological dimension $\leq n$.

We use obstruction theory, in this case using a version of the Postnikov tower in $\mathbb{A}^1$-homotopy theory, to deduce Theorem 1 and Corollary 2; this is explained in Section 6. Indeed, at least over algebraically closed fields, the classification results can be deduced from the computations of $\mathbb{A}^1$-homotopy sheaves above by establishing certain vanishing theorems for cohomology of certain constituents of these sheaves. In particular, our approach necessitates understanding cohomology of $\mathbf{K}_2^{\mathrm{MW}}, \mathbf{K}_3^Q, \mathbf{K}_4^M/6, \mathbf{K}_4^M/12, \mathbf{I}^5$ and $\mathbf{K}_3^{Sp}$. The cohomology of $\mathbf{K}_3^Q$, $\mathbf{K}_4^M/6$ (or $\mathbf{K}_4^M/12$), and $\mathbf{I}^5$ can be studied by means of Bloch's formula [Blo86] and the Gersten resolution; the relevant vanishing theorems are established in Section 5. The cohomology of $\mathbf{K}_3^{Sp}$ can be studied by a careful analysis of the Gersten-Grothendieck-Witt spectral sequence (see, e.g., [FS09]), and this constitutes the bulk of Section 4; the techniques of that section can be used to study the cohomology of $\mathbf{K}_2^{\mathrm{MW}}$ as well.

These observations are just the beginning of the story. The moral we draw is: additional information about unstable $\mathbb{A}^1$-homotopy groups of GL_n can be directly translated into results about vector bundles on smooth affine schemes. To keep the length of this paper reasonable, we have deferred the discussion of some natural questions to subsequent work; we mention just two points here. For example, in [AF12a], we complement Theorem 5 by providing a description of $\boldsymbol{\pi}_{2n-1}^{\mathbb{A}^1}(SL_{2n})$; we also discuss compatibility of our computations with computations in classical homotopy theory by means of realization functors. These comparison results demonstrate that the factor of $n!$ or 12 appearing in the computations above is an algebro-geometric manifestation of results regarding the classical unstable homotopy groups of spheres or special linear groups. We also study vector bundles on smooth affine schemes that have the $\mathbb{A}^1$-homotopy types of motivic spheres.

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2 $\mathbb{A}^1$-homotopy theory of SL_n and Sp_n: the stable range

In this section, we review some preliminaries from $\mathbb{A}^1$-homotopy theory, especially some results regarding classifying spaces in $\mathbb{A}^1$-homotopy theory, some results from the theory of $\mathbb{A}^1$-fiber sequences, due to Morel and Wendt, and Morel's classification theorem for vector bundles over smooth affine schemes. We then recall some stabilization results for $\mathbb{A}^1$-homotopy sheaves of linear and symplectic groups; these results are also due to Morel and Wendt. The ultimate goal of this section is to define the stable range, and understand the $\mathbb{A}^1$-homotopy sheaves in this range.

Preliminaries from $\mathbb{A}^1$-homotopy theory

Assume k is a field. Write $\mathcal{Sm}_k$ for the category of schemes that are smooth, separated and have finite type over $\operatorname{Spec} k$. Set $\mathcal{Spc}_k := \Delta^\circ \mathcal{Shv}_{\mathrm{Nis}}(\mathcal{Sm}_k)$ (resp. $\mathcal{Spc}_{k,\bullet}$) for the category of (pointed) simplicial sheaves on the site of smooth schemes equipped with the Nisnevich topology; objects of this category will be referred to as *(pointed) k-spaces*, or simply as *(pointed) spaces* if k is clear from context. Write $\mathcal{H}_s^{\mathrm{Nis}}(k)$ (resp $\mathcal{H}_{s,\bullet}^{\mathrm{Nis}}(k)$) for the (pointed) Nisnevich simplicial homotopy category: this category can be obtained as the homotopy category of, e.g., the injective local model structure on $\mathcal{Spc}_k$ (see, e.g., [MV99] for details). Write $\mathcal{H}(k)$ (resp. $\mathcal{H}_\bullet(k)$) for the associated $\mathbb{A}^1$-homotopy category, which is constructed as a Bousfield localization of $\mathcal{H}_s^{\mathrm{Nis}}(k)$ (resp. $\mathcal{H}_{s,\bullet}^{\mathrm{Nis}}(k)$).

Given two (pointed) spaces $\mathcal{X}$ and $\mathcal{Y}$, we set $[\mathcal{X},\mathcal{Y}]_s := \operatorname{Hom}_{\mathcal{H}_s^{\mathrm{Nis}}(k)}(\mathcal{X},\mathcal{Y})$ and $[\mathcal{X},\mathcal{Y}]_{\mathbb{A}^1} := \operatorname{Hom}_{\mathcal{H}(k)}(\mathcal{X},\mathcal{Y})$; morphisms in pointed homotopy categories will be denoted similarly with base-points explicitly written if it is not clear from context. We write S_s^i for the constant sheaf on $\mathcal{Sm}_k$ associated with the simplicial i-sphere, and $\mathbf{G}_m$ will always be pointed by 1. The $\mathbb{A}^1$-homotopy sheaves of a pointed space $(\mathcal{X},x)$, denoted $\boldsymbol{\pi}_i^{\mathbb{A}^1}(\mathcal{X},x)$ are defined as the Nisnevich sheaves associated with the presheaves $U \mapsto [S_s^i \wedge U_+, (\mathcal{X},x)]_{\mathbb{A}^1}$. We also write $\boldsymbol{\pi}_{i,j}^{\mathbb{A}^1}(\mathcal{X},x)$ for the Nisnevich sheafification of the presheaf $U \mapsto [S_s^i \wedge {\mathbf{G}_m}^{\wedge j} \wedge U_+, (\mathcal{X},x)]_{\mathbb{A}^1}$.

A presheaf of sets $\mathcal{F}$ on $\mathcal{Sm}_k$ is called *$\mathbb{A}^1$-invariant* if for any smooth k-scheme U the morphism $\mathcal{F}(U) \to \mathcal{F}(U \times \mathbb{A}^1)$ induced by pullback along the projection $U \times \mathbb{A}^1 \to U$ is a bijection. A Nisnevich sheaf of groups $\mathcal{G}$ is called *strongly $\mathbb{A}^1$-invariant* if the cohomology presheaves $H^i_{\mathrm{Nis}}(\cdot,\mathcal{G})$ are $\mathbb{A}^1$-invariant for $i = 0,1$. A Nisnevich sheaf of abelian groups $\mathbf{A}$ is called *strictly $\mathbb{A}^1$-invariant* if the cohomology presheaves $H^i_{\mathrm{Nis}}(\cdot,\mathbf{A})$ are $\mathbb{A}^1$-invariant for every $i \geq 0$.

A review of the theory of $\mathbb{A}^1$-fiber sequences

If K is a compact Lie group, principal K-bundles are standard examples of Serre fibrations. Associated with a Serre fibration is a corresponding long exact sequence in homotopy groups. Constructing $\mathbb{A}^1$-fibrations is more delicate and not so many examples are known. If G is a (smooth) algebraic group over a field F, then in general, G-torsors are only locally trivial in the étale topology. This observation and the failure of homotopy invariance for the functor "isomorphism classes of G-torsors" make attaching fibrations in $\mathbb{A}^1$-homotopy theory to G-torsors somewhat delicate. Nevertheless, if G is a special group in the sense of Grothendieck-Serre, i.e., if all G-torsors are Zariski locally trivial, G-torsors give rise to $\mathbb{A}^1$-fiber sequences in a sense we now explain.

We will use the general theory of fibrations in model categories, for which we refer the reader to [Hov99, §6.2]. Given a morphism $f : (\mathcal{E},x) \to (\mathcal{B},y)$ of pointed spaces that is an $\mathbb{A}^1$-fibration in the sense of the $\mathbb{A}^1$-local model structure, we write $\mathcal{F}$ for the $\mathbb{A}^1$-homotopy fiber of f. Given this setup, there is an induced action of $\mathbf{R}\Omega_s^1\mathcal{B}$ (the simplicial loop space of a fibrant model of $\mathcal{B}$) on $\mathcal{F}$; this action is specified functorially, i.e., given an arbitrary space $\mathcal{A}$ one constructs an action of $[\mathcal{A},\mathbf{R}\Omega_s^1\mathcal{B}]_{\mathbb{A}^1}$ on $[\mathcal{A},\mathcal{F}]_{\mathbb{A}^1}$ by means of the homotopy lifting property of fibrations. In other words, given an $\mathbb{A}^1$-fibration, we obtain a sequence of pointed spaces and morphisms of the form

$$(\mathcal{F},x_0) \longrightarrow (\mathcal{E},x) \xrightarrow{f} (\mathcal{B},y)$$

together with an action of $\mathbf{R}\Omega_s^1\mathcal{B}$ on $\mathcal{F}$. An $\mathbb{A}^1$-*fiber sequence* is then a sequence of morphisms of pointed spaces $\mathcal{Z} \to \mathcal{X} \to \mathcal{Y}$, together with an action of $\mathbf{R}\Omega^1_s\mathcal{Y}$ on $\mathcal{Z}$ that is isomorphic in $\mathcal{H}_\bullet(k)$ to a sequence constructed from an $\mathbb{A}^1$-fibration as above.

Morphisms of fiber sequences are sequences of morphisms in $\mathcal{H}_\bullet(k)$ that respect the actions of loop spaces. The main result about fiber sequences we will use is summarized in the following statement, which is quoted from [Wen11, Proposition 5.1, Proposition 5.2, and Theorem 5.3]; in any situation in this paper where a sequence of spaces is asserted to be an $\mathbb{A}^1$-fiber sequence (and for which no auxiliary reference is given), the sequence has this property because of the following result.

Theorem 2.1 (Morel, Moser, Wendt)**.** *Assume F is a field, and (X, x) is a pointed smooth F-scheme. If $P \to X$ is a G-torsor for $G = \mathbf{G}_m, SL_n, GL_n$ or Sp_{2n}, then there is an $\mathbb{A}^1$-fiber sequence of the form*

$$G \longrightarrow P \longrightarrow X.$$

If, moreover, Y is a pointed smooth quasi-projective F-scheme equipped with a left action of G, then the associated fiber space, i.e., the quotient $P \times^G Y$, exists as a smooth scheme, and there is an $\mathbb{A}^1$-fiber sequence of the form

$$Y \longrightarrow P \times^G Y \longrightarrow X.$$

Comments on the proof. As regards attribution: Morel proved the above result for $\mathbf{G}_m, SL_n$ or GL_n ($n \geq 3$) in [Mor12], and Wendt extended his result to treat a rather general class of reductive groups; the case where $G = SL_2$ requires the results of Moser [Mos11]. In [Wen11, Proposition 5.1], this result is stated under the apparently additional hypothesis that F be infinite. However, the assumption that F is infinite is only used by way of Proposition 4.1 of *ibid* to guarantee Nisnevich local triviality of G-torsors that are trivial upon restriction to the base point. In particular, since SL_n and Sp_n are special groups (in the sense of Grothendieck-Serre), G-torsors for such groups are automatically Zariski locally trivial over any base.

In the second statement, quasi-projectivity of Y is only used to guarantee that the quotient $P \times^G Y$ exists as a smooth scheme and that this quotient coincides with the Nisnevich sheaf quotient of the functor represented by $P \times Y$ by the functor represented by G. $\square$

By the general theory of fiber sequences [Hov99, §6.2 and Proposition 6.5.3] together with a sheafification argument, an $\mathbb{A}^1$-local fiber sequence as above gives rise to an associated long exact sequence in $\mathbb{A}^1$-homotopy *sheaves*; we summarize this in the next statement; we will use this result without mention in the sequel.

Proposition 2.2. *If $(\mathcal{F}, x_0) \to (\mathcal{E}, x) \to (\mathcal{B}, y)$ is an $\mathbb{A}^1$-fiber sequence, then for any pair of integers i, j, there is a long exact sequence of the form*

$$\cdots \longrightarrow \boldsymbol{\pi}^{\mathbb{A}^1}_{i+1,j}(\mathcal{B}, y) \xrightarrow{\delta} \boldsymbol{\pi}^{\mathbb{A}^1}_{i,j}(\mathcal{F}, x_0) \longrightarrow \boldsymbol{\pi}^{\mathbb{A}^1}_{i,j}(\mathcal{E}, x) \longrightarrow \boldsymbol{\pi}^{\mathbb{A}^1}_{i,j}(\mathcal{B}, y) \longrightarrow \cdots,$$

where all the unmarked arrows are induced by covariant functoriality of homotopy sheaves, and the connecting homomorphism δ is defined by the composite $\mathbf{R}\Omega^1_s\mathcal{B} \to \mathcal{F} \times \mathbf{R}\Omega^1_s\mathcal{B} \to \mathcal{F}$, where the first map is given by inclusion of the base-point and the second map is given by the action of the simplicial loop space of the base on the fiber.

Classifying spaces and vector bundles

Suppose G is a Nisnevich sheaf of groups. Throughout the paper, we will always assume G is pointed by the identity $\mathrm{Spec}(k) \to G$, and we will suppress the base-point. We write $EG_\bullet$ for the usual Čech-simplicial object associated with the morphism $G \to \mathrm{Spec}(k)$, i.e., $EG_n = G^{\times n+1}$ and the simplicial structures are induced by projections and partial diagonals. The sheaf G acts on $EG_\bullet$ (on the right), and the quotient $EG_\bullet/G = BG_\bullet$ gives the usual simplicial bar construction [MV99, §4.1]. The space $BG_\bullet$ is a reduced simplicial sheaf (i.e., its sheaf of 0-simplices is the constant sheaf $\mathrm{Spec}(k)$), and so $BG_\bullet$ has a canonical base-point.

Morel and Voevodsky show [MV99, §4 Proposition 1.15] that if $\mathcal{X}$ is a space, then there is a canonical bijection

$$[\mathcal{X}, BG_\bullet]_s \xrightarrow{\sim} H^1_{\mathrm{Nis}}(\mathcal{X}, G);$$

to be clear, we are taking maps in the *unpointed* simplicial homotopy category here. In particular, if G is a linear algebraic group that is special, then it follows that isomorphism classes of G-torsors are in bijection with elements of $[\mathcal{X}, BG_\bullet]_s$.

Write $Gr_{n,n+N}$ for the grassmannian parameterizing n-dimensional subspaces of an $n+N$-dimensional vector space. We let $Gr_{n,\infty}$ be $\mathrm{colim}_N\, Gr_{n,n+N}$ for the morphisms induced by standard inclusions. The universal vector bundle on $Gr_{n,n+N}$ induces a simplicial homotopy class of morphisms $Gr_{n,n+N} \to BGL_{n,\bullet}$, and Morel and Voevodsky observe that the induced morphism $Gr_{n,\infty} \to BGL_n$ is an $\mathbb{A}^1$-weak equivalence [MV99, §4 Proposition 3.7]. Morel proves the following fact.

Theorem 2.3 (Morel, Moser [Mor12, Theorem 8.1])**.** *If k is a perfect field, and if X is a smooth affine k-scheme, then there is a canonical bijection*

$$[X, Gr_{n,\infty}]_{\mathbb{A}^1} \cong \mathcal{V}_n(X),$$

where $\mathcal{V}_n(X)$ is the set of isomorphism classes of rank n vector bundles on X.

In a number of situations below, only the $\mathbb{A}^1$-homotopy type of $BG_\bullet$ plays a role. For that reason, we make the following convention.

Notation 2.4. Write BG for any space that has the $\mathbb{A}^1$-homotopy type of $BG_\bullet$.

Stabilization sequences

We now apply the results on fiber sequences above to the $\mathbb{A}^1$-fiber sequences

$$SL_{n-1} \hookrightarrow SL_n \longrightarrow SL_n/SL_{n-1}$$

and

$$Sp_{2n-2} \hookrightarrow Sp_{2n} \longrightarrow Sp_{2n}/Sp_{2n-2}.$$

Each of these fiber sequences gives rise to a long exact sequence in $\mathbb{A}^1$-homotopy sheaves. Taken together, the next pair of results, observed by Morel and Wendt, shows that the quotients that appear are highly $\mathbb{A}^1$-connected.

Proposition 2.5. *The "projection onto the first column" morphism $SL_n \to \mathbb{A}^n \setminus 0$ (resp. $Sp_{2n} \to \mathbb{A}^{2n} \setminus 0$) factors through an $\mathbb{A}^1$-weak equivalence $SL_n/SL_{n-1} \to \mathbb{A}^n \setminus 0$ (resp. $Sp_{2n}/Sp_{2n-2} \to \mathbb{A}^{2n} \setminus 0$).*

Proof. In the first case, SL_n acts transitively on $\mathbb{A}^n \setminus 0$ and the stabilizer of a point can be identified with an extension of SL_{n-1} by a unipotent group. Furthermore, there is a Zariski locally trivial morphism $SL_n/SL_{n-1} \to \mathbb{A}^n \setminus 0$ with affine space fibers. The case of the symplectic group is similar. ☐

The stable range

Theorem 2.6 (Morel)**.** *For any integer $n \geq 2$, the space $\mathbb{A}^n \setminus 0$ is $(n-2)$-$\mathbb{A}^1$-connected, and there is a canonical isomorphism $\boldsymbol{\pi}_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0) \cong \mathbf{K}_n^{\mathrm{MW}}$.*

Remark 2.7. Explicit generators and relations for the sections of the sheaves $\mathbf{K}_n^{\mathrm{MW}}$ are given in [Mor12, §3]. A number of basic properties of the sheaves we use will be quoted from this source.

Corollary 2.8 (Morel, Wendt)**.** *The morphisms $\boldsymbol{\pi}_i^{\mathbb{A}^1}(SL_{n-1}) \to \boldsymbol{\pi}_i^{\mathbb{A}^1}(SL_n)$ are epimorphisms for $i \leq n-2$ and isomorphisms for $i \leq n-3$. The morphisms $\boldsymbol{\pi}_i^{\mathbb{A}^1}(Sp_{2n-2}) \to \boldsymbol{\pi}_i^{\mathbb{A}^1}(Sp_{2n})$ are epimorphisms for $i \leq 2n-2$ and isomorphisms for $i \leq 2n-3$.*

Re-indexing slightly, the sheaves $\boldsymbol{\pi}_i^{\mathbb{A}^1}(SL_n)$ coincide with the stable groups $\boldsymbol{\pi}_i^{\mathbb{A}^1}(SL_\infty)$ for $i \leq n-2$ and homotopy sheaves in this range of indices will be said to be in the stable range. Likewise, the sheaves $\boldsymbol{\pi}_i^{\mathbb{A}^1}(Sp_{2n})$ coincide with $\boldsymbol{\pi}_i^{\mathbb{A}^1}(Sp_\infty)$ for $i \leq 2n-1$ and homotopy sheaves in this range of indices will again be said to be in the stable range.

Homotopy sheaves of GL_n in the stable range

We quickly review the computation of the homotopy sheaves of GL_n in the stable range, which is due to Morel. Forming a colimit in the index n there are spaces $Gr_{\infty,\infty}$ and $BGL_{\infty,\bullet}$ together with an $\mathbb{A}^1$-weak equivalence $Gr_{\infty,\infty} \to BGL_{\infty,\bullet}$. By [MV99, Theorem 3.13], the space $\mathbb{Z} \times Gr_{\infty,\infty}$ represents Quillen K-theory for smooth k-schemes.

Write $\mathbf{K}_i^Q$ for the Nisnevich sheaf associated with the presheaf $U \mapsto K_i(U)$, where K_i denotes Quillen K-theory; these sheaves are called Quillen K-theory sheaves. The next result describes the $\mathbb{A}^1$-homotopy sheaves of SL_n or GL_n in the stable range in terms of Quillen K-theory.

Theorem 2.9. *For any integers $i > 0$ and any $n > 1$ there are canonical isomorphisms*

$$\boldsymbol{\pi}_i^{\mathbb{A}^1}(SL_n) \cong \boldsymbol{\pi}_i^{\mathbb{A}^1}(GL_n) \cong \boldsymbol{\pi}_{i+1}^{\mathbb{A}^1}(BGL_{n,\bullet}).$$

If furthermore, $0 \leq i \leq n-2$, there are canonical isomorphisms of the form

$$\boldsymbol{\pi}_{i+1}^{\mathbb{A}^1}(BGL_{n,\bullet}) \cong \boldsymbol{\pi}_{i+1}^{\mathbb{A}^1}(BGL_{\infty,\bullet}) \cong \mathbf{K}_{i+1}^Q.$$

Proof. There are $\mathbb{A}^1$-fiber sequences of the form

$$\begin{aligned} \mathbf{G}_m &\longrightarrow GL_n \longrightarrow SL_n \\ GL_n &\longrightarrow EGL_{n,\bullet} \longrightarrow BGL_{n,\bullet}. \end{aligned}$$

Since $\mathbf{G}_m$ is $\mathbb{A}^1$-rigid [MV99, §4 Example 2.4], we have $\boldsymbol{\pi}_i^{\mathbb{A}^1}(\mathbf{G}_m) = 1$ for $i \geq 1$. It is straightforward to check that the inclusion $\mathbf{G}_m \hookrightarrow GL_n$ induces an isomorphism $\mathbf{G}_m = \boldsymbol{\pi}_0^{\mathbb{A}^1}(\mathbf{G}_m) \xrightarrow{\sim} \boldsymbol{\pi}_0^{\mathbb{A}^1}(GL_n)$. Combining these two observations gives the first isomorphism.

For the second isomorphism, note that $EGL_{n,\bullet} \to BGL_{n,\bullet}$ is a GL_n-torsor (it can even be realized as a colimit of GL_n-torsors over smooth schemes) and thus gives rise to an $\mathbb{A}^1$-fiber sequence. Since EGL_n is $\mathbb{A}^1$-contractible, the second isomorphism is immediate.

By representability of algebraic K-theory, $\mathbf{K}_i^Q$ can also be described as $\boldsymbol{\pi}_i^{\mathbb{A}^1}(\mathbb{Z} \times BGL_\infty)$. Moreover, for $i > 0$, the only contribution to this sheaf comes from the $\mathbb{A}^1$-connected component of the base-point, so $\boldsymbol{\pi}_i^{\mathbb{A}^1}(\mathbb{Z} \times BGL_{\infty,\bullet}) \cong \boldsymbol{\pi}_i^{\mathbb{A}^1}(BGL_{\infty,\bullet})$. The final statement can then be deduced from Corollary 2.8. $\square$

Homotopy sheaves of Sp_{2n} in the stable range

Replacing the general (or special) linear group by the symplectic group, there are analogous stability statements. Let $HGr(2n, 2(n+N))$ be the open subscheme of $Gr_{2n,2(n+N)}$ parameterizing $2n$-dimensional subspaces of a $2(n+N)$-dimensional symplectic vector space to which the symplectic form restricts non-degenerately. One can give a more functorial description of this space, but let us note that, upon choice of a base-point, $HGr(2n, 2(n+N))$ becomes isomorphic to the homogeneous space $Sp_{2(n+N)}/(Sp_{2N} \times Sp_{2n})$.

The morphism $Sp_{2(n+N)}/Sp_{2N} \to HGr(2n, 2(n+N))$ is an Sp_{2n}-torsor and, as mentioned above, is therefore classified by a simplicial homotopy class of maps $HGr(2n, 2(n+N)) \to BSp_{2n,\bullet}$. Taking an appropriate colimit over N, there is an induced morphism $HGr(2n, \infty) \to BSp_{2n,\bullet}$. Likewise, taking a colimit over n, there is an induced morphism $HGr(\infty, \infty) \to BSp_{\infty,\bullet}$. Panin and Walter show [PW10, Theorem 8.2] that the space $\mathbb{Z} \times HGr(\infty, \infty)$ represents symplectic K-theory.

Theorem 2.10. *For any integers $i > 0$ and any $n > 1$ there are canonical isomorphisms*

$$\boldsymbol{\pi}_i^{\mathbb{A}^1}(Sp_{2n}) \cong \boldsymbol{\pi}_{i+1}^{\mathbb{A}^1}(BSp_{2n,\bullet}).$$

If $0 \leq i \leq 2n-1$ and, furthermore, the base field k is assumed to have characteristic unequal to 2, there are canonical isomorphisms of the form

$$\boldsymbol{\pi}_{i+1}^{\mathbb{A}^1}(BSp_{2n,\bullet}) \cong \boldsymbol{\pi}_{i+1}^{\mathbb{A}^1}(BSp_{\infty,\bullet}) \cong \mathbf{K}_{i+1}^{Sp}.$$

Proof. This result is proven in a fashion formally analogous to that for SL_n. The first identification comes from the $\mathbb{A}^1$-fiber sequence associated with the torsor $Sp_{2(n+N)}/Sp_{2N} \to HGr(2n, 2(n+N))$ together with a colimit argument. The second isomorphism results by applying the stabilization isomorphisms of Corollary 2.8. This result was stated by Wendt in a slightly different form in [Wen11, Theorem 6.8 and Remark 6.12]. The hypothesis on the characteristic of the base-field is required to apply the results of [PW10]. $\square$

Remark 2.11. In [MV99, §4], a different geometric model $B_{gm}Sp_{2n}$ for the classifying space Sp_{2n} is constructed; this model is essentially Totaro's model. This model can also be used to construct a space representing symplectic K-theory as explained in [Hor05, Remark 3.8]. While the spaces

$B_{gm}Sp_{2n}$ are $\mathbb{A}^1$-weakly equivalent to $HGr(2n,\infty)$, and both spaces are given as colimits of finite-dimensional approximations, the space $B_{gm}Sp_{2n}$ is not well-adapted to our needs since it is not a colimit of homogeneous spaces. The main technical difference between the finite dimensional smooth varieties approximating $HGr(2n,\infty)$ and those approximating $B_{gm}Sp_{2n}$ is that the former do not form an admissible gadget in the sense of [MV99, §4 Definition 2.1]. Furthermore, the proof that the spaces $\mathbb{Z} \times HGr(\infty,\infty)$ represent symplectic K-theory is very similar to that given for algebraic K-theory in [MV99, §4].

3 $\mathbb{A}^1$-homotopy theory of SL_n and Sp_n: some non-stable results

The main goal of this section is to describe the first non-stable $\mathbb{A}^1$-homotopy sheaf for SL_n (resp. GL_n) for $n \geq 2$. This result is broken into two largely independent parts. The case $n \geq 3$ is treated first. In this range, the groups in question are *meta-stable* in the following loose sense: at least if n is odd, the sheaf $\boldsymbol{\pi}^{\mathbb{A}^1}_{n-1}(SL_n)$ takes a form that depends in a uniform fashion on n. The first non-stable $\mathbb{A}^1$-homotopy sheaf of SL_2 was, as explained in the introduction, computed by Morel (see Theorem 2.6 and use the fact that $SL_2 \to \mathbb{A}^2 \setminus 0$ is an $\mathbb{A}^1$-weak equivalence). The next non-stable $\mathbb{A}^1$-homotopy sheaf of SL_2 is treated, extending an idea of Wendt [Wen11, Proposition 6.11], by means of the exceptional isomorphism $SL_2 \cong Sp_2$ and stabilization results for symplectic K-theory. Some of the results are proven in greater generality than necessary since we expect they will be useful in understanding $\boldsymbol{\pi}^{\mathbb{A}^1}_{n-1}(SL_n)$ when $n \geq 2$ is an even integer.

A short exact sequence describing $\boldsymbol{\pi}^{\mathbb{A}^1}_{n-1}(SL_n)$, $n \geq 2$

The long exact sequence in $\mathbb{A}^1$-homotopy sheaves associated with the $\mathbb{A}^1$-fiber sequence

$$SL_{n-1} \longrightarrow SL_n \longrightarrow SL_n/SL_{n-1}$$

gives rise to an exact sequence of the form

$$\boldsymbol{\pi}^{\mathbb{A}^1}_{n-1}(SL_n) \longrightarrow \boldsymbol{\pi}^{\mathbb{A}^1}_{n-1}(SL_n/SL_{n-1}) \longrightarrow \boldsymbol{\pi}^{\mathbb{A}^1}_{n-2}(SL_{n-1}) \longrightarrow \boldsymbol{\pi}^{\mathbb{A}^1}_{n-2}(SL_n) \longrightarrow 0.$$

In case $n = 3$, we furthermore observe that $\boldsymbol{\pi}^{\mathbb{A}^1}_1(SL_2)$ and $\boldsymbol{\pi}^{\mathbb{A}^1}_1(SL_3)$ are known to be sheaves of abelian groups and are therefore strictly $\mathbb{A}^1$-invariant [Mor12, Corollary 6.2], i.e., the sequence above is always a sequence of strictly $\mathbb{A}^1$-invariant sheaves of groups.

Theorem 2.6 and Theorem 2.9 (the groups $\boldsymbol{\pi}^{\mathbb{A}^1}_{n-2}(SL_n)$ are in the stable range) allow us to rewrite this sequence as

$$\boldsymbol{\pi}^{\mathbb{A}^1}_{n-1}(SL_n) \xrightarrow{q_{n-1}} \mathbf{K}^{MW}_n \xrightarrow{\delta_{n-1}} \boldsymbol{\pi}^{\mathbb{A}^1}_{n-2}(SL_{n-1}) \longrightarrow \mathbf{K}^Q_{n-1} \longrightarrow 0.$$

Our goal is to understand the image of $\boldsymbol{\pi}^{\mathbb{A}^1}_{n-1}(SL_n) \to \mathbf{K}^{MW}_n$.

The connecting homomorphism δ_{n-1} gives a homomorphism $\boldsymbol{\pi}^{\mathbb{A}^1}_{n-1}(SL_n/SL_{n-1}) \to \boldsymbol{\pi}^{\mathbb{A}^1}_{n-2}(SL_{n-1})$. The composite homomorphism $q_{n-2} \circ \delta_{n-1}$ therefore gives a map $\mathbf{K}^{MW}_n \to \mathbf{K}^{MW}_{n-1}$. Since SL_n/SL_{n-1} is $\mathbb{A}^1$-$(n-2)$-connected by Proposition 2.5 and Theorem 2.6, if $\mathbf{A}$ is any strictly $\mathbb{A}^1$-invariant sheaf, [AD09, Theorem 3.30] gives a canonical bijection

$$H^{n-1}_{\mathrm{Nis}}(SL_n/SL_{n-1}, \mathbf{A}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{Ab}^{\mathbb{A}^1}_k}(\boldsymbol{\pi}^{\mathbb{A}^1}_{n-1}(SL_n/SL_{n-1}), \mathbf{A}).$$

Applying these observations with $\mathbf{A} = \mathbf{K}^{\mathrm{MW}}_{n-1}$, the morphism $q_{n-2} \circ \delta_{n-1}$ is determined by an element of $H^{n-1}_{\mathrm{Nis}}(SL_n/SL_{n-1}, \mathbf{K}^{\mathrm{MW}}_{n-1})$.

The connecting homomorphism in the long exact sequence is obtained (up to simplicial homotopy) by applying simplicial loops to the classifying morphism $SL_n/SL_{n-1} \to BSL_{n-1,\bullet}$ of SL_{n-1}-torsor $SL_n \to SL_n/SL_{n-1}$. The composite morphism $\Omega^1_s SL_n/SL_{n-1} \to SL_{n-1} \to SL_{n-1}/SL_{n-2}$ comes from the action of SL_{n-1} on SL_{n-1}/SL_{n-2}. There is an induced morphism from SL_{n-1}/SL_{n-2} to the $\mathbb{A}^1$-homotopy fiber of the morphism $BSL_{n-2,\bullet} \to BSL_{n-1,\bullet}$ and this morphism is an $\mathbb{A}^1$-weak equivalence. As a consequence, the cohomology class determined by the homomorphism $q_{n-2} \circ \delta_{n-1}$ is precisely the primary obstruction to lifting the classifying map of the SL_{n-1}-torsor $SL_n \to SL_n/SL_{n-1}$ to a map $SL_n/SL_{n-1} \to BSL_{n-2,\bullet}$. By definition, the resulting class is therefore precisely Morel's Euler class of the SL_{n-1}-torsor in question.

The $\mathbb{A}^1$-weak equivalence $SL_n/SL_{n-1} \sim \mathbb{A}^n \setminus 0$ also gives an identification $SL_n/SL_{n-1} \cong \Sigma^{n-1}_s \mathbf{G}_m{}^{\wedge n}$. By means of the suspension isomorphism, the group $H^{n-1}_{\mathrm{Nis}}(SL_n/SL_{n-1}, \mathbf{K}^{\mathrm{MW}}_{n-1})$ is then canonically isomorphic to $H^0_{\mathrm{Nis}}(\mathbf{G}_m{}^{\wedge n}, \mathbf{K}^{\mathrm{MW}}_{n-1})$. The group on the right hand side can be described in terms of contractions (see Proposition 5.4), and one obtains a canonical identification $H^{n-1}_{\mathrm{Nis}}(SL_n/SL_{n-1}, \mathbf{K}^{\mathrm{MW}}_{n-1}) \cong \mathbf{K}^{\mathrm{MW}}_{-1}(k)$. By [Mor12, Lemma 3.10], $\mathbf{K}^{\mathrm{MW}}_{-1}(k) \cong W(k)$, and every element of this group is of the form ηs for $s \in GW(k)$. The next lemma gives a precise description of this Euler class.

Lemma 3.1. *The Euler class of the SL_{n-1}-torsor $SL_n \to SL_n/SL_{n-1}$, which is an element of $H^{n-1}_{\mathrm{Nis}}(SL_n/SL_{n-1}, \mathbf{K}^{\mathrm{MW}}_{n-1})$, is the class of η if n is odd and 0 if n is even.*

Proof. Let $A_{2n-1} := k[x_1, \ldots, x_n, y_1, \ldots, y_n]/\langle \sum x_i y_i - 1\rangle$ and $Q_{2n-1} = \mathrm{Spec}(A_{2n-1})$. Projecting a matrix to its first row and the first column of its inverse yields a SL_{n-1}-equivariant morphism $\pi_n : SL_n \to Q_{2n-1}$, where SL_{n-1} acts trivially on the right-hand term; abusing notation, this morphism induces an isomorphism $\pi_n : SL_n/SL_{n-1} \to Q_{2n-1}$ for any integer $n \geq 2$.

The vector bundle given by the morphism $SL_n/SL_{n-1} \to BSL_{n-1,\bullet}$ can be described as follows. Let V_n be the standard n-dimensional representation of SL_n. As usual, if V is a k-vector space, we write $\mathbb{A}(V) := \mathrm{Spec}\,\mathrm{Sym} V^\vee$, where $V^\vee$ is the k-vector space dual. We view $\mathbb{A}(V_n)$ as an SL_n-scheme with the induced right action. Let $i : SL_{n-1} \to SL_n$ be the closed immersion group homomorphism given by

$$i(G) = \begin{pmatrix} 1 & 0 \\ 0 & G \end{pmatrix}.$$

View SL_n as an SL_{n-1}-scheme by means of left multiplication by $i(G)$. The quotient map $SL_n \to SL_n/SL_{n-1}$ is an SL_{n-1}-bundle and we can form the associated geometric vector bundle:

$$E_n := \mathbb{A}(V_{n-1}) \times^{SL_{n-1}} SL_n,$$

i.e., the quotient of $\mathbb{A}(V_{n-1}) \times SL_n$ by the diagonal SL_{n-1}-action, where we view SL_{n-1} as a subgroup of SL_n.

Claim. *Under the identification $\pi_n : SL_n/SL_{n-1} \to Q_{2n-1}$, E_n is the total space associated with the stably free module P_n of rank $n-1$ defined by the following (split) exact sequence:*

$$0 \longrightarrow A_{2n-1} \xrightarrow{(x_1,\ldots,x_n)} (A_{2n-1})^n \longrightarrow P_n \longrightarrow 0.$$

Viewing V_n as an SL_{n-1}-module by restriction via i, it splits as a direct sum $k \oplus V_{n-1}$. The short exact sequence of SL_{n-1}-modules

$$0 \longrightarrow k \longrightarrow V_n \longrightarrow V_{n-1} \longrightarrow 0$$

gives rise, by faithfully flat descent, to an exact sequence of geometric vector bundles

$$0 \longrightarrow \mathbb{A}^1 \times^{SL_{n-1}} SL_n \longrightarrow \mathbb{A}(V_n) \times^{SL_{n-1}} SL_n \longrightarrow E_n \longrightarrow 0.$$

Since SL_{n-1} acts trivially on $\mathbb{A}^1$, the first vector bundle is simply the trivial bundle $\mathbb{A}^1 \times SL_n/SL_{n-1}$. Since the morphism on the right hand side is split, it follows that this is a split short exact sequence.

Next, define a morphism $\phi_n : \mathbb{A}(V_n) \times SL_n \to \mathbb{A}^n \times Q_{2n-1}$ by $\phi_n(v, M) = (vM, \pi_n(M))$. This morphism is SL_{n-1}-equivariant for the action of SL_{n-1} on $\mathbb{A}(V_n) \times SL_n$ specified above and for the trivial SL_{n-1}-action on $\mathbb{A}^n \times Q_{2n-1}$ and, once again abusing notation slightly, therefore descends to a morphism $\phi_n : \mathbb{A}(V_n) \times^{SL_{n-1}} SL_n \to \mathbb{A}^n \times Q_{2n-1}$.

Combining these facts, we get a commutative diagram whose vertical morphisms are isomorphisms

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{A}^1 \times^{SL_{n-1}} SL_n \arrow[r, "j"] \arrow[d, equal] & \mathbb{A}(V_n) \times^{SL_{n-1}} SL_n \arrow[r, "q"] \arrow[d, "\phi_n"] & E_n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{A}^1 \times Q_{2n-1} \arrow[r, "j'"'] & \mathbb{A}^n \times Q_{2n-1} \arrow[r] & E_n \arrow[r] & 0
\end{tikzcd}$$

It suffices to check that j' is the announced morphism to prove the claim.

We now proceed to the computation of the Euler class of E_n. If n is even, then the Euler class of E_n is trivial since a stably free module given by a unimodular row of even length always has a free factor of rank one and thus a trivial Euler class. In case n is odd, the Euler class is computed in [Fas12, Proposition 3.2]. ☐

Lemma 3.2. *For $n \geq 3$ and odd, there is a short exact sequence of the form*

$$0 \longrightarrow \mathbf{S}_{n+1} \longrightarrow \boldsymbol{\pi}^{\mathbb{A}^1}_{n-1}(SL_n) \longrightarrow \mathbf{K}^Q_n \longrightarrow 0,$$

where $\mathbf{S}_{n+1}$ is a quotient of $\mathbf{K}^M_{n+1}$.

Proof. We combine the long exact sequences in $\mathbb{A}^1$-homotopy sheaves associated with the fibrations $SL_{n-1} \to SL_n \to SL_n/SL_{n-1}$ and $SL_{n-2} \to SL_{n-1} \to SL_{n-1}/SL_{n-2}$ to get a diagram of the form

$$\begin{tikzcd}
 & \mathbf{K}^{\mathrm{MW}}_n \arrow[d, "\delta_{n-1}"] & & & & \\
\boldsymbol{\pi}^{\mathbb{A}^1}_{n-2}(SL_{n-2}) \arrow[r] & \boldsymbol{\pi}^{\mathbb{A}^1}_{n-2}(SL_{n-1}) \arrow[r, "q_{n-2}"] \arrow[d] & \mathbf{K}^{\mathrm{MW}}_{n-1} \arrow[r] & \boldsymbol{\pi}^{\mathbb{A}^1}_{n-3}(SL_{n-2}) \arrow[r] & \mathbf{K}^Q_{n-2} \arrow[r] & 0 \\
 & \mathbf{K}^Q_{n-1} \arrow[d] & & & & \\
 & 0 & & & &
\end{tikzcd}$$

Now, note that $\mathrm{coker}(\cdot\eta : \mathbf{K}^{\mathrm{MW}}_n \to \mathbf{K}^{\mathrm{MW}}_{n-1}) = \mathbf{K}^M_{n-1}$. Under the hypotheses, the composite map $q_{n-2} \circ \delta_{n-1} : \mathbf{K}^{\mathrm{MW}}_n \to \mathbf{K}^{\mathrm{MW}}_{n-1}$ is multiplication by η by Lemma 3.1, a diagram chase shows that there is an isomorphism $\mathrm{coker}(\mathbf{K}^Q_{n-1} \to \mathbf{K}^M_{n-1}) \to \mathrm{coker}(q_{n-2})$. □

Homological stabilization for GL_n and the sheaf $\mathbf{S}_{n+1}$

By Lemma 3.2, the sheaf $\mathbf{S}_{n+1}$ is a quotient of $\mathbf{K}^M_{n+1}$ by means of a homomorphism $\mathbf{K}^Q_{n+1} \to \mathbf{K}^M_{n+1}$. As the proof of the aforementioned result makes clear, the factor of $\mathbf{K}^Q_{n+1}$ that appears comes from the stabilization homomorphism $SL_n \to SL_{n+1}$. We now attempt to obtain a more concrete description of the image of this homomorphism. Because the sheaf $\mathbf{S}_{n+1}$ is strictly $\mathbb{A}^1$-invariant, which follows from the fact that the category of strictly $\mathbb{A}^1$-invariant sheaves over a field F is abelian [Mor05, §6], to describe $\mathbf{S}_{n+1}$, it suffices to describe its sections over any finitely generated separable extension L/F. The goal, which is realized in Lemma 3.8 after a number of preliminaries, is to connect the sections of the morphism $\mathbf{K}^Q_{n+1} \to \mathbf{K}^M_{n+1}$ over L to some results of Suslin, which we recall below.

The sections of the simplicial classifying space $BGL_{n,\bullet}$ over any field L give a simplicial set whose homology is precisely the standard bar complex used to compute group homology. The usual homomorphism $GL_{n-1} \hookrightarrow GL_n$ induces a morphism $BGL_{n-1,\bullet} \to BGL_{n,\bullet}$. In this context, we can state the result that we will refer to as Suslin's stabilization theorem in the sequel.

Theorem 3.3 ([Sus84, Theorem 3.4]). *If L is an infinite field, the stabilization homomorphism*

$$s_{m,n} : H_m(BGL_{n-1,\bullet}(L), \mathbb{Z}) \longrightarrow H_m(BGL_{n,\bullet}(L), \mathbb{Z})$$

is an isomorphism if $m \leq n-1$, and $s_{n,n}$ has cokernel $K^M_n(L)$.

Using the above stabilization result, Suslin constructed a homomorphism from Quillen K-theory to Milnor K-theory [Sus84, §4]. Consider the sequence

(3.1)
$$\begin{aligned} K^Q_n(L) := \pi_n(BGL_{\infty,\bullet}(L)^+) &\longrightarrow H_n(BGL_{\infty,\bullet}(L)^+, \mathbb{Z}) \\ &\xrightarrow{\sim} H_n(BGL_{\infty,\bullet}(L), \mathbb{Z}) \xrightarrow{\sim} H_n(BGL_{n,\bullet}(L), \mathbb{Z}) \longrightarrow K^M_n(L), \end{aligned}$$

where the first homomorphism is the Hurewicz homomorphism, the second homomorphism comes from the definition of the plus construction, the third homomorphism is the inverse to the composites of the maps coming from the stabilization theorem, and the fourth homomorphism is the projection morphism coming from Theorem 3.3.

First, let us reinterpret the short exact sequence from Lemma 3.2. We observed before that the canonical morphism $Gr_{n,\infty} \to BGL_{n,\bullet}$ is an $\mathbb{A}^1$-weak equivalence, so by means of the isomorphisms of Theorem 2.9, for $i \geq 2$ the homotopy sheaves $\boldsymbol{\pi}^{\mathbb{A}^1}_i(SL_n)$ can be replaced by homotopy sheaves of $\boldsymbol{\pi}^{\mathbb{A}^1}_{i+1}(Gr_{n,\infty})$. Furthermore, the inclusion $SL_n \hookrightarrow GL_n$ induces an isomorphism $SL_n/SL_{n-1} \xrightarrow{\sim} GL_n/GL_{n-1}$.

If we identify $Gr_{n,\infty}$ as a quotient of an $\mathbb{A}^1$-contractible space $V_{n,\infty}$ by a free action of GL_n, then we get an $\mathbb{A}^1$-fiber sequence of the form (*cf.* [Mor12, Proposition 8.11])

$$GL_n/GL_{n-1} \longrightarrow V_{n,\infty} \times^{GL_n} GL_n/GL_{n-1} \longrightarrow Gr_{n,\infty}.$$

The space in the middle is $\mathbb{A}^1$-weakly equivalent to $Gr_{n-1,\infty}$ by a standard argument. The connecting homomorphism in the long exact sequence of homotopy sheaves attached to this $\mathbb{A}^1$-fiber sequence fits into the exact sequence:

$$\boldsymbol{\pi}_i^{\mathbb{A}^1}(Gr_{n-1,\infty}) \longrightarrow \boldsymbol{\pi}_i^{\mathbb{A}^1}(Gr_{n,\infty}) \longrightarrow \boldsymbol{\pi}_{i-1}^{\mathbb{A}^1}(GL_n/GL_{n-1}). \tag{3.2}$$

and one checks that the first homomorphism in the above sequence is precisely the stabilization homomorphism $\boldsymbol{\pi}_{i-1}^{\mathbb{A}^1}(SL_{n-1}) \to \boldsymbol{\pi}_{i-1}^{\mathbb{A}^1}(SL_n)$ for $i \geq 2$.

For any space $\mathcal{X}$, recall that the functor $Sing_*^{\mathbb{A}^1}(\mathcal{X})$ is obtained as the diagonal of the bisimplicial space $(i,j) \to \mathrm{Hom}(\Delta^i, \mathcal{X}_j)$, where Δ^i is the algebraic i-simplex. There is a canonical morphism $\mathcal{X} \to Sing_*^{\mathbb{A}^1}(\mathcal{X})$, which is an $\mathbb{A}^1$-weak equivalence, and $Sing_*^{\mathbb{A}^1}(\cdot)$ commutes with formation of finite limits [MV99, p. 87 and §2 Corollary 3.8]. Furthermore, the augmentation map $\mathrm{Spec}\, k \to \Delta^\bullet$ induces a morphism $Sing_*^{\mathbb{A}^1}(\mathcal{X}) \to \mathcal{X}$.

The spaces $Sing_*^{\mathbb{A}^1}(GL_n)$, $Sing_*^{\mathbb{A}^1}(GL_n/GL_{n-1})$, and $Sing_*^{\mathbb{A}^1}(Gr_{n,\infty})$ all have a version of the so-called affine BG property (see [Mor12, Appendix A.1] for the relevant definitions, and [Mor12, Theorems 8.1, 8.9, and 9.21] for the results). The main consequence of this that we use is that if L is a finitely generated separable extension of F, there are canonical isomorphisms

$$\begin{aligned}
\boldsymbol{\pi}_i^{\mathbb{A}^1}(Gr_{n,\infty})(L) &\cong \pi_i(Sing_*^{\mathbb{A}^1}(Gr_{n,\infty})(L)),\\
\boldsymbol{\pi}_i^{\mathbb{A}^1}(GL_n)(L) &\cong \pi_i(Sing_*^{\mathbb{A}^1}(GL_n)(L)), \text{ and}\\
\boldsymbol{\pi}_i^{\mathbb{A}^1}(GL_n/GL_{n-1})(L) &\cong \pi_i(Sing_*^{\mathbb{A}^1}(GL_n/GL_{n-1})(L)),
\end{aligned}$$

where the π_i on the right hand side denotes the ordinary i-th homotopy group of a simplicial set.

Remark 3.4. To be more precise, Morel proved that $Sing_*^{\mathbb{A}^1}(GL_n)$ and $Sing_*^{\mathbb{A}^1}(GL_n/GL_{n-1})$ have the affine BG property in the Nisnevich topology for $n \geq 3$. Moser [Mos11] extended this to treat the case $n = 2$ as well. The space $Sing_*^{\mathbb{A}^1}(Gr_{n,\infty})$ has the affine BG property in the Zariski topology, and the statement we use is then a consequence of [Mor12, Theorem A.19].

The $\mathbb{A}^1$-weak equivalence $GL_n/GL_{n-1} \to \mathbb{A}^n \setminus 0$ of Theorem 2.6 says that GL_n/GL_{n-1} is $\mathbb{A}^1$-$(n-2)$-connected, and thus the simplicial set $Sing_*^{\mathbb{A}^1}(GL_n/GL_{n-1})(F)$ is $(n-2)$-connected for an arbitrary field F. Suslin's homomorphism is defined in terms of a different model of the classifying space of GL_n, and to this end, we will replace $Gr_{n,\infty}$ by a different model. We saw above that the canonical morphism $Gr_{n,\infty} \to BGL_{n,\bullet}$ is an $\mathbb{A}^1$-weak equivalence. We now establish a slightly stronger version of this fact.

Lemma 3.5. *The morphism $Sing_*^{\mathbb{A}^1}(Gr_{n,\infty}) \to Sing_*^{\mathbb{A}^1}(BGL_{n,\bullet})$ induced by the classifying morphism $Gr_{n,\infty} \to BGL_{n,\bullet}$ is a simplicial weak equivalence. In particular, $Sing_*^{\mathbb{A}^1}(BGL_{n,\bullet})$ is $\mathbb{A}^1$-local.*

Proof. The space $Gr_{n,\infty}$ is a quotient of $V_{n,\infty}$ by GL_n, where $V_{n,\infty}$ is a colimit of open subschemes of affine spaces. Consider the Čech simplicial object $\check{C}(p)$ obtained from the morphism $p : V_{n,\infty} \to Gr_{n,\infty}$. By [MV99, §2 Lemma 2.14], the morphism $\check{C}(p) \to Gr_{n,\infty}$ is a simplicial weak equivalence. It follows that the map $Sing_*^{\mathbb{A}^1}(\check{C}(p)) \to Sing_*^{\mathbb{A}^1}(Gr_{n,\infty})$ is a simplicial weak equivalence (use [MV99, §2 Lemma 1.8]). Furthermore, the n-th term of this simplicial scheme is the $n+1$-fold fiber product of $V_{n,\infty}$ with itself over $Gr_{n,\infty}$. In this case, the n-th term is isomorphic to $V_{n,\infty} \times GL_n^{\times n}$. In particular, $\check{C}(p)$ can be described as the quotient $(V_{n,\infty} \times EGL_{n,\bullet})/GL_n$, where $EGL_\bullet$ is the Čech simplicial scheme associated with the projection $GL_n \to \operatorname{Spec} k$. Projection onto the factor $EGL_\bullet$ then defines a morphism $\check{C}(p) \to BGL_{n,\bullet}$ and therefore also a morphism $Sing_*^{\mathbb{A}^1}(\check{C}(p)) \to Sing_*^{\mathbb{A}^1}(BGL_{n,\bullet})$

By construction $Sing_*^{\mathbb{A}^1}(\cdot)$ commutes with formation of finite products. The proof of [MV99, §4 Proposition 2.3] shows that the structure morphism $Sing_*^{\mathbb{A}^1}(V_{n,\infty}) \to \operatorname{Spec} k$ is a simplicial weak equivalence. It follows that the induced morphism $Sing_*^{\mathbb{A}^1}(\check{C}(p)) \to Sing_*^{\mathbb{A}^1}(BGL_{n,\bullet})$ is a termwise weak-equivalence of simplicial sheaves and therefore a simplicial weak equivalence by [MV99, §2 Corollary 1.21]. $\square$

As a consequence of this lemma, we have $\boldsymbol{\pi}_i^{\mathbb{A}^1}(Gr_{n,\infty})(L) \cong \pi_i(Sing_*^{\mathbb{A}^1}(BGL_{n,\bullet})(L))$ for any n and any integer $i \geq 0$. Taking sections of the exact sequence in Equation 3.2 over L thus yields an exact sequence of the form

$$\pi_i(Sing_*^{\mathbb{A}^1}(BGL_{n-1,\bullet})(L)) \longrightarrow \pi_i(Sing_*^{\mathbb{A}^1}(BGL_{n,\bullet})(L)) \longrightarrow \pi_{i-1}(Sing_*^{\mathbb{A}^1}(GL_n/GL_{n-1})(L)). \tag{3.3}$$

for any integer $i > 0$.

Because $\mathbb{A}^n \backslash 0$ is $\mathbb{A}^1$-$(n-2)$-connected, and because $Sing_*^{\mathbb{A}^1}(GL_n/GL_{n-1})$ is $\mathbb{A}^1$-local and $\mathbb{A}^1$-weakly equivalent to $\mathbb{A}^n \backslash 0$, it follows that $Sing_*^{\mathbb{A}^1}(GL_n/GL_{n-1})$ is simplicially $(n-2)$-connected. As a consequence, taking sections over any field L, the Hurewicz theorem (for simplicial sets) gives a canonical isomorphism

$$\pi_{n-1}(Sing_*^{\mathbb{A}^1}(GL_n/GL_{n-1})(L)) \xrightarrow{\sim} H_{n-1}(Sing_*^{\mathbb{A}^1}(GL_n/GL_{n-1})(L), \mathbb{Z}). \tag{3.4}$$

The Hurewicz homomorphism is functorial and therefore gives a commutative square of the form

$$\begin{tikzcd}
\pi_i(Sing_*^{\mathbb{A}^1}(BGL_{n-1,\bullet})(L)) \arrow[r] \arrow[d] & \pi_i(Sing_*^{\mathbb{A}^1}(BGL_{n,\bullet})(L)) \arrow[d] \\
H_i(Sing_*^{\mathbb{A}^1}(BGL_{n-1,\bullet})(L), \mathbb{Z}) \arrow[r] & H_i(Sing_*^{\mathbb{A}^1}(BGL_{n,\bullet})(L), \mathbb{Z}).
\end{tikzcd}$$

These two homomorphisms are compatible by the following result.

Lemma 3.6. *If $n \geq 3$, there is a commutative diagram of the form*

$$\begin{tikzcd}
\pi_n(Sing_*^{\mathbb{A}^1}(BGL_{n-1,\bullet})(L)) \arrow[r] \arrow[d] & \pi_n(Sing_*^{\mathbb{A}^1}(BGL_{n,\bullet})(L)) \arrow[r] \arrow[d] & K_n^{\mathrm{MW}}(L) \arrow[d] \\
H_n(Sing_*^{\mathbb{A}^1}(BGL_{n-1,\bullet})(L), \mathbb{Z}) \arrow[r] & H_n(Sing_*^{\mathbb{A}^1}(BGL_{n,\bullet})(L), \mathbb{Z}) \arrow[r] & K_n^{\mathrm{MW}}(L),
\end{tikzcd}$$

where all the vertical morphisms are Hurewicz homomorphisms and the farthest right arrow is the isomorphism of Equation 3.4.

Proof. If $n \geq 3$, because $Sing_*^{\mathbb{A}^1}(GL_n/GL_{n-1})(L)$ is $(n-2)$-connected, the first non-trivial differential for the homological Serre spectral sequence d^n of the fibration $Sing_*^{\mathbb{A}^1}(BGL_{n-1,\bullet}) \to Sing_*^{\mathbb{A}^1}(BGL_{n,\bullet})$ fits into the short exact sequence of the stated form. $\square$

Lemma 3.7. *The augmentation $Sing_*^{\mathbb{A}^1}(\mathcal{X}) \to \mathcal{X}$ induces a commutative diagram of the form*

$$\begin{CD}
H_n(Sing_*^{\mathbb{A}^1}(BGL_{n-1,\bullet})(L),\mathbb{Z}) @>>> H_n(Sing_*^{\mathbb{A}^1}(BGL_{n,\bullet})(L),\mathbb{Z}) @>>> K_n^{\mathrm{MW}}(L)\\
@VVV @VVV @VVV\\
H_n(BGL_{n-1,\bullet}(L),\mathbb{Z}) @>>> H_n(BGL_{n,\bullet}(L),\mathbb{Z}) @>>> K_n^M(L),
\end{CD}$$

where the two vertical arrows on the left are split surjections.

Proof. By Suslin's stabilization theorem 3.3, the homotopy fiber of the map $BGL_{n-1,\bullet}(L) \to BGL_{n,\bullet}(L)$ is homologically $(n-2)$-connected. In particular, the same argument as above using the homological Serre spectral sequence together with the other part of Suslin's theorem identifies the cokernel of the lower left horizontal map with $K_n^M(L)$. That the two vertical arrows on the left are split surjections follows from the splitting of the augmentation given by the natural transformation $Id \to Sing_*^{\mathbb{A}^1}(\cdot)$. □

Finally, we can prove the main result we need.

Lemma 3.8. *For any finitely generated separable extension L/F, the morphism $\mathbf{K}_{n-1}^Q(L) \to \mathbf{K}_{n-1}^M(L)$ from* Lemma 3.2 *factors through Suslin's stabilization morphism 3.1.*

Proof. Consider the following commutative diagram

$$\begin{CD}
\pi_n(Sing_*^{\mathbb{A}^1}(BGL_{n,\bullet})(L)) @>>> \pi_n(Sing_*^{\mathbb{A}^1}(BGL_{\infty,\bullet})(L))\\
@VVV @VVV\\
H_n(Sing_*^{\mathbb{A}^1}(BGL_{n,\bullet})(L),\mathbb{Z}) @>>> H_n(Sing_*^{\mathbb{A}^1}(BGL_{\infty,\bullet})(L),\mathbb{Z})\\
@VVV @VVV\\
H_n(BGL_{n,\bullet}(L),\mathbb{Z}) @>>> H_n(BGL_{\infty,\bullet}(L),\mathbb{Z}),
\end{CD}$$

where the vertical arrows emanating from the top row are Hurewicz homomorphisms, and the vertical arrows incident on the bottom row are split surjections. Furthermore, the lowest horizontal arrow is an isomorphism by Suslin's stabilization theorem.

As we observed at the beginning of this section, the homomorphism $\boldsymbol{\pi}_n^{\mathbb{A}^1}(BGL_{n,\bullet}) \longrightarrow \mathbf{K}_n^Q$ factors through the map $\boldsymbol{\pi}_n^{\mathbb{A}^1}(BGL_{n,\bullet}) \longrightarrow \boldsymbol{\pi}_n^{\mathbb{A}^1}(BGL_{\infty,\bullet})$. The bottom horizontal arrow is the isomorphism from Suslin's stabilization theorem. Finally, the first sequence of composites in Suslin's homomorphism 3.1 is precisely the composite of the two vertical arrows on the right hand side with the inverse to the isomorphism given by the bottom horizontal arrow.

Now, combining Lemmas 3.6 and 3.7 we get a commutative diagram with three rows. We start with an element of $K_n^{\mathrm{MW}}(L)$ lying in the image of the map from $\pi_n(Sing_*^{\mathbb{A}^1}(BGL_{n,\bullet})(L))$. By the discussion of the two previous paragraphs, that element factors through $\mathbf{K}_n^Q(L)$. Since the image factors through to $K_n^M(L)$, commutativity of the diagram described in the first line of this paragraph shows that the morphism in question factors through $H_n(BGL_{n,\bullet},\mathbb{Z})$. Then, by commutativity of the diagram two paragraphs above and the definition of Suslin's homomorphism, it follows that our morphism factors through Equation 3.1. □

By Theorem 2.9, there are canonical isomorphisms $\boldsymbol{\pi}_n^{\mathbb{A}^1}(BGL_{n,\bullet}) \cong \boldsymbol{\pi}_n^{\mathbb{A}^1}(BSL_{n,\bullet}) \cong \boldsymbol{\pi}_{n-1}^{\mathbb{A}^1}(SL_n)$. Using this fact, the next result implies Theorem 5, and taking $n = 3$ the second exact sequence of Theorem 4.

Theorem 3.9. *If F is an infinite perfect field, then for any odd integer $n \geq 3$, there is a short exact sequence of the form*

$$0 \longrightarrow \mathbf{S}_{n+1} \longrightarrow \boldsymbol{\pi}_n^{\mathbb{A}^1}(BGL_n) \longrightarrow \mathbf{K}_n^Q \longrightarrow 0,$$

together with an epimorphism $\mathbf{K}_{n+1}^M/n! \to \mathbf{S}_{n+1}$.

Proof. The assumption that the base field F is perfect stems from our implicit use of Morel's result that a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups is strictly $\mathbb{A}^1$-invariant.

By Lemma 3.2 there is a short exact sequence where $\mathbf{S}_{n+1}$ is the cokernel of a morphism $\mathbf{K}_{n+1}^Q \to \mathbf{K}_{n+1}^M$. By Lemma 3.8, the morphism of the previous homomorphism factors through the Suslin's homomorphism $\mathbf{K}_{n+1}^Q \to \mathbf{K}_{n+1}^M$ of Equation 3.1. By [Sus84, Corollary 4.4], the image of this homomorphism on sections over fields contains $n!\mathbf{K}_{n+1}^M(F)$, which gives the epimorphism. $\square$

Remark 3.10. Suslin's stabilization theorem 3.3 and [Sus84, Corollary 4.4] were extended to local rings with infinite residue field in [NS89], and independently, to all rings of stable rank 1 in [Gui89]. While this is unnecessary for our analysis, these results imply that maps on stalks induced by the homomorphism $\mathbf{K}_{n+1}^Q \to \mathbf{K}_{n+1}^M$ we constructed in Lemma 3.2 are understood.

Comparing fiber sequences via homogeneous spaces

Proposition 3.11. *For any integers $m, n \geq 1$, let $i_{2n} : Sp_{2n} \to SL_{2n}$ be the obvious closed immersion group homomorphism (obtained by picking a symplectic form on an n-dimensional vector space), $j_m : SL_m \to SL_{m+1}$ be defined by $j_m(M) = \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}$ and $l_{2n} : Sp_{2n} \to Sp_{2n+2}$ defined by $l_{2n}(N) = \begin{pmatrix} Id & 0 \\ 0 & N \end{pmatrix}$. The following diagram is cartesian*

$$\begin{tikzcd} Sp_{2n} \arrow[r, "l_{2n}"] \arrow[d, "j_{2n}i_{2n}"'] & Sp_{2n+2} \arrow[d, "i_{2n+2}"] \\ SL_{2n+1} \arrow[r, "j_{2n+1}"'] & SL_{2n+2} \end{tikzcd}$$

and it induces a diagram

$$\begin{tikzcd} Sp_{2n} \arrow[r, "l_{2n}"] \arrow[d, "j_{2n}i_{2n}"'] & Sp_{2n+2} \arrow[r] \arrow[d, "i_{2n+2}"] & Sp_{2n+2}/Sp_{2n} \arrow[d] \\ SL_{2n+1} \arrow[r, "j_{2n+1}"'] \arrow[d] & SL_{2n+2} \arrow[r] \arrow[d] & SL_{2n+2}/SL_{2n+1} \\ SL_{2n+1}/Sp_{2n} \arrow[r] & SL_{2n+2}/Sp_{2n+2} & \end{tikzcd}$$

where the lines and columns are exact sequences of étale (representable) sheaves. Moreover, the induced morphisms $Sp_{2n+2}/Sp_{2n} \to SL_{2n+2}/SL_{2n+1}$ and $SL_{2n+1}/Sp_{2n} \to SL_{2n+2}/Sp_{2n+2}$ are isomorphisms.

Proof. We first check that the square

$$\begin{tikzcd} Sp_{2n} \arrow[r, "l_{2n}"] \arrow[d, "j_{2n}i_{2n}"'] & Sp_{2n+2} \arrow[d, "i_{2n+2}"] \\ SL_{2n+1} \arrow[r, "j_{2n+1}"'] & SL_{2n+2} \end{tikzcd}$$

is cartesian. Let R be a ring, $M \in SL_{2n+1}(R)$ and $N \in Sp_{2n+2}(R)$ such that $j_{2n+1}(M) = i_{2n+2}(N)$, i.e., $\begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} \in Sp_{2n+2}(R)$. Write $M = \begin{pmatrix} a & v \\ w & M' \end{pmatrix}$ with $a \in R$, $v \in M_{1,2n}(R)$, $w \in M_{2n,1}(R)$ and $M' \in M_{2n}(R)$. Expressing the condition for $\begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}$ to be symplectic, we see that $a = 1$, $v, w = 0$ and $M' \in Sp_{2n}(R)$. That the diagram is cartesian follows because all the maps in the diagram are injective. The quotients in the diagram

$$\begin{tikzcd} Sp_{2n} \arrow[r, "l_{2n}"] \arrow[d, "j_{2n}i_{2n}"'] & Sp_{2n+2} \arrow[r] \arrow[d, "i_{2n+2}"] & Sp_{2n+2}/Sp_{2n} \arrow[d] \\ SL_{2n+1} \arrow[r, "j_{2n+1}"'] \arrow[d] & SL_{2n+2} \arrow[r] \arrow[d] & SL_{2n+2}/SL_{2n+1} \\ SL_{2n+1}/Sp_{2n} \arrow[r] & SL_{2n+2}/Sp_{2n+2} & \end{tikzcd}$$

exist by [SGA70, exposé VIA, Théorème 3.2, p311] and they represent the étale sheaves associated with the quotient presheaves.

The isomorphism of the quotients $SL_{2n+2}/SL_{2n+1} \to Sp_{2n+2}/Sp_{2n}$ is classical. Moreover, as a consequence, there is a transitive action of Sp_{2n+2} on SL_{2n+2}/SL_{2n+1}. Equivalently, there is a transitive action of SL_{2n+1} on SL_{2n+2}/Sp_{2n+2} and the computations above identifies the stabilizer of the identity coset with Sp_{2n}; the required isomorphism of homogeneous spaces follows directly from this observation. $\square$

Corollary 3.12. *The rows and columns of the pullback square of* Proposition 3.11 *give rise to commutative diagrams of long exact sequences of homotopy sheaves associated with a fibration.*

Proof. We prove the result for rows; the result for columns is established in a formally identical fashion. Essentially, this is a consequence of properness of the $\mathbb{A}^1$-local model structure. The functor $Sing_*^{\mathbb{A}^1}(\cdot)$ is compatible with the formation of limits [MV99, p. 87]. As a consequence, given any pullback square of schemes, upon application of $Sing_*^{\mathbb{A}^1}(\cdot)$ one obtains a corresponding pullback square of spaces. Applying this observation to Proposition 3.11 we obtain a pullback square of the form

$$\begin{tikzcd} Sing_*^{\mathbb{A}^1}(Sp_{2n-2}) \arrow[r] \arrow[d] & Sing_*^{\mathbb{A}^1}(Sp_{2n}) \arrow[d] \\ Sing_*^{\mathbb{A}^1}(SL_{2n-1}) \arrow[r] & Sing_*^{\mathbb{A}^1}(SL_{2n}). \end{tikzcd}$$

The induced morphisms of spaces $Sing_*^{\mathbb{A}^1}(Sp_{2n}) \to Sing_*^{\mathbb{A}^1}(Sp_{2n}/Sp_{2n-2})$ and $Sing_*^{\mathbb{A}^1}(SL_{2n}) \to Sing_*^{\mathbb{A}^1}(SL_{2n}/SL_{2n+1})$ give rise to a diagram of the form

$$\begin{tikzcd} Sing_*^{\mathbb{A}^1}(Sp_{2n-2}) \arrow[r] \arrow[d] & Sing_*^{\mathbb{A}^1}(Sp_{2n}) \arrow[r] \arrow[d] & Sing_*^{\mathbb{A}^1}(Sp_{2n}/Sp_{2n-2}) \arrow[d] \\ Sing_*^{\mathbb{A}^1}(SL_{2n-1}) \arrow[r] & Sing_*^{\mathbb{A}^1}(SL_{2n}) \arrow[r] & Sing_*^{\mathbb{A}^1}(SL_{2n}/SL_{2n-1}). \end{tikzcd}$$

where the right vertical map is the identity. It suffices to show that this commutative diagram of spaces is a morphism of simplicial fiber sequences by [Wen11, Proposition 5.1] and [Wen11, Theorem 5.3].

The action of Sp_{2n-2} on Sp_{2n} is compatible with the action of SL_{2n-1} on SL_{2n} by the commutativity of the diagram in Proposition 3.11. We can assume all spaces in question are simplicially fibrant since the fibrant replacement functor commutes with formation of limits as well. Then, the morphism $Sing_*^{\mathbb{A}^1}(Sp_{2n}) \to Sing_*^{\mathbb{A}^1}(Sp_{2n}/Sp_{2n-2})$ is a simplicial $Sing_*^{\mathbb{A}^1}(Sp_{2n-2})$-torsor and is classified by a morphism $Sing_*^{\mathbb{A}^1}(Sp_{2n}/Sp_{2n-2}) \to BSing_*^{\mathbb{A}^1}(Sp_{2n-2})$, and likewise, there is a classifying morphism $Sing_*^{\mathbb{A}^1}(SL_{2n}/SL_{2n-1}) \to BSing_*^{\mathbb{A}^1}(SL_{2n-1})$. Furthermore, the morphism $Sp_{2n-2} \hookrightarrow SL_{2n-1}$ induces a morphism $BSing_*^{\mathbb{A}^1}(Sp_{2n-2}) \to BSing_*^{\mathbb{A}^1}(SL_{2n-1})$.

Now, to show that we have a morphism of fiber sequences, it suffices to observe that the action of the $\mathbf{R}\Omega_s^1 BSing_*^{\mathbb{A}^1}(Sp_{2n-2})$ (resp. $\mathbf{R}\Omega_s^1 BSing_*^{\mathbb{A}^1}(SL_{2n-1})$) on $Sing_*^{\mathbb{A}^1}(Sp_{2n})$ (resp. $Sing_*^{\mathbb{A}^1}(SL_{2n})$) is precisely given by the induced action of $Sing_*^{\mathbb{A}^1}(Sp_{2n-2})$ on $Sing_*^{\mathbb{A}^1}(Sp_{2n})$ (resp. the action of $Sing_*^{\mathbb{A}^1}(SL_{2n-1})$ on $Sing_*^{\mathbb{A}^1}(SL_{2n})$). □

Remark 3.13. When $n = 2$, one can refine Proposition 3.11 using the isomorphism $SL_2 \cong Sp_2$. In that case one also knows that $SL_4/Sp_4 \cong SL_3/SL_2$ is a 5-dimensional smooth affine quadric that is $\mathbb{A}^1$-1-connected.

The $\mathbb{A}^1$-homotopy type of SL_{2n}/Sp_{2n}

The inclusion morphism $Sp_{2n} \to GL_{2n}$ induces a morphism $BSp_{2n} \to BGL_{2n}$. Stabilizing this morphism with respect to n and taking $\mathbb{A}^1$-homotopy sheaves produces a morphism $\mathbf{K}_i^{Sp} \to \mathbf{K}_i^Q$. This morphism, which will be studied in greater detail at the beginning of Section 4, is precisely the map induced by "forgetting" the symplectic structure. In the stable range, our understanding of this homomorphism can be translated into understanding of the $\mathbb{A}^1$-homotopy theory of SL_{2n}/Sp_{2n}. Indeed, there is an $\mathbb{A}^1$-fiber sequence of the form

$$SL_{2n}/Sp_{2n} \longrightarrow BSp_{2n} \longrightarrow BSL_{2n}.$$

We now construct a stabilization morphism that will allow us to compare the spaces SL_{2n}/Sp_{2n} for different values of n.

Lemma 3.14. *There is a pullback diagram of the form*

$$\begin{tikzcd} Sp_{2n-2} \arrow[r] \arrow[d] & SL_{2n-2} \arrow[d] \\ Sp_{2n} \arrow[r] & SL_{2n}. \end{tikzcd}$$

Next, consider the sequence of inclusions $Sp_{2n-2} \hookrightarrow SL_{2n-2} \hookrightarrow SL_{2n}$. These inclusions induce morphisms of homogeneous spaces

$$SL_{2n-2}/Sp_{2n-2} \hookrightarrow SL_{2n}/Sp_{2n-2} \longrightarrow SL_{2n}/SL_{2n-2},$$

where the second morphism is Zariski locally trivial with fibers SL_{2n-2}/Sp_{2n-2}. Indeed, the second morphism is the projection onto the first factor $SL_{2n} \times^{SL_{2n-2}} SL_{2n-2}/Sp_{2n-2} \to SL_{2n}/SL_{2n-2}$.

Next, consider the sequence of inclusions $Sp_{2n-2} \hookrightarrow Sp_{2n} \hookrightarrow SL_{2n}$. These inclusions induce morphisms of homogeneous spaces

$$Sp_{2n}/Sp_{2n-2} \hookrightarrow SL_{2n}/Sp_{2n-2} \longrightarrow SL_{2n}/Sp_{2n}.$$

Again, the second morphism is a Zariski locally trivial smooth morphism with fibers isomorphic to Sp_{2n}/Sp_{2n-2}. In this case, the second morphism is the projection onto the first factor $SL_{2n} \times^{Sp_{2n}} Sp_{2n}/Sp_{2n-2} \to SL_{2n}/Sp_{2n}$.

Composing the inclusion in the first sequence with the projection in the second sequence we obtain a morphism

$$SL_{2n-2}/Sp_{2n-2} \longrightarrow SL_{2n}/Sp_{2n}, \tag{3.5}$$

and composing the inclusion in the second sequence with the projection in the first sequence we obtain a morphism

$$Sp_{2n}/Sp_{2n-2} \longrightarrow SL_{2n}/SL_{2n-2}$$

Furthermore, in both sequences above, as the associated spaces of an SL_{2n}-torsor (resp. Sp_{2n}-torsor), these sequences are again $\mathbb{A}^1$-fiber sequences. Using Lemma 3.14 as before, there are morphisms between the resulting $\mathbb{A}^1$-fiber sequences as well; we summarize this in the next result.

Corollary 3.15. *The pullback diagram of* Lemma 3.14 *induces a morphism of fiber sequences*

$$\begin{tikzcd} Sp_{2n-2} \arrow[r] \arrow[d] & SL_{2n-2} \arrow[r] \arrow[d] & SL_{2n-2}/Sp_{2n-2} \arrow[d] \\ Sp_{2n} \arrow[r] & SL_{2n} \arrow[r] & SL_{2n}/Sp_{2n} \end{tikzcd}$$

where the right vertical morphism is that of 3.5.

From the discussion above, we have a morphism $Sp_{2n}/Sp_{2n-2} \to SL_{2n}/SL_{2n-2}$. Composing with the projection $SL_{2n}/SL_{2n-2} \to SL_{2n}/SL_{2n-1}$, we get the isomorphism $Sp_{2n}/Sp_{2n-2} \to SL_{2n}/SL_{2n-1}$ of the previous section. We summarize these two observations in the following result.

Lemma 3.16. *The morphism $Sp_{2n}/Sp_{2n-2} \to SL_{2n}/SL_{2n-2}$ admits a retraction, and the projection morphism $SL_{2n}/SL_{2n-2} \to SL_{2n}/SL_{2n-1}$ splits.*

There is one further relationship between the quotients SL_{2n}/Sp_{2n} and SL_{2n-2}/Sp_{2n-2}. By the isomorphism we wrote down before, we can identify $SL_{2n}/Sp_{2n} \cong SL_{2n-1}/Sp_{2n-2}$. There is then a sequence of the form

$$SL_{2n-2}/Sp_{2n-2} \hookrightarrow SL_{2n-1} \times^{SL_{2n-2}} SL_{2n-2}/Sp_{2n-2} \longrightarrow SL_{2n-1}/SL_{2n-2},$$

where the middle term is isomorphic to SL_{2n-1}/Sp_{2n-2} and the second morphism is Zariski locally trivial with fibers isomorphic to SL_{2n-2}/Sp_{2n-2}. Again, this is an associated space of an SL_{2n-2}-torsor and so gives rise to an $\mathbb{A}^1$-fiber sequence. Note that, in this case, SL_{2n-1}/SL_{2n-2} is $\mathbb{A}^1$-$(2n-3)$-connected.

A short exact sequence describing $\boldsymbol{\pi}_2^{\mathbb{A}^1}(SL_2)$

The fibre sequence

$$Sp_2 \longrightarrow Sp_4 \longrightarrow Sp_4/Sp_2$$

and the $\mathbb{A}^1$-equivalence $Sp_4/Sp_2 \simeq \mathbb{A}^4 \setminus 0$ induce an exact sequence

$$\boldsymbol{\pi}_3^{\mathbb{A}^1}(Sp_4) \longrightarrow \mathbf{K}_4^{MW} \longrightarrow \boldsymbol{\pi}_2^{\mathbb{A}^1}(SL_2) \longrightarrow \boldsymbol{\pi}_2^{\mathbb{A}^1}(Sp_4) \longrightarrow 0.$$

Both sheaves $\boldsymbol{\pi}_3^{\mathbb{A}^1}(Sp_4)$ and $\boldsymbol{\pi}_2^{\mathbb{A}^1}(Sp_4)$ are in the stable range and so we know by Theorem 2.10 that $\boldsymbol{\pi}_2^{\mathbb{A}^1}(Sp_4) = \mathbf{K}_3^{Sp}$ and $\boldsymbol{\pi}_3^{\mathbb{A}^1}(Sp_4) = \mathbf{K}_4^{Sp}$. Thus the above sequence reads as

$$\mathbf{K}_4^{Sp} \longrightarrow \mathbf{K}_4^{MW} \longrightarrow \boldsymbol{\pi}_2^{\mathbb{A}^1}(SL_2) \longrightarrow \mathbf{K}_3^{Sp} \longrightarrow 0.$$

If we write

$$\mathbf{S}_4'' := \operatorname{coker}(\mathbf{K}_4^{Sp} \longrightarrow \mathbf{K}_4^{MW}),$$

where the map on the right is from the exact sequence just above, and we identify $SL_2 \cong Sp_2$, then in order to understand $\boldsymbol{\pi}_2^{\mathbb{A}^1}(SL_2)$, it remains to describe $\mathbf{S}_4''$.

To this end, let

$$\varphi_4' : \mathbf{K}_4^{Sp} \longrightarrow \mathbf{K}_4^{M}$$

be defined as the composite of the forgetful morphism $\mathbf{K}_4^{Sp} \to \mathbf{K}_4^{Q}$ and the morphism $\varphi_4 : \mathbf{K}_4^{Q} \to \mathbf{K}_4^{M}$ of Lemma 3.8. We write $\mathbf{S}_4'$ for the cokernel of φ_4'. The next result is a refinement of [Wen11, Proposition 6.11].

Lemma 3.17. *The composite $\mathbf{I}^5 \to \mathbf{K}_4^{MW} \to \mathbf{S}_4''$ yields an exact sequence*

$$\mathbf{I}^5 \longrightarrow \mathbf{S}_4'' \longrightarrow \mathbf{S}_4' \longrightarrow 0.$$

Proof. Consider the cartesian square

$$\begin{tikzcd} Sp_4 \arrow[r] \arrow[d] & SL_4 \arrow[d] \\ Sp_6 \arrow[r] & SL_6 \end{tikzcd}$$

of Lemma 3.14. The vertical quotients are respectively Sp_6/Sp_4 and SL_6/SL_4. We know that Sp_6/Sp_4 has the $\mathbb{A}^1$-homotopy type of $\mathbb{A}^6 \setminus 0$, which is $\mathbb{A}^1$-4-connected again by Theorem 2.6.

The columns of the cartesian square in the previous paragraph give rise to a morphism of $\mathbb{A}^1$-fiber sequences by Corollary 3.15 and a corresponding morphism of associated long exact sequences in $\mathbb{A}^1$-homotopy sheaves. By the connectivity results just mentioned, the portion of these long exact sequences regarding degree 3 homotopy sheaves yields a commutative diagram with exact columns of the form:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \boldsymbol{\pi}_4^{\mathbb{A}^1}(SL_6/SL_4) \arrow[d] \\ \boldsymbol{\pi}_3^{\mathbb{A}^1}(Sp_4) \arrow[r] \arrow[d] & \boldsymbol{\pi}_3^{\mathbb{A}^1}(SL_4) \arrow[d] \\ \boldsymbol{\pi}_3^{\mathbb{A}^1}(Sp_6) \arrow[r] \arrow[d] & \boldsymbol{\pi}_3^{\mathbb{A}^1}(SL_6) \arrow[d] \\ 0 & 0. \end{tikzcd}$$

The space SL_6/SL_4 also fits into an $\mathbb{A}^1$-fiber sequence of the form

$$SL_5/SL_4 \longrightarrow SL_6/SL_4 \longrightarrow SL_6/SL_5,$$

where the fiber has the $\mathbb{A}^1$-homotopy type of $\mathbb{A}^5 \setminus 0$ and the base has the $\mathbb{A}^1$-homotopy type of $\mathbb{A}^6 \setminus 0$. Moreover, by Lemma 3.16 the morphism $SL_6/SL_4 \to SL_6/SL_5$ is split by the morphism $Sp_6/Sp_4 \to SL_6/SL_4$ and therefore the corresponding long exact sequence in $\mathbb{A}^1$-homotopy sheaves is also split; this splitting induces an isomorphism $\mathbf{K}_5^{\mathrm{MW}} \xrightarrow{\sim} \boldsymbol{\pi}_4^{\mathbb{A}^1}(SL_6/SL_4)$.

The connecting homomorphism in the $\mathbb{A}^1$-fiber sequence $SL_4 \to SL_6 \to SL_6/SL_4$ is induced by the classifying map $SL_6/SL_4 \to BSL_4$, while the connecting homomorphism in the $\mathbb{A}^1$-fiber sequence $SL_4 \to SL_5 \to SL_5/SL_4$ is induced by the classifying map $SL_5/SL_4 \to BSL_4$. These two homomorphisms are compatible by means of the inclusion $SL_4 \hookrightarrow SL_5 \hookrightarrow SL_6$, and it follows that the composition $\mathbf{K}_5^{\mathrm{MW}} \to \boldsymbol{\pi}_4^{\mathbb{A}^1}(SL_6/SL_4) \to \boldsymbol{\pi}_3^{\mathbb{A}^1}(SL_4)$ is induced by the connecting homomorphism in the fibre sequence

$$SL_4 \longrightarrow SL_5 \longrightarrow SL_5/SL_4.$$

There are stabilization isomorphisms $\boldsymbol{\pi}_3^{\mathbb{A}^1}(SL_5) \to \boldsymbol{\pi}_3^{\mathbb{A}^1}(SL_6) = \mathbf{K}_4^Q$ and $\boldsymbol{\pi}_3^{\mathbb{A}^1}(Sp_4) = \boldsymbol{\pi}_3^{\mathbb{A}^1}(Sp_6) =$

$\mathbf{K}_4^{Sp}$. We then obtain a commutative diagram with exact columns of the form:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbf{K}_5^{\mathrm{MW}} \arrow[r, equal] \arrow[d] & \mathbf{K}_5^{\mathrm{MW}} \arrow[d, "\eta"] \\
\boldsymbol{\pi}_3^{\mathbb{A}^1}(Sp_4) \arrow[r] \arrow[d] & \boldsymbol{\pi}_3^{\mathbb{A}^1}(SL_4) \arrow[r] \arrow[d] & \mathbf{K}_4^{\mathrm{MW}} \arrow[d] \\
\mathbf{K}_4^{Sp} \arrow[r] \arrow[d] & \mathbf{K}_4^{Q} \arrow[r] \arrow[d] & \mathbf{K}_4^{M} \arrow[d] \\
0 & 0 & 0.
\end{tikzcd}$$

The morphism $\mathbf{K}_4^Q \to \mathbf{K}_4^M$ in the bottom row agrees with Suslin's homomorphism upon taking sections over finitely generated extensions of the base field by Lemma 3.8 (while the quoted lemma appears to implicitly assume that n is odd, that assumption was only necessary in order to define the homomorphism; in the situation under consideration we observe explicitly a factorization through η and the same proof then applies here as well). Similarly, we already observed that the forgetful homomorphism is induced by the morphism $BSp_\infty \hookrightarrow BGL_\infty$ coming from the inclusion $Sp_{2n} \hookrightarrow GL_{2n}$ and therefore, since the morphism $\mathbf{K}_4^{Sp} \to \mathbf{K}_4^Q$ in the bottom row is induced by stabilization, it follows that this morphism is precisely the forgetful homomorphism.

By performing a diagram chase, we observe that $\mathbf{S}_4''$ is an extension of $\mathbf{S}_4'$ by the image of $\eta : \mathbf{K}_5^{MW} \to \mathbf{K}_4^{MW}$. To finish, observe that it follows from the description of [Mor04, Theorem 5.3] that the image of $\eta : \mathbf{K}_5^{MW} \to \mathbf{K}_4^{MW}$ is precisely $\mathbf{I}^5$. $\square$

Remark 3.18. Our guess is that the morphism $\mathbf{I}^5 \to \mathbf{S}_4''$ is injective.

Lemma 3.19. *If the base field k is assumed to have characteristic unequal to 2, there is a surjective morphism $\mathbf{K}_4^M/12 \to \mathbf{S}_4'$.*

Proof. In Section 4 we write $f_{4,2} : \mathbf{K}_4^{Sp} \to \mathbf{K}_4^Q$ for the forgetful homomorphism. With this notation, by definition, we have an exact sequence of sheaves

$$\mathbf{K}_4^{Sp} \xrightarrow{\varphi_4 \circ f_{4,2}} \mathbf{K}_4^M \longrightarrow \mathbf{S}_4' \longrightarrow 0.$$

Now, for any field F, there is a natural homomorphism of groups $K_4^M(F) \to K_4^Q(F)$ induced by the isomorphism $K_1^M(F) \xrightarrow{\sim} K_1^Q(F)$ and the ring structures. This yields a morphism of sheaves $\xi_4 : \mathbf{K}_4^M \to \mathbf{K}_4^Q$. In Section 4, we will also introduce the hyperbolic morphism $H_{4,2} : \mathbf{K}_4^Q \to \mathbf{K}_4^{Sp}$. To prove the lemma, it suffices to show that there is an epimorphism of $\mathbf{K}_4^M/12$ to the cokernel of the composition

$$\varphi_4 \circ f_{4,2} \circ H_{4,2} \circ \xi_4 : \mathbf{K}_4^M \to \mathbf{K}_4^M.$$

In view of [Sus84, Corollary 4.4], this follows from the commutative diagram

$$\begin{tikzcd}
\mathbf{K}_4^Q \arrow[r, "f_{4,2}\circ H_{4,2}"] & \mathbf{K}_4^Q \\
\mathbf{K}_4^M \arrow[u, "\xi_4"] \arrow[r, "\cdot 2"'] & \mathbf{K}_4^M \arrow[u, "\xi_4"']
\end{tikzcd}$$

induced by Lemma 4.3. □

The next result, which gives the second part of Theorem 4, follows immediately by combining Theorem 2.9, and Lemmas 3.17 and 3.19.

Theorem 3.20. *If the base field k is assumed to be infinite perfect and to have characteristic unequal to 2, then there are short exact sequences of the form*

$$0 \longrightarrow \mathbf{S}_4'' \longrightarrow \boldsymbol{\pi}_2^{\mathbb{A}^1}(GL_2) \longrightarrow \mathbf{K}_3^{Sp} \longrightarrow 0$$

and

$$\mathbf{I}^5 \longrightarrow \mathbf{S}_4'' \longrightarrow \mathbf{S}_4' \longrightarrow 0,$$

where $\mathbf{S}_4'$ is a quotient of $\mathbf{K}_4^M/12$.

Remark 3.21. The stabilization sequence for Sp_{2n} gives the exact sequence

$$\begin{aligned}\boldsymbol{\pi}_{2n}^{\mathbb{A}^1}(BSp_{2n-2}) \longrightarrow \boldsymbol{\pi}_{2n}^{\mathbb{A}^1}(BSp_{2n}) \longrightarrow \boldsymbol{\pi}_{2n-1}^{\mathbb{A}^1}(Sp_{2n}/Sp_{2n-2}) \\ \longrightarrow \boldsymbol{\pi}_{2n-1}^{\mathbb{A}^1}(BSp_{2n-2}) \longrightarrow \boldsymbol{\pi}_{2n-1}^{\mathbb{A}^1}(BSp_{2n}) \longrightarrow 0.\end{aligned}$$

We know $\boldsymbol{\pi}_{2n-1}^{\mathbb{A}^1}(Sp_{2n}/Sp_{2n-2}) \cong \mathbf{K}_{2n}^{\mathrm{MW}}$ and $\boldsymbol{\pi}_{2n}^{\mathbb{A}^1}(BSp_{2n}) \cong \mathbf{K}_{2n}^{Sp}$. By the compatibility of fiber sequences of Corollary 3.15, we know that the morphism $\boldsymbol{\pi}_{2n}^{\mathbb{A}^1}(BSp_{2n}) \to \boldsymbol{\pi}_{2n-1}^{\mathbb{A}^1}(Sp_{2n}/Sp_{2n-2})$ fits into a commutative diagram of the form

$$\begin{tikzcd}
\boldsymbol{\pi}_{2n}^{\mathbb{A}^1}(BSp_{2n-2}) \arrow[r] \arrow[d] & \boldsymbol{\pi}_{2n}^{\mathbb{A}^1}(BSp_{2n}) \arrow[r] \arrow[d] & \boldsymbol{\pi}_{2n-1}^{\mathbb{A}^1}(Sp_{2n}/Sp_{2n-2}) \arrow[d] \\
\boldsymbol{\pi}_{2n}^{\mathbb{A}^1}(BSL_{2n-1}) \arrow[r] & \boldsymbol{\pi}_{2n}^{\mathbb{A}^1}(BSL_{2n}) \arrow[r] & \boldsymbol{\pi}_{2n-1}^{\mathbb{A}^1}(SL_{2n}/SL_{2n-1})
\end{tikzcd}$$

Results of Hutchinson-Tao [HT10, Theorem 1.1] give stabilization results for the homology of the special linear group which are very similar to Suslin's stabilization theorem 3.3. In addition to only being proven for fields having characteristic 0, one main difference between these results and Suslin's involves the appearance of K_n^{MW}. In particular, Hutchinson and Tao construct an exact sequence of the form

$$H_{2n}(BSL_{2n-1,\bullet}(L), \mathbb{Z}) \longrightarrow H_{2n}(BSL_{2n,\bullet}(L), \mathbb{Z}) \longrightarrow K_n^{\mathrm{MW}}(L) \longrightarrow 0$$

One can then construct a homomorphism $K_{2n}^{Sp}(L) \to K_{2n}^{\mathrm{MW}}(L)$ as follows. Consider the composite

$$
\begin{aligned}
K_{2n}^{Sp}(L) := \pi_{2n}(BSp_\infty^+) &\longrightarrow H_{2n}(BSp_\infty, \mathbb{Z}) \longrightarrow H_{2n}(BSL_\infty, \mathbb{Z}) \\
&\longrightarrow H_{2n}(BSL_{2n}, \mathbb{Z}) \longrightarrow K_{2n}^{\mathrm{MW}}(L)
\end{aligned}
\tag{3.6}
$$

An argument similar to that of Lemma 3.8, will then show that the induced homomorphism $\mathbf{K}_{2n}^{Sp} \to \mathbf{K}_{2n}^{\mathrm{MW}}$ factors through a Hurewicz style homomorphism just like Suslin's [Sus84, §4]. Unfortunately, we do not know a result analogous to Suslin's result [Sus84, Corollary 4.4] regarding the image of this map.

4 Grothendieck-Witt groups

In this section, we begin by recalling some basic facts about Grothendieck-Witt groups. These are a Waldhausen-style version of hermitian K-theory. The general reference here is the work of M. Schlichting ([Sch10a], [Sch10b]). The main goal is to prove Theorem 4.11, which will give a description of the third cohomology of the sheaf $\mathbf{K}_3^{Sp}$ (which appears as one term in the exact sequence of Theorem 3.20).

Definitions

Let X be a smooth scheme with $2 \in \mathcal{O}_X(X)^\times$ (we keep these assumptions throughout the section, though it is not necessary for some of the arguments). Let $\mathcal{P}(X)$ be the category of coherent locally free $\mathcal{O}_X$-modules and $Ch^b(X)$ be the category of bounded complexes of objects in $\mathcal{P}(X)$. It carries the structure of an exact category, by saying that an exact sequence of complexes is exact if it is exact in $\mathcal{P}(X)$ degreewise.

For any line bundle on $\mathcal{L}$ on X, the duality $\mathrm{Hom}_{\mathcal{O}_X}(\_, \mathcal{L})$ on $\mathcal{P}(X)$ induces a duality $\sharp_{\mathcal{L}}$ on $Ch^b(X)$ and the canonical identification of a coherent locally free module with its double dual gives a natural isomorphism of functors $\varpi_{\mathcal{L}} : 1 \to \sharp_{\mathcal{L}}\sharp_{\mathcal{L}}$. One can also define a weak-equivalence in $Ch^b(X)$ to be a quasi-isomorphism of complexes. This shows that $(Ch^b(X), qis, \sharp_{\mathcal{L}}, \varpi_{\mathcal{L}})$ is an *exact category with weak-equivalences and (strong) duality* in the sense of [Sch10b, §2.3] (see also *loc. cit.*, §6.1]). The (left) translation functor $T : Ch^b(R) \to Ch^b(R)$ yields new dualities $\sharp_{\mathcal{L}}^n := T^n \circ \sharp_{\mathcal{L}}$ and canonical isomorphisms $\varpi_{\mathcal{L}}^n := (-1)^{n(n+1)/2}\varpi_{\mathcal{L}}$.

To any exact category with weak-equivalences and duality, Schlichting associates a space $\mathcal{GW}$ and defines the (higher) Grothendieck-Witt groups to be the homotopy groups of that space [Sch10b, §2.11]. More precisely:

Definition 4.1. For $i \geq 0$, we denote by $GW_i^j(X, \mathcal{L})$ the group $\pi_i\mathcal{GW}(Ch^b(X), qis, \sharp_{\mathcal{L}}^j, \varpi_{\mathcal{L}}^j)$. If $\mathcal{L} = \mathcal{O}_X$, we write $GW_i^j(X)$ for $GW_i^j(X, \mathcal{O}_X)$.

One can extend further the definition of Grothendieck-Witt groups by considering a spectrum $\mathbb{G}W(Ch^b(X), qis, \sharp_{\mathcal{L}}^j, \varpi_{\mathcal{L}}^j)$ [Sch10b, §10]. The negative Grothendieck-Witt groups are defined as $GW_{-i}^j(X, \mathcal{L}) := \pi_{-i}\mathbb{G}W(Ch^b(X), qis, \sharp_{\mathcal{L}}^j, \varpi_{\mathcal{L}}^j)$ for $i > 0$.

For any $j \in \mathbb{Z}$, the group $GW^j_0(X, \mathcal{L})$ coincides with the Grothendieck-Witt group defined by Balmer-Walter of the triangulated category $D^b(\mathcal{P}(X))$ of bounded complexes of coherent locally free $\mathcal{O}_X$-modules endowed with the corresponding duality ([Sch11, Lemma 8.2], [Wal03, Theorem 5.1]), and negative Grothendieck-Witt groups coincide with triangular Witt groups as defined by P. Balmer (see, e.g., [Bal05]) under the formula $GW^j_{-i}(X, \mathcal{L}) = W^{i+j}(X, \mathcal{L})$.

The Grothendieck-Witt groups defined above coincide with hermitian K-theory as defined by M. Karoubi ([Kar73], [Kar80]) in the case of affine schemes, at least when 2 is invertible (see [Sch10a, Remark 4.16], see also [Hor02]). In particular, given a smooth k-algebra R we have the identifications

$$
\begin{aligned}
GW^0_i(R) &= K_i O(R) \\
GW^2_i(R) &= K_i Sp(R).
\end{aligned}
$$

There are identifications $GW^1_i(R) = {}_{-1}U_i(R)$ and $GW^3_i(R) = U_i(R)$, where the groups $U_i(R)$ and ${}_{-1}U_i(R)$ are Karoubi's U groups, and GW^n_i is 4-periodic in n. Comparing [PW10, Theorem 8.2] with the above definitions yields a description of the sheaves $\mathbf{K}^{Sp}_i$ from the previous section. We summarize this observation in the following result.

Proposition 4.2. *The sheaf $\mathbf{K}^{Sp}_i$ is the Nisnevich sheafification of the functor on $\mathcal{S}m_k$ defined by $X \mapsto GW^2_i(X)$.*

Functoriality

Let k be a field having characteristic unequal to 2, and suppose X is a smooth k-scheme. By definition, these groups are contravariantly functorial in the input space, i.e., given a morphism $f : X \to Y$ of smooth schemes and a line bundle $\mathcal{L}$ on Y, there are pullback homomorphisms

$$
f^* : GW^j_i(Y, \mathcal{L}) \longrightarrow GW^j_i(X, f^*\mathcal{L}).
$$

These pullback morphisms satisfy a number of the "usual" properties, which we now discuss.

If $i : U \to X$ is an open immersion with closed complement $Z := X \setminus U$, then one defines the Grothendieck-Witt groups with support on Z using the exact category $CH^b(X)_Z$ of complexes supported on Z. In this setup, there is an associated long exact localization sequence:

$$
\ldots \longrightarrow GW^j_{i,Z}(X, \mathcal{L}) \longrightarrow GW^j_i(X, \mathcal{L}) \longrightarrow GW^j_i(U, \mathcal{L}) \longrightarrow GW^j_{i-1,Z}(X, \mathcal{L}) \longrightarrow \ldots .
$$

Note, however, that in general there is no "dévissage" isomorphism comparing the Grothendieck-Witt theory of Z with the theory supported on Z. We will return to this issue when we discuss transfers.

The higher Grothendieck-Witt groups also are $\mathbb{A}^1$-homotopy invariant. More precisely, given a vector bundle $p : E \to X$ (or, more generally, a Nisnevich locally trivial morphism with affine space fibers), the induced morphism p^* is an isomorphism.

One can compare Quillen K-theory with higher Grothendieck-Witt groups with the hyperbolic morphisms $H_{i,j} : K_i(X) \to GW^j_i(X, \mathcal{L})$ and forgetful morphisms $f_{i,j} : GW^j_i(X, \mathcal{L}) \to K_i(X)$

defined for any $i, j \in \mathbb{N}$ and any line bundle $\mathcal{L}$ over X. The hyperbolic and forgetful morphisms are connected by means of the *Karoubi periodicity* exact sequences

$$\ldots \longrightarrow K_i(X) \xrightarrow{H_{i,j}} GW_i^j(X,\mathcal{L}) \xrightarrow{\eta_{i,j}} GW_{i-1}^{j-1}(X,\mathcal{L}) \xrightarrow{f_{i-1,j-1}} K_{i-1}(X) \xrightarrow{H_{i-1,j}} GW_{i-1}^j(X,\mathcal{L}) \longrightarrow \ldots,$$

where $\eta_{i,j}$ are certain connecting homomorphisms.

The composition $f_{i,j} \circ H_{i,j}$ is in general difficult to understand, but the situation is slightly better when X is taken to be a field. For any field F, the identification $\xi_1 : K_1^M(F) \to K_1^Q(F)$, induces a (functorial in F) homomorphism $\xi_i : K_i^M(F) \to K_i^Q(F)$ using the ring structures on both sides.

Lemma 4.3. *For any field F having characteristic unequal 2, and for any integers $i, j \geq 0$, the following diagram commutes:*

$$\begin{tikzcd} K_i^Q(F) \arrow[r, "f_{i,j}\circ H_{i,j}"] & K_i^Q(F) \\ K_i^M(F) \arrow[u, "\xi_i"] \arrow[r, "(1+(-1)^{i+j})Id"'] & K_i^M(F). \arrow[u, "\xi_i"'] \end{tikzcd}$$

Proof. Let $(\mathcal{E}, \omega, \sharp, \eta)$ be an exact category with weak-equivalences and duality in the sense of [Sch10b, §2.3]. With any exact category with weak-equivalences, one can associate the hyperbolic category $(\mathcal{HE}, \omega)$ ([Sch10b, §2.15]). Its objects are pairs (X, Y) of objects of $\mathcal{C}$, a morphism $(X, Y) \to (X', Y')$ is a pair (a, b) of morphisms of $\mathcal{C}$ with $a : X \to X'$ and $b : Y' \to Y$. Such a morphism is a weak-equivalence if a and b are. The switch $(X, Y) \mapsto (Y, X)$ yields a duality * on $\mathcal{HE}$ and there is an obvious identification $id : 1 \to^{\star\star}$. Thus $(\mathcal{HE}, \omega, ^\star, id)$ is an exact category with weak-equivalences and duality. The Grothendieck-Witt space $\mathcal{GW}(\mathcal{HE}, \omega, ^\star, id)$ is naturally homotopic to the K-theory space $\mathcal{K}(\mathcal{E}, \omega)$ [Sch10b, Proposition 2.17]. In this context, the forgetful functor F reads as $F(X) = (X, X^\sharp)$ for any X in $\mathcal{E}$. On the other hand the hyperbolic functor $H : \mathcal{HE} \to \mathcal{E}$ is defined by $H(X, Y) = X \oplus Y^\sharp$ ([Sch10a, §3.9]). The composition $FH : (\mathcal{HE}, \omega, ^\star, id) \to (\mathcal{HE}, \omega, ^\star, id)$ is then given by $FH(X, Y) = (X \oplus Y^\sharp, X^\sharp \oplus Y^{\sharp\sharp})$. In particular, this composition is the same for $(\mathcal{E}, \omega, \sharp, \eta)$ and $(\mathcal{E}, \omega, \sharp, -\eta)$.

Consider now $(Ch^b(F), qis, \sharp^j, \varpi^j)$. We have an involution on $GL(F)$ defined by $G \mapsto (G^t)^{-1}$. This involution induces a ring homomorphism $\tau : K_i(F) \to K_i(F)$ (which is the identity on K_0). Using [Sch10b, Proposition 8.4] and our description of FH, we see that the compositions

$$K_i(F) \xrightarrow{H_{i,j}} GW_i^j(F) \xrightarrow{f_{i,j}} K_i(F)$$

are equal to $1 + (-1)^j\tau$. Now τ corresponds to Id on $K_0(F)$ and multiplication by -1 on $K_1(F)$. □

The Gersten-Grothendieck-Witt spectral sequences

In analogy with Quillen K-theory, one can define a coniveau spectral sequence on Grothendieck-Witt groups [FS09], but the situation is a bit more complicated since one has to take into account the relevant dualities. Nevertheless, there exists for any $n \in \mathbb{N}$ and any line bundle $\mathcal{L}$ over X a spectral sequence $E(n,\mathcal{L})^{p,q}$ converging to $GW^n_{n-p-q}(X,\mathcal{L})$ whose terms at page 2 are

$$E(n,\mathcal{L})^{p,q}_2 := \bigoplus_{x_p \in X^{(p)}} GW^{n-p}_{n-p-q}(k(x_p), \omega^{\mathcal{L}}_{x_p}).$$

Here, $\omega^{\mathcal{L}}_{x_p}$ denotes the duality on $Ch^b(k(x_p))$ defined on $k(x_p)$-vector spaces by

$$\omega^{\mathcal{L}}_{x_p}(V) = \mathrm{Hom}_{k(x_p)}(V, \mathrm{Ext}^p_{\mathcal{O}_{X,x_p}}(k(x_p), \mathcal{L} \otimes \mathcal{O}_{X,x_p})).$$

Since the Gersten conjecture holds for Grothendieck-Witt groups [FS09, Remark 27], any line $q \in \mathbb{Z}$ at page 2 gives a flasque resolution of the sheaf associated with the presheaf

$$U \mapsto \ker(GW^n_{n-q}(k(U), \omega^{\mathcal{L}}_{x_0}) \longrightarrow \bigoplus_{x_1 \in U^{(1)}} GW^{n-1}_{n-1-q}(k(x_1), \omega^{\mathcal{L}}_{x_1}));$$

one typically refers to elements of this kernel as unramified elements in $GW^n_{n-q}(k(U), \omega^{\mathcal{L}}_{x_0})$.

Notation 4.4. If X is a smooth scheme, and $\mathcal{L}$ is a line bundle on X, for any $i, j \in \mathbb{N}$, write $\mathbf{GW}^j_i(\mathcal{L})$ for the Zariski (or Nisnevich) sheaf associated with the presheaf of unramified elements in $GW^j_i(k(X), \omega^{\mathcal{L}}_{x_0})$.

In particular, taking $n = 2$ and $q = -1$, we see that the line $q = -1$ in the spectral sequence $E(2)^{p,q}$ is a flasque resolution of the sheaf $\mathbf{K}^{Sp}_3 = \mathbf{GW}^2_3$.

Remark 4.5. One useful fact about the Gersten-Grothendieck-Witt spectral sequences is that the forgetful homomorphisms $f_{i,j} : GW^j_i(X,\mathcal{L}) \to K_i(X)$ and hyperbolic homomorphisms $H_{i,j} : K_i(X) \to GW^j_i(X,\mathcal{L})$ induce morphisms of spectral sequences between the Gersten-Grothendieck-Witt spectral sequence and the Brown-Gersten-Quillen spectral sequence in K-theory (and conversely).

Transfers

In this section, we refer to [Gil07a] for more information on the category of complexes of quasi-coherent $\mathcal{O}_X$-modules with coherent and bounded homology, and the duality on it.

Let X be a smooth scheme with 2 invertible and let $\mathcal{M}(X)$ be the category of quasi-coherent $\mathcal{O}_X$-modules. We denote by $Ch^b_c(\mathcal{M}(\mathcal{O}_X))$ the category of complexes of objects in $\mathcal{M}(\mathcal{O}_X)$ whose homology is bounded and coherent. We can define a structure of exact category on $Ch^b_c(\mathcal{M}(\mathcal{O}_X))$ by saying that a sequence of complexes is exact if it is degreewise exact. A weak-equivalence of complexes is a quasi-isomorphism.

If $\mathcal{L}$ is a line bundle over X, we fix an injective resolution $\mathcal{I}$ of $\mathcal{L}$

$$0 \longrightarrow \mathcal{L} \longrightarrow I_0 \longrightarrow I_{-1} \longrightarrow \dots \longrightarrow I_{-d} \longrightarrow 0$$

and we consider for any complex $M \in Ch^b_c(\mathcal{M}(X))$ the complex $\sharp_{\mathcal{I}}(M) := \mathrm{Hom}_{\mathcal{O}_X}(M, \mathcal{I})$. There is a canonical isomorphism $\varpi_{\mathcal{I}} : 1 \to \sharp_{\mathcal{I}}\sharp_{\mathcal{I}}$ and then $(Ch^b_c(\mathcal{M}(X)), qis, \sharp_{\mathcal{I}}, \varpi_{\mathcal{I}})$ is an exact category with weak-equivalences and duality. As seen in the previous section, the translation functor yields new dualities $\sharp^n_{\mathcal{I}}$ and $\varpi^n_{\mathcal{I}}$ and we can define the *coherent Grothendieck-Witt groups* as the homotopy groups of the space $\mathcal{GW}$ associated with $(Ch^b_c(\mathcal{M}(X)), qis, \sharp_{\mathcal{I}}, \varpi_{\mathcal{I}})$.

Definition 4.6. For $i \geq 0$, we denote by $\widetilde{GW}_i^j(X, \mathcal{L})$ the group $\pi_i \mathcal{GW}(Ch^b_c(\mathcal{M}(X)), qis, \sharp_{\mathcal{I}}, \varpi_{\mathcal{I}})$. If $\mathcal{L} = \mathcal{O}_X$, we simply put $\widetilde{GW}_i^j(X)$ instead of $\widetilde{GW}_i^j(X, \mathcal{O}_X)$.

The embedding $\mathcal{P}(X) \subset \mathcal{M}(X)$ yields a functor $\iota : Ch^b(X) \to Ch^b_c(\mathcal{M}(X))$ and the choice of an injective resolution $\mathcal{I}$ of $\mathcal{L}$ induces a natural transformation $\sharp_{\mathcal{I}} \circ \iota \to \iota \circ \sharp_{\mathcal{L}}$ which is a duality preserving functor in the sense of [Sch10b, §2.1]. The functor ι is moreover non-singular and exact and then induces a morphism of spaces

$$\iota : \mathcal{GW}(Ch^b(X), qis, \sharp_{\mathcal{L}}, \varpi_{\mathcal{L}}) \longrightarrow \mathcal{GW}(Ch^b_c(\mathcal{M}(X)), qis, \sharp_{\mathcal{I}}, \varpi_{\mathcal{I}})$$

which is a weak-equivalence by means of [Sch10b, Lemma 2].

Let X, Y be smooth schemes, and $f : Y \to X$ be a finite morphism. We will assume that X and Y are integral and set $r = \dim(X) - \dim(Y)$. Let $\mathcal{L}$ be a line bundle on X and $\mathcal{I}$ be an injective resolution of $\mathcal{L}$

$$0 \longrightarrow \mathcal{L} \longrightarrow I_0 \longrightarrow I_{-1} \longrightarrow \ldots \longrightarrow I_{-d} \longrightarrow 0.$$

Let $\overline{f} : (Y, \mathcal{O}_Y) \to (X, f_*\mathcal{O}_Y)$ be the morphism of ringed spaces induced by f. We let $f^\sharp\mathcal{I}$ be the complex $\overline{f}^*(\mathrm{Hom}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{I}))$ and we observe that $f^\sharp\mathcal{I}$ induces a duality on $Ch^b_c(\mathcal{M}(Y))$ [Gil07a, §2.4]. The trace map induces a duality preserving functor $f_* \circ \sharp_{f^\sharp\mathcal{I}} \to \sharp_{\mathcal{I}} \circ f_*$ and therefore we get a morphism of spaces

$$f_* : \mathcal{GW}(Ch^b_c(\mathcal{M}(Y)), qis, \sharp_{f^\sharp\mathcal{I}}, \varpi_{f^\sharp\mathcal{I}}) \longrightarrow \mathcal{GW}(Ch^b_c(\mathcal{M}(X)), qis, \sharp_{\mathcal{I}}, \varpi_{\mathcal{I}}).$$

Let $\mathcal{N}$ be the invertible $\mathcal{O}_Y$-module $\overline{f}^*\mathrm{Ext}^r_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{L})$ [Gil07a, §4.3]. Then $\sharp_{f^\sharp\mathcal{I}}$ is an injective resolution of $\mathcal{N}$ (shifted $-r$ times) by [Gil07a, §4.3] and therefore f_* induces a morphism of spaces

$$f_* : \mathcal{GW}(Ch^b_c(\mathcal{M}(Y)), qis, \sharp^{-r}_{f^\sharp\mathcal{I}}, \varpi^{-r}_{f^\sharp\mathcal{I}}) \longrightarrow \mathcal{GW}(Ch^b_c(\mathcal{M}(X)), qis, \sharp_{\mathcal{I}}, \varpi_{\mathcal{I}})$$

giving homomorphisms $f_* : GW_i^{j-r}(Y, \mathcal{N}) \to GW_i^j(X, \mathcal{L})$ for any $i \geq 0$ and any $j \in \mathbb{Z}$. If $f : Y \to X$ is a closed immersion, observe that, by construction, f_* factorizes through the groups on X supported on Y.

A finite morphism preserves the filtration by codimension of support, and then induces morphisms of Gersten-Grothendieck-Witt spectral sequences (see, for instance, [Fas08, Lemma 5.3.2]). These observations allow one to prove the following "dévissage" result:

Proposition 4.7. *Let X be a smooth scheme and let $Y \subset X$ be a closed smooth subscheme. Let $\mathcal{L}$ be a line bundle over X and $\mathcal{N} := \overline{f}^*\mathrm{Ext}^r_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{L})$. Let $r = \dim(X) - \dim(Y)$. Then the transfer homomorphisms*

$$f_* : GW_i^{j-r}(Y, \mathcal{N}) \longrightarrow GW_{i,Y}^j(X, \mathcal{L})$$

are isomorphisms for any $i, j \in \mathbb{N}$.

Proof. Our argument is along the same lines as [Gil02, §4]. We know that f_* induces a morphism between the corresponding Gersten-Grothendieck-Witt spectral sequences and it suffices to prove that it is an isomorphism at page 2 to prove the result. After dévissage [FS09, Proposition 28], f_* induces a homomorphism

$$f_* : GW_{n-p}^{n-p}(k(y_p), \omega_{y_p}^{\mathcal{N}}) \longrightarrow GW_{n-p}^{n-p}(k(y_p), \omega_{y_p}^{\mathcal{L}})$$

for any $y_p \in Y^{(p)}$, where $\omega_{y_p}^{\mathcal{N}} = \mathrm{Hom}_{k(y_p)}(V, \mathrm{Ext}^p_{\mathcal{O}_{Y,y_p}}(k(y_p), \mathcal{N} \otimes \mathcal{O}_{Y,y_p}))$ for any vector space V and $\omega_{y_p}^{\mathcal{L}} = \mathrm{Hom}_{k(y_p)}(V, \mathrm{Ext}^{p+r}_{\mathcal{O}_{X,y_p}}(k(y_p), \mathcal{L} \otimes \mathcal{O}_{X,y_p}))$. The morphism f_* is induced by the canonical identification

$$\mathrm{Ext}^p_{\mathcal{O}_{Y,y_p}}(k(y_p), \mathcal{N} \otimes \mathcal{O}_{Y,y_p}) \simeq \mathrm{Ext}^{p+r}_{\mathcal{O}_{X,y_p}}(k(y_p), \mathcal{L} \otimes \mathcal{O}_{X,y_p})$$

and is therefore an isomorphism. □

Computation of $H^3_{\mathrm{Nis}}(X, \mathbf{K}_3^{Sp})$

Our aim in this section is the computation of $H^3_{\mathrm{Nis}}(X, \mathbf{K}_3^{Sp})$. As seen in the section on the Gersten-Grothendieck-Witt spectral sequences, the line $q = -1$ at page 2 in the spectral sequence $E(2)^{p,q}$ provides a flasque resolution of $\mathbf{K}_3^{Sp}$ in the Zariski topology. Since Grothendieck-Witt groups of smooth schemes are homotopy invariant, the sheaf $\mathbf{K}_3^{Sp}$ is strictly $\mathbb{A}^1$-invariant and thus its Zariski cohomology coincides with its Nisnevich cohomology by, e.g., [Mor12, Corollary 5.43]. We are thus reduced to the study of the Gersten resolution of $\mathbf{K}_3^{Sp}$ whose last terms look as follows:

$$\bigoplus_{x_2 \in X^{(2)}} GW_1^0(k(x_2), \omega_{x_2}) \longrightarrow \bigoplus_{x_3 \in X^{(3)}} GW_0^3(k(x_3), \omega_{x_3}) \longrightarrow H^3(X, \mathbf{K}_3^{Sp}) \longrightarrow 0.$$

To understand the right-hand cohomology group, we first compute the groups $GW_1^0(F)$ and $GW_0^3(F)$ for any field F (again, we assume that F has characteristic unequal to 2). Observe that by definition $GW_1^0(F) = K_1O(F)$.

Lemma 4.8 ([FS08, Lemma 4.1])**.** *Let F be a field with $2 \in F^\times$. The hyperbolic functor $H_{0,3}$: $K_0(F) \to GW^3(F)$ yields an isomorphism $\mathbb{Z}/2 \to GW_0^3(F)$.*

Lemma 4.9 ([Bas74, Corollary 4.7.7])**.** *Let F be a field with $2 \in F^\times$. The determinant $\det$: $K_1O(F) \to \mathbb{Z}/2$ and the spinor norm $\mathrm{Sn} : K_1O(F) \to F^\times/(F^\times)^2$ induce an isomorphism*

$$K_1O(F) \xrightarrow{\sim} \mathbb{Z}/2 \oplus F^\times/(F^\times)^2$$

More precisely, the factor $F^\times/(F^\times)^2$ is the image of the hyperbolic functor

$$H_{1,0} : K_1(F) \longrightarrow GW_1^0(F)$$

while the factor $\mathbb{Z}/2$ is the image of the homomorphism $\eta_{1,0} : GW_1^0(F) \to GW^3(F)$.

Notation 4.10. We will denote by $Ch^n(X)$ the group $CH^n(X)/2$, where $CH^n(X)$ is the Chow groups of codimension n cycles in X.

By means of functoriality of the Gersten resolutions, we obtain a commutative diagram of the form

$$\begin{tikzcd}
\bigoplus_{x_2\in X^{(2)}} K_1(k(x_2))/2 \arrow[r] \arrow[d, "H"] & \bigoplus_{x_3\in X^{(3)}} K_0(k(x_3))/2 \arrow[r] \arrow[d, "H"] & Ch^3(X) \arrow[r] \arrow[d, dashed] & 0\, , \\
\bigoplus_{x_2\in X^{(2)}} GW_1^0(k(x_2),\omega_{x_2}) \arrow[r] \arrow[d] & \bigoplus_{x_3\in X^{(3)}} GW_0^3(k(x_3),\omega_{x_3}) \arrow[r] \arrow[d] & H^3(X,\mathbf{K}_3^{Sp}) \arrow[r] \arrow[d] & 0 \\
\bigoplus_{x_2\in X^{(2)}} \mathbb{Z}/2 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

and combining Lemmas 4.9 and 4.8 with the discussion of the last paragraph, the first two columns are short exact sequences. A diagram chase then yields an exact sequence of the form

$$\bigoplus_{x_2\in X^{(2)}} \mathbb{Z}/2 \xrightarrow{\chi} Ch^3(X) \longrightarrow H^3(X,\mathbf{K}_3^{Sp}) \longrightarrow 0.$$

Now, the commutative diagram

$$\begin{tikzcd}
\bigoplus_{x_1\in X^{(1)}} GW_2^1(k(x_1),\omega_{x_1}) \arrow[r] \arrow[d, "\eta_{2,1}"'] & \bigoplus_{x_2\in X^{(2)}} GW_1^0(k(x_2),\omega_{x_2}) \arrow[d, "\eta_{1,0}"] \\
\bigoplus_{x_1\in X^{(1)}} GW_1^0(k(x_1),\omega_{x_1}) \arrow[r] & \bigoplus_{x_2\in X^{(2)}} GW_0^3(k(x_2),\omega_{x_2}) \\
\bigoplus_{x_1\in X^{(1)}} K_1(k(x_1))/2 \arrow[r] \arrow[u, "H_{1,0}"] & \bigoplus_{x_2\in X^{(2)}} K_0(k(x_2))/2 \arrow[u, "H_{0,3}"']
\end{tikzcd}$$

shows that the map $\bigoplus_{x_2\in X^{(2)}} \mathbb{Z}/2 \to Ch^3(X)$ actually factors through a map $Ch^2(X) \to Ch^3(X)$ that we still denote by χ.

Next, recall from [Bro03, §8] that one can define Steenrod square operations $Sq^2 : Ch^n(X) \to Ch^{n+1}(X)$ for any $n \in \mathbb{N}$ satisfying reasonable functorial properties. In particular, if $f : Y \to X$ is a proper morphism of smooth connected schemes, we have [Bro03, 8.10, 8.11, 9.4]:

$$Sq^2(f_*[Y]) = c_1(\omega_{X/k})f_*([Y]) - f_*(c_1(\omega_{Y/k})),$$

where $\omega_{X/k}$ (resp. $\omega_{Y/k}$) is the canonical sheaf of X over $\operatorname{Spec} k$ (resp. Y over $\operatorname{Spec} k$).

Theorem 4.11. *If X is a smooth scheme of dimension 3 over a field k having characteristic different from 2, then there is an exact sequence of the form*

$$Ch^2(X) \xrightarrow{Sq^2} Ch^3(X) \longrightarrow H^3(X, \mathbf{K}_3^{Sp}) \longrightarrow 0.$$

Proof. Let A be a Dedekind domain, and $\mathcal{L}$ be an invertible A-module. We compute $GW_0^0(A, \mathcal{L})$ using two different methods. Karoubi periodicity yields an exact sequence of groups for any $n \in \mathbb{N}$

$$GW_0^n(A, \mathcal{L}) \xrightarrow{f_{0,n}} K_0(A) \xrightarrow{H_{0,n+1}} GW_0^{n+1}(A, \mathcal{L}) \longrightarrow W^{n+1}(A, \mathcal{L}) \longrightarrow 0$$

where the last term on the right is the triangular Witt group defined by P. Balmer (see [Bal05]). Using the fact that $W^3(A, \mathcal{L}) = 0$ because A is of dimension 1 [BW02, Theorem 10.1], we get a surjective map $H_{0,3} : K_0(A) \to GW_0^3(A, \mathcal{L})$. Using the exact sequence once again, we get an exact sequence

$$K_0(A) \xrightarrow{f_{0,3} \circ H_{0,3}} K_0(A) \xrightarrow{H_{0,0}} GW_0^0(A, \mathcal{L}) \longrightarrow W^0(A, \mathcal{L}) \longrightarrow 0.$$

To compute $f_{0,3} \circ H_{0,3}$, recall that there is an isomorphism $\varphi : \mathbb{Z} \oplus Pic(A) \to K_0(A)$ defined by $\varphi(m, N) = (m-1)[A] + [N]$ for any $m \in \mathbb{N}$ and $N \in Pic(A)$. If M is a projective A-module, then $(f_{0,3} \circ H_{0,3})([M]) = [M] - [M^\vee \otimes \mathcal{L}]$. Using φ, we see that the composite morphism

$$f_{0,3} \circ H_{0,3} : \mathbb{Z} \oplus Pic(A) \longrightarrow \mathbb{Z} \oplus Pic(A)$$

is given by $(f_{0,3} \circ H_{0,3})(m, N) = (0, N^{\otimes 2} \otimes (\mathcal{L}^\vee)^{\otimes m})$. We therefore get an exact sequence

$$0 \longrightarrow \mathbb{Z} \oplus Ch^1(A)/\langle \mathcal{L} \rangle \xrightarrow{H_{0,0}} GW_0^0(A, \mathcal{L}) \longrightarrow W^0(A, \mathcal{L}) \longrightarrow 0.$$

We now use the Grothendieck-Witt spectral sequence $E(0)^{p,q}$ to compute $GW_0^0(A, \mathcal{L})$. Setting $Y = \mathrm{Spec}(A)$ and writing K for the field of fractions of A, we see that the line $q = 0$ at page 1 takes the form

$$GW_0^0(K, \mathcal{L}) \xrightarrow{d} \bigoplus_{y_1 \in Y^{(1)}} W(k(y_1), \omega_{y_1}^{\mathcal{L}}) \ ,$$

while the line $q = -1$ takes the form

$$GW_1^0(K, \mathcal{L}) \xrightarrow{d} \bigoplus_{y_1 \in Y^{(1)}} GW_0^3(k(y_1), \omega_{y_1}^{\mathcal{L}}).$$

We first analyze the line $q = 0$. Since K is a field, there is an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{H} GW_0^0(K, \mathcal{L}) \longrightarrow W(K, \mathcal{L}) \longrightarrow 0 \ ,$$

which yields a commutative diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[rr] \arrow[d, "H"'] & & 0 \arrow[d] \\
GW_0^0(K,\mathcal{L}) \arrow[r, "d"] \arrow[d] & \bigoplus_{y_1\in Y^{(1)}} & W(k(y_1),\omega_{y_1}^{\mathcal{L}}) \arrow[d, equal] \\
W(K,\mathcal{L}) \arrow[r, "d"] & \bigoplus_{y_1\in Y^{(1)}} & W(k(y_1),\omega_{y_1}^{\mathcal{L}})
\end{tikzcd}$$

where the columns are short exact sequences, the middle line is the line $q = 0$ in the spectral sequence $E(0)^{p,q}$ and the bottom line is the line $q = 0$ in the Gersten-Witt spectral sequence. The kernel of the bottom map is $W^0(A,\mathcal{L})$ by [BW02, Corollary 10.3] and we get an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow H^0(A,\mathbf{GW}_0^0(\mathcal{L})) \longrightarrow W^0(A,\mathcal{L}) \longrightarrow 0.$$

We now analyze the line $E(0)_2^{p,-1}$. Lemmas 4.8 and 4.9 yield a commutative diagram

$$\begin{tikzcd}
K_1(K)/2 \arrow[r] \arrow[d, "H"] & \bigoplus_{y_1\in Y^{(1)}} K_0(k(y_1))/2 \arrow[d, "H"] \\
GW_1^0(K,\mathcal{L}) \arrow[r, "d"] \arrow[d] & \bigoplus_{y_1\in Y^{(1)}} GW^3(k(y_1),\omega_{y_1}^{\mathcal{L}}) \arrow[d] \\
GW^3(K,\mathcal{L}) \arrow[r] & 0
\end{tikzcd}$$

where the columns are short exact sequences. Thus, just as in the discussion subsequent to Notation 4.10, we obtain an exact sequence of the form

$$\mathbb{Z}/2 \xrightarrow{g} Pic(A)/2 \longrightarrow H^1(A,\mathbf{GW_1^0}(\mathcal{L})) \longrightarrow 0$$

where the map g has yet to be identified. To understand g, observe that the spectral sequence yields an extension (since $\dim(A) = 1$, the spectral sequence collapses at page 2)

$$0 \longrightarrow H^1(A,\mathbf{GW}_1^0(\mathcal{L})) \longrightarrow GW_0^0(A,\mathcal{L}) \longrightarrow H^0(A,\mathbf{GW}_0^0(\mathcal{L})) \longrightarrow 0.$$

Using again our computation of $GW^3(k(y_1),\omega_{y_1}^{\mathcal{L}})$, we see that the composition

$$Pic(A)/2 \longrightarrow H^1(A,\mathbf{GW}_1^0(\mathcal{L})) \longrightarrow GW_0^0(A,\mathcal{L})$$

is equal to $H_{0,0}$ (restricted to the Picard group). Looking at our first computation of $GW_0^0(A,\mathcal{L})$, we therefore see that g is such that $g(1) = [\mathcal{L}]$.

With these results in hand, we now return to the original problem. Let $x_2 \in X^{(2)}$ and let Y be the normalization of the closure Z of x_2 in X. Observe that the composition $f : Y \to Z \subset X$ is a finite morphism. To compute the composition

$$GW_1^0(k(x_2),\omega_{x_2}) \longrightarrow \bigoplus_{x_2\in X^{(2)}} GW_1^0(k(x_2),\omega_{x_2}) \xrightarrow{d} \bigoplus_{x_3\in X^{(3)}} GW_0^3(k(x_3),\omega_{x_3})$$

it suffices by definition to compute the component corresponding to the summand $GW_0^3(k(x_3),\omega_{x_3})$ for any $x_3 \in X^{(3)}$. We can then assume that $X = \operatorname{Spec}(\mathcal{O}_{X,x_3})$ and that Y is an essentially smooth curve with a finite number of closed points. Moreover, we have $k(Y) = k(x_2)$ by definition, and we let $\mathcal{L} := \omega_{Y/k} \otimes f^*\omega_{X/k}^\vee = \overline{f}^*\operatorname{Ext}^2_{\mathcal{O}_X}(f_*\mathcal{O}_Y,\mathcal{O}_X)$. The morphism f_* induces a commutative diagram

$$\begin{tikzcd}
GW_1^0(k(x_2),\omega_{x_2}) \arrow[r, "d_X"] \arrow[d, equal] & GW_0^3(k(x_3),\omega_{x_3}) \\
GW_1^0(k(Y),\omega_{y_0}^{\mathcal{L}}) \arrow[r, "d_Y"'] & \bigoplus_{y_1\in Y^{(1)}} GW_0^3(k(y_1),\omega_{y_1}^{\mathcal{L}}) \arrow[u, "f_*"']
\end{tikzcd}$$

and the differential d_Y on the component $\mathbb{Z}/2$ of $GW_1^0(k(Y),\omega_{y_0}^{\mathcal{L}})$ can be computed using our analysis in the case of Dedekind rings. Projecting $\bigoplus_{y_1\in Y^{(1)}} GW_0^3(k(y_1),\omega_{y_1}^{\mathcal{L}})$ onto $CH^1(Y)$, we find $d_Y(\overline{1}) = c_1(\mathcal{L})$ in $Ch^1(Y)$.

We find therefore

$$f_*d_Y(\overline{1}) = f_*c_1(\omega_{Y/k} \otimes f^*\omega_{X/k}^\vee) = Sq^2(\overline{1})$$

and the theorem is proved. $\square$

Remark 4.12. Let X be a smooth scheme of dimension d over a field k with $2 \in k^\times$. The same proof as above shows that $H^d(X,\mathbf{GW}_d^{d-1}) = \operatorname{coker}(Sq^2 : Ch^{d-1}(X) \to Ch^d(X))$.

5 Vanishing theorems

In this section, we review some basic properties of the contraction construction, which is useful in giving explicit descriptions of the terms of Gersten resolutions. Together with these facts, we prove a number of cohomological vanishing results that will be used in Section 6 to provide explicit descriptions of sets of isomorphism classes of vector bundles.

Contractions

Suppose $\mathcal{G}$ is a strongly $\mathbb{A}^1$-invariant sheaf of groups. For any smooth k-scheme U, the unit of $\mathbf{G}_m$ defines a morphism $\mathbf{G}_m \to \mathbf{G}_m \times U$. Recall that $\mathcal{G}_{-1}$ is the sheaf

$$\mathcal{G}_{-1}(U) = \ker(\mathcal{G}(\mathbf{G}_m \times U) \to \mathcal{G}(U))$$

Iterating this construction one defines $\mathcal{G}_{-i}$.

Remark 5.1. The projection map $\mathbf{G}_m \times U \to U$ is split by the inclusion $U \to \mathbf{G}_m \times U$ given by taking the product with the identity $\operatorname{Spec} k \to \mathbf{G}_m$. Suppose $\mathbf{A}$ is a strictly $\mathbb{A}^1$-invariant sheaf of groups. Applying $\mathbf{A}$ to the projection gives a homomorphism $\mathbf{A}(U) \to \mathbf{A}(\mathbf{G}_m \times U)$, which is split injective since the composition $U \to \mathbf{G}_m \times U \to U$ is the identity on U. This observation allows us to identify its cokernel with the contraction just mentioned. We will use this alternative presentation for contractions later.

If we restrict $()_{-1}$ to the category of strictly $\mathbb{A}^1$-invariant sheaves of groups, it is an exact functor (see, e.g., [Mor12, Lemma 7.33] or, more precisely, its proof).

Theorem 5.2 ([Mor12, Theorem 6.13]). *If $(\mathcal{X}, x)$ is a pointed $\mathbb{A}^1$-connected space, then for every pair of integers $i, j \geq 1$,*

$$\boldsymbol{\pi}_i^{\mathbb{A}^1}(\mathbf{R}\Omega^j_{\mathbf{G}_m}\mathcal{X}) = \boldsymbol{\pi}_i^{\mathbb{A}^1}(\mathcal{X})_{-j}.$$

Lemma 5.3. *For any integers $i, j \geq 0$ and any integer $n > 0$, there are canonical isomorphisms*

$$(\mathbf{K}_i^M/n)_{-j} \cong \begin{cases} \mathbf{K}^M_{i-j}/n & \text{if } j \leq i \\ 0 & \text{if } j > i \end{cases}, \text{ and}$$

$$(\mathbf{K}_i^Q)_{-j} \cong \begin{cases} \mathbf{K}^Q_{i-j} & \text{if } j \leq i \\ 0 & \text{if } j > i \end{cases}.$$

Remarks on the proof. The proofs of these two statements can be obtained by the same method as that of Proposition 5.4, which is a bit more delicate, so we will prove that statement instead. □

To describe the contractions of $\mathbf{GW}_i^j$, it is more convenient to identify $(\mathbf{GW}_i^j)_{-1}$ as the cokernel of the morphism (see Remark 5.1)

$$p^* : \mathbf{GW}_i^j \longrightarrow \mathbf{GW}_i^j(- \times \mathbf{G}_m)$$

where p^* is induced by the projection $p : X \times \mathbf{G}_m \to X$.

Proposition 5.4. *For any $i, j \in \mathbb{N}$, we have $(\mathbf{GW}_i^j)_{-1} = \mathbf{GW}_{i-1}^{j-1}$.*

Proof. It suffices to prove that for any local ring A we have an exact sequence of groups

$$0 \longrightarrow \mathbf{GW}_i^j(X) \xrightarrow{p^*} \mathbf{GW}_i^j(X \times \mathbf{G}_m) \longrightarrow \mathbf{GW}_{i-1}^{j-1}(X) \longrightarrow 0$$

where $X = \operatorname{Spec}(A)$. Denote by $\mathcal{C}$ the line $q = j - i$ at page 2 of the Gersten-Grothendieck-Witt spectral sequence $E(j)^{pq}$ and by $\mathcal{C}'$ the line $q = j - i$ at page 2 of the spectral sequence $E(j-1)^{pq}$. Since the Gersten conjecture holds for Grothendieck-Witt groups, $\mathcal{C}$ provides a flasque resolution of the sheaf $\mathbf{GW}_i^j$ while $\mathcal{C}'$ provides a flasque resolution of $\mathbf{GW}_{i-1}^{j-1}$. Arguing as [Mor12, Theorem 5.38], we see that the projection $p' : X \times \mathbb{A}^1 \to X$ induces for any $n \in \mathbb{N}$ an isomorphism $(p')^* : H^n(X, \mathcal{C}) \to H^n(X \times \mathbb{A}^1, \mathcal{C})$. Since X is local, we then have $H^1(X \times \mathbb{A}^1, \mathcal{C}) = 0$ and the long exact sequence in cohomology associated with the open embedding $i : X \times \mathbf{G}_m \to X \times \mathbb{A}^1$ reads as

$$0 \longrightarrow H^0(X \times \mathbb{A}^1, \mathcal{C}) \xrightarrow{i^*} H^0(X \times \mathbf{G}_m, \mathcal{C}) \longrightarrow H^1_{X \times \{0\}}(X \times \mathbb{A}^1, \mathcal{C}) \longrightarrow 0.$$

The proof of Proposition 4.7 shows that the closed embedding $j : X \times \{0\} \to X \times \mathbb{A}^1$ induces isomorphism $H^n(X, \mathcal{C}') \to H^{n+1}_{X \times \{0\}}(X \times \mathbb{A}^1, \mathcal{C})$ for any $n \in \mathbb{N}$ (here we trivialize the invertible module $\overline{j}^* \mathrm{Ext}^1_{\mathcal{O}_{X \times \mathbb{A}^1}}(j_* \mathcal{O}_X, \mathcal{O}_{X \times \mathbb{A}^1})$ using the Koszul complex associated with the global section $t \in k[t]$). Thus we get an exact sequence

$$0 \longrightarrow \mathbf{GW}^j_i(X) \xrightarrow{p^*} \mathbf{GW}^j_i(X \times \mathbf{G}_m) \longrightarrow \mathbf{GW}^{j-1}_{i-1}(X) \longrightarrow 0$$

and therefore $(\mathbf{GW}^j_i)_{-1} = \mathbf{GW}^{j-1}_{i-1}$. $\square$

Cohomology of $\mathbf{K}^{\mathrm{MW}}_j$

For any $n \in \mathbb{Z}$, we denote by $\mathbf{I}^n$ the unramified sheaf (in the Nisnevich or Zariski topology) of the n-th power of the fundamental ideal as considered for instance in [Fas09]. If $\mathcal{L}$ is a line bundle over a smooth k-scheme X, we denote by $\mathbf{I}^n(\mathcal{L})$ the sheaf twisted by $\mathcal{L}$ (denoted by $I^n_{\mathcal{L}}$ in [Fas09]). The next result, which uses the affirmation of the Milnor conjecture on quadratic forms [OVV07], follows from [Mor04, Theorem 5.3].

Theorem 5.5 (Morel)**.** *Suppose X is a smooth k-scheme. For any $n \in \mathbb{Z}$ and any line bundle $\mathcal{L}$ on X, there is a short exact sequence of sheaves on the small Nisnevich site of X of the form*

$$0 \longrightarrow \mathbf{I}^{n+1}(\mathcal{L}) \longrightarrow \mathbf{K}^{\mathrm{MW}}_n(\mathcal{L}) \longrightarrow \mathbf{K}^M_n \longrightarrow 0.$$

Proposition 5.6. *Let X be a smooth scheme of dimension d over a field k with $\mathrm{cd}_2(k) = r < \infty$ and let $\mathcal{L}$ be a line bundle on X. Then the Zariski sheaf $\mathbf{I}^n(\mathcal{L}) = 0$ for any $n \geq r + d + 1$.*

Proof. By definition of $\mathbf{I}^n(\mathcal{L})$ it is sufficient to prove that $I^n(k(X), \mathcal{L} \otimes k(X)) = 0$. Choosing a generator of $\mathcal{L} \otimes k(X)$ yields an isomorphism $I^n(k(X)) \simeq I^n(k(X), \mathcal{L} \otimes k(X))$ and we can thus suppose that $\mathcal{L}$ is trivial. Consider the quotient group $\overline{I}^n(k(X)) := I^n(k(X))/I^{n+1}(k(X))$. The affirmation of the Milnor conjecture yields an isomorphism $\overline{I}^n(k(X)) \simeq H^n_{\mathrm{Gal}}(k(X), \mu_2^{\otimes n})$. The latter is trivial since $\mathrm{cd}_2(k(X)) \leq r + d$ by [Ser94, §4.2, Proposition 11]. It follows then from [AP71, Korollar 2] that $I^n(k(X)) = 0$. $\square$

Corollary 5.7. *Let X be a smooth scheme of dimension d over a field k with $\mathrm{cd}_2(k) = r < \infty$ and let $\mathcal{L}$ be a line bundle on X. Then $H^i_{\mathrm{Nis}}(X, \mathbf{I}^n(\mathcal{L})) = 0$ for any $i \in \mathbb{N}$ and any $n \geq r + d + 1$.*

Proof. The sheaf $\mathbf{I}^n(\mathcal{L})$ admits a Gersten resolution by [Gil07b, Corollary 7.7]. It follows that its Nisnevich cohomology coincides with its Zariski cohomology, and therefore the result follows from the above proposition. $\square$

We now prove a yet stronger vanishing statement for $\mathbf{I}^{d+r}(\mathcal{L})$.

Proposition 5.8. *Let X be a smooth affine scheme of dimension d over a field k with $\mathrm{cd}_2(k) = r < \infty$ and let $\mathcal{L}$ be a line bundle on X. If $d \geq 1$, then $H^d_{\mathrm{Nis}}(X, \mathbf{I}^j(\mathcal{L})) = 0$ for any $j \geq d + r$. If $d \geq 2$ then we have $H^{d-1}_{\mathrm{Nis}}(X, \mathbf{I}^j(\mathcal{L})) = 0$ for any $j \geq d + r$.*

Proof. Once again, the cohomology of the sheaf $\mathbf{I}^j(\mathcal{L})$ computed in the Zariski topology coincides with the corresponding computation in the Nisnevich topology. We therefore prove the result for cohomology computed in the Zariski topology. By Proposition 5.6, we are reduced to the case $j = d + r$. The exact sequence of sheaves [Fas09, §2.1]

$$0 \longrightarrow \mathbf{I}^{d+r+1}(\mathcal{L}) \longrightarrow \mathbf{I}^{d+r}(\mathcal{L}) \longrightarrow \overline{\mathbf{I}}^{d+r} \longrightarrow 0$$

yields a long exact sequence in cohomology and Proposition 5.6 shows that $H^i(X, \mathbf{I}^{d+r+1}(\mathcal{L})) = 0$ for any $i \in \mathbb{N}$. Therefore, for any $i \in \mathbb{N}$, one obtains isomorphisms

$$H^i(X, \mathbf{I}^{d+r}(\mathcal{L})) \longrightarrow H^i(X, \overline{\mathbf{I}}^{d+r})$$

and it suffices to prove the result for $H^i(X, \overline{\mathbf{I}}^{d+r})$.

For any smooth scheme X and any $q \in \mathbb{N}$, let $\mathcal{H}^q$ be the sheaf associated with the presheaf $U \mapsto H^q_{\text{ét}}(U, \mu_2^{\otimes q})$. The Bloch-Ogus spectral sequence ([BO74]) converges to the étale cohomology groups $H^*_{\text{ét}}(X, \mu_2^{\otimes q})$ and its groups at page 2 are the groups $H^p_{Zar}(X, \mathcal{H}^q)$. These are computed via the Gersten complex

$$H^q(k(X), \mu_2^{\otimes q}) \xrightarrow{d_0} \bigoplus_{x_1 \in X^{(1)}} H^{q-1}(k(x_1), \mu_2^{\otimes q-1}) \longrightarrow \ldots .$$

The affirmation of Milnor's conjecture on quadratic forms [OVV07] shows that this complex is isomorphic to the complex

$$\overline{I}^q(k(X)) \xrightarrow{d_0} \bigoplus_{x_1 \in X^{(1)}} \overline{I}^{q-1}(k(x_1)) \longrightarrow \ldots ,$$

which is a flasque resolution of the sheaf $\overline{\mathbf{I}}^q$. It follows that the two sheaves are isomorphic and therefore that $H^i(X, \overline{\mathbf{I}}^q) \simeq H^i(X, \mathcal{H}^q)$ for any $i \in \mathbb{N}$. The proof of Proposition 5.6 shows that the lines $q \geq r + d + 1$ are trivial in the Bloch-Ogus spectral sequence. This shows that we have an isomorphism $H^d_{\mathrm{Zar}}(X, \mathcal{H}^{d+r}) \simeq H^{2d+r}_{\text{ét}}(X, \mu_2)$ and a surjective homomorphism $H^{2d+r-1}_{\text{ét}}(X, \mu_2) \to H^{d-1}_{\mathrm{Zar}}(X, \mathcal{H}^{d+r})$. The result therefore follows if we can show that $H^i_{\text{ét}}(X, \mu_2) = 0$ for $i \geq d+r+1$. If k is separably closed, this is [Mil80, Chapter VI, Theorem 7.2]. In general, it suffices to use the Hochschild-Serre spectral sequence ([Mil80, Chapter III, Theorem 2.20], see also [Mil80, Remark 2.21(b)]) and the result for separably closed fields. □

Corollary 5.9. *If k is a quadratically closed field, X is a smooth affine k-scheme of dimension $d \geq 2$, and $\mathcal{L}$ is a line bundle on X, for any pair of integers $i, j \geq d - 1$, there are isomorphisms*

$$H^i_{\mathrm{Nis}}(X, \mathbf{K}^{\mathrm{MW}}_j(\mathcal{L})) \xrightarrow{\sim} H^i_{\mathrm{Nis}}(X, \mathbf{K}^M_j).$$

Proof. Use the long exact sequence in cohomology associated with the short exact sequence of sheaves
$$0 \longrightarrow \mathbf{I}^{j+1}(\mathcal{L}) \longrightarrow \mathbf{K}_j^{\mathrm{MW}}(\mathcal{L}) \longrightarrow \mathbf{K}_j^M \longrightarrow 0$$
and Proposition 5.8. □

A vanishing result for cohomology of $\mathbf{K}_n^M/m$

Proposition 5.10. *Let X be a smooth affine variety of dimension d over a field k. If there exists an integer $m > 0$ such that for any closed point $x \in X$ the group $k(x)^\times$ is m-divisible, then $H^d(X, \mathbf{K}_{d+1}^M/m) = 0$.*

Proof. The Gersten resolution for the sheaf $\mathbf{K}_{d+1}^M/m$ gives an exact sequence of the form
$$\bigoplus_{x \in X^{(d)}} (\mathbf{K}_{d+1}^M/m)_{-d}(k(x)) \longrightarrow H^d_{\mathrm{Nis}}(X, \mathbf{K}_{d+1}^M/m).$$
Furthermore $(\mathbf{K}_{d+1}^M/m)_{-d} = \mathbf{K}_1^M/m$, and $\mathbf{K}_1^M/m(k(x)) = k(x)^\times/(k(x)^\times)^m = 0$ by assumption. □

6 Obstruction theory and classification results

In this section, we begin by reviewing aspects of obstruction theory involving the Postnikov tower in $\mathbb{A}^1$-homotopy theory. We combine the results of the previous sections with obstruction theory for the Postnikov tower of BGL_n to obtain information about vector bundles. The section closes with some additional information. Specifically, using the discussion of contractions, we provide some statements relating our computations of $\mathbb{A}^1$-homotopy sheaves from Section 3 to ordinary homotopy groups of the unitary groups by means of the complex realization functor.

The Postnikov tower in $\mathbb{A}^1$-homotopy theory

If $\mathcal{G}$ is a (Nisnevich) sheaf of groups, and $\mathbf{A}$ is a (Nisnevich) sheaf of abelian groups on which $\mathcal{G}$ acts, there is an induced action of $\mathcal{G}$ on the Eilenberg-Mac Lane space $K(\mathbf{A}, n)$ that fixes the base-point. In that case, we set $K^{\mathcal{G}}(A, n) := E\mathcal{G} \times^{\mathcal{G}} K(\mathbf{A}, n)$. The projection onto the first factor defines a morphism $K^{\mathcal{G}}(\mathbf{A}, n) \to B\mathcal{G}$ that is split by the inclusion of the base-point.

Just as simplicial homotopy classes of maps $[\mathcal{X}, K(\mathbf{A}, n)]_s$ are in bijection with elements of $H^n_{\mathrm{Nis}}(X, \mathbf{A})$, there is a corresponding classification theorem in this "twisted" setting. A map $\mathcal{X} \to K^{\mathcal{G}}(\mathbf{A}, n)$ gives, by composition, a morphism $\mathcal{X} \to B\mathcal{G}$, which yields a $\mathcal{G}$-torsor $\mathcal{P} \to \mathcal{X}$ by pullback. Then, the morphism of the previous sentence can be interpreted as a $\mathcal{G}$-equivariant map $\mathcal{P} \to K(A, n)$, i.e., a $\mathcal{G}$-equivariant degree n cohomology class on X with coefficients in $\mathbf{A}$. The following result summarizes the form of the Postnikov tower we will use; this result is collated from a collection of sources including [GJ09, Chapter VI.5], [MV99] and [Mor12, Appendix B].

Theorem 6.1. *If $(\mathcal{Y}, y)$ is any pointed $\mathbb{A}^1$-connected space, then there are a sequence of pointed spaces $(\mathcal{Y}^{(i)}, y)$, morphisms $p_i : \mathcal{Y} \to \mathcal{Y}^{(i)}$, and morphisms $f_i : \mathcal{Y}^{(i+1)} \to \mathcal{Y}^{(i)}$ such that* *i)* $\mathcal{Y}^{(i)}$ *is* $i+1$*-truncated, i.e.,* $\boldsymbol{\pi}_j^{\mathbb{A}^1}(\mathcal{Y}, y) = 0$ *for* $j > i$*,*

ii) the morphism p_i *induces an isomorphism on homotopy sheaves in degree* $\leq i$*,*

iii) the morphism f_i *is an* $\mathbb{A}^1$*-fibration, and the homotopy fiber of* f_i *is a* $K(\boldsymbol{\pi}_{i+1}^{\mathbb{A}^1}(\mathcal{Y}), i+1)$*,*

iv) the induced morphism $\mathcal{Y} \to \operatorname{holim}_i \mathcal{Y}^{(i)}$ *is an* $\mathbb{A}^1$*-weak equivalence.*

Furthermore, f_i *is a* twisted $\mathbb{A}^1$-principal fibration, *i.e., there is a unique (up to* $\mathbb{A}^1$*-homotopy)*

$$k_{i+1} : \mathcal{Y}^{(i)} \longrightarrow K^{\boldsymbol{\pi}_1^{\mathbb{A}^1}(\mathcal{Y})}(\boldsymbol{\pi}_{i+1}^{\mathbb{A}^1}(\mathcal{Y}), i+2)$$

such that $\mathcal{Y}^{(i+1)}$ *is the* $\mathbb{A}^1$*-homotopy fiber of this morphism, and the action of* $\boldsymbol{\pi}_1^{\mathbb{A}^1}(\mathcal{Y})$ *on the higher* $\mathbb{A}^1$*-homotopy sheaves is the usual conjugation action induced by change of base-points.*

Remark 6.2. When we apply this theorem, $\boldsymbol{\pi}_1^{\mathbb{A}^1}(\mathcal{Y}, y)$ will be a sheaf of abelian groups.

If $\mathcal{G} = 1$, then the word "twisted" can be dropped in the above statement. In that case, given an $\mathbb{A}^1$-principal fibration $\mathcal{E} \to \mathcal{B}$ classified by a morphism $\mathcal{B} \to \mathcal{F}'$ (here $\mathcal{F}'$ is an Eilenberg-Mac Lane space), a morphism $\mathcal{X} \to \mathcal{B}$ lifts to $\mathcal{E}$ if and only if the composite morphism $\mathcal{X} \to \mathcal{F}'$ is homotopically constant. Moreover, the simplicial function object preserves fibrations [MV99, §2 Lemma 1.8.3], so there is a fibration

$$S(\mathcal{X}, \mathcal{E}) \longrightarrow S(\mathcal{X}, \mathcal{B})$$

whose fiber is $S(\mathcal{X}, \Omega_s^1 \mathcal{F}')$. Thus, the space of lifts over a given map $\mathcal{X} \to \mathcal{B}$ is isomorphic to $S(\mathcal{X}, \Omega_s^1 \mathcal{F}')$.

In the special case where $\mathcal{F}'$ is an Eilenberg-MacLane sheaf, the obstruction to lifting is the pullback of the "universal" class on $\mathcal{B}$ given by $\mathcal{B} \to \mathcal{F}'$ to $\mathcal{X}$. Furthermore, the loop space in question is again an Eilenberg-MacLane sheaf, and the space of lifts of a given map $\mathcal{X} \to \mathcal{B}$ is parameterized (as a set) by a corresponding cohomology set.

When $\mathcal{G}$ acts non-trivially, the setup is similar, but one works $\mathcal{G}$-equivariantly. In that case, the obstruction to lifting is given by an *equivariant* cohomology class on $\mathcal{X}$, which is pulled back from the "universal" class $\mathcal{B} \to \mathcal{F}'$. Note that, in this case, the homotopy fiber is an ordinary Eilenberg-Mac Lane space (rather than a twisted one). In practice, we will use the Postnikov tower to factor a space as a sequence of twisted $\mathbb{A}^1$-principal fibrations and then deduce an (inductively defined) sequence of obstructions to lifting: each subsequent obstruction is defined after choosing a lift, whose existence is guaranteed by vanishing of the previous obstruction.

The universal primary obstruction vanishes

The primary obstruction to existence of vector bundles can be analyzed by means of the discussion of the previous section: in this case, the situation is particularly simple. To begin, recall that BSL_n is $\mathbb{A}^1$-1-connected for any $n \geq 2$ by Theorem 2.9. We also know that $\boldsymbol{\pi}_2^{\mathbb{A}^1}(BSL_n) = \boldsymbol{\pi}_1^{\mathbb{A}^1}(SL_n)$ and the latter is $\mathbf{K}_2^{\mathrm{MW}}$ for $n = 2$, and $\mathbf{K}_2^M$ for $n > 2$.

The second stage of the Postnikov tower for BSL_n gives rise to a (principal) fiber sequence of the form

$$BSL_n^{(2)} \longrightarrow BSL_n^{(1)} \longrightarrow K(\boldsymbol{\pi}_2^{\mathbb{A}^1}(BSL_n), 3).$$

Since $BSL_{n,\bullet}$ is $\mathbb{A}^1$-1-connected, $BSL_{n,\bullet}^{(1)} = *$, and the map $BSL_{n,\bullet}^{(1)} \longrightarrow K(\boldsymbol{\pi}_2^{\mathbb{A}^1}(BSL_n), 3)$ is trivial. We summarize this in the following result.

Lemma 6.3. *The universal obstruction class $BSL_n^{(1)} \to K(\boldsymbol{\pi}_2^{\mathbb{A}^1}(BSL_n), 3)$ is trivial.*

We know that the map $BSL_n \to BGL_n$ induced by the inclusion of SL_n into GL_n is, up to $\mathbb{A}^1$-homotopy, a $\mathbf{G}_m$-torsor and consequently an $\mathbb{A}^1$-covering space [Mor12, Definition 7.1 and Lemma 7.5] (we can replace BSL_n with the model $B_{gm}SL_n$ of [MV99, §4.2] using the standard representation of SL_n, and this space is evidently a $\mathbf{G}_m$-torsor over $Gr_{n,\infty}$). This presentation allows us to deduce an action of $\boldsymbol{\pi}_1^{\mathbb{A}^1}(BGL_n) = \mathbf{G}_m$ on $\boldsymbol{\pi}_i^{\mathbb{A}^1}(BSL_n)$ for $i \geq 2$. In this case, the twisted Eilenberg-MacLane space $K^{\mathbf{G}_m}(\boldsymbol{\pi}_i^{\mathbb{A}^1}(BGL_n), j)$ is the quotient sheaf

$$K^{\mathbf{G}_m}(\boldsymbol{\pi}_i^{\mathbb{A}^1}(BGL_n), j) := E\mathbf{G}_m \times^{\mathbf{G}_m} K(\boldsymbol{\pi}_i^{\mathbb{A}^1}(BGL_n), j),$$

as discussed in [Mor12, §B.2]; this furthermore completely determines the second stage of the $\mathbb{A}^1$-Postnikov tower for BGL_n.

Remark 6.4. The action of $\mathbf{G}_m$ on $\boldsymbol{\pi}_2^{\mathbb{A}^1}(BGL_2) \cong \mathbf{K}_2^{\mathrm{MW}}$ is non-trivial and gives rise to an action of $\mathbf{G}_m$ on the space $K(\mathbf{K}_2^{\mathrm{MW}}, 2)$. The homotopy fiber of the map $BGL_2^{(2)} \to B\mathbf{G}_m$ is also a $K(\mathbf{K}_2^{\mathrm{MW}}, 2)$.

Proposition 6.5. *The universal obstruction class $BGL_n^{(1)} \to K^{\mathbf{G}_m}(\boldsymbol{\pi}_2^{\mathbb{A}^1}(BGL_n), 3)$ factors through the constant map $B\mathbf{G}_m \to K^{\mathbf{G}_m}(\boldsymbol{\pi}_2^{\mathbb{A}^1}(BGL_n), 3)$ induced by inclusion of the base-point of $K(\boldsymbol{\pi}_2^{\mathbb{A}^1}(BGL_n), 3)$.*

Proof. First, identify $BGL_n^{(1)} = B\mathbf{G}_m$. The map $BGL_n^{(1)} \to K^{\mathbf{G}_m}(\boldsymbol{\pi}_2^{\mathbb{A}^1}(BGL_n), 3)$ is then a map $B\mathbf{G}_m \to K^{\mathbf{G}_m}(\boldsymbol{\pi}_2^{\mathbb{A}^1}(BGL_n), 3)$. However, this map comes from a $\mathbf{G}_m$-equivariant map $BSL_n^{(1)} \to K(\boldsymbol{\pi}_2^{\mathbb{A}^1}(BSL_n), 3)$. The aforementioned map is homotopically trivial by Lemma 6.3. As a consequence of this, the action of $\mathbf{G}_m$ on a representing class is also trivial. □

Corollary 6.6. *If X is any smooth scheme over a field k, then the primary $\mathbb{A}^1$-homotopy theoretic obstruction to existence of a rank n vector bundle on X with given determinant line bundle ξ lies in cohomological degree > 3.*

Proof. We want to build a map $X \to BGL_n$ by inductively working up the (twisted) Postnikov tower. We begin with a constant map $X \to BGL_n^{(0)} = *$. We then choose a lift $X \to BGL_n^{(1)}$, which, since $BGL_n^{(1)} = B\mathbf{G}_m$, corresponds to fixing a line bundle ξ on X. The primary obstruction to lifting this class to a map $X \to BGL_n^{(2)}$ is the pullback of the universal obstruction $BGL_n^{(1)} \to K^{\mathbf{G}_m}(\mathbf{K}_n^{\mathrm{MW}}, 3)$; since the latter map is homotopically the constant map $B\mathbf{G}_m \to B\mathbf{G}_m$ by Proposition 6.5, it follows that precomposing with the map $X \to BGL_n^{(1)}$ is simply the map ξ. We may therefore fix a lift of this class to the second stage of the Postnikov tower of BGL_n, and the next potentially non-trivial obstruction lies in degree ≥ 4. □

Lifting classes versus Chern classes

As we saw above, the $\mathbb{A}^1$-Postnikov tower for BGL_n gives rise to a sequence of morphisms $BGL_n \to BGL_n^{(i)}$. If we consider the identity map $BGL_n \to BGL_n$, since each induced map

$BGL_n \to BGL_n^{(i-1)}$ lifts to a morphism $BGL_n \to BGL_n^{(i)}$, the identity map factors through a morphism $BGL_{n,\bullet} \to K(\boldsymbol{\pi}_i^{\mathbb{A}^1}(BGL_n), i)$ for each i. Since for $i < n$ we have identifications $\boldsymbol{\pi}_i^{\mathbb{A}^1}(BGL_n) \cong \mathbf{K}_i^Q$, these classes can be identified with elements of $[BGL_n, K(\mathbf{K}_i^Q, i)]_{\mathbb{A}^1}$. These classes admit the following geometric description.

First, using the $\mathbb{A}^1$-weak equivalence $Gr_{n,\infty} \to BGL_n$, we can view these classes as canonical elements in $H^i(Gr_{n,\infty}, \mathbf{K}_i^Q)$. The space $Gr_{n,\infty}$ is a filtering colimit of finite-dimensional grassmannians $Gr_{n,n+N}$. By Bloch's formula, $H^i(Gr_{n,n+N}, \mathbf{K}_i^Q) \cong CH^i(Gr_{n,n+N})$. In particular, these groups are isomorphic to $\mathbb{Z}$ independent of N for $i \leq n$. Therefore, the limit $CH^i(Gr_{n,n+N})$ only depends on n and, as a consequence, $H^i(Gr_{n,\infty}, \mathbf{K}_i^Q) = \mathbb{Z}$. The calculation of the cohomology of the grassmannian gives us a canonical generator c_i of $H^i(Gr_{n,\infty}, \mathbf{K}_i^Q)$. It follows that our obstruction class is a multiple of c_i.

When $i = n$, the situation is just a bit more complicated. In that case, repeating the discussion of the previous paragraph, one obtains a canonical class in $o_{n,n} \in H^n(BGL_n, \boldsymbol{\pi}_n^{\mathbb{A}^1}(BGL_n))$. If n is odd, then Theorem 3.9 gives rise to a long exact sequence of the form

$$H^n_{\mathrm{Nis}}(Gr_{n,\infty}, \boldsymbol{\pi}_n^{\mathbb{A}^1}(BGL_{n,\infty})) \longrightarrow H^n_{\mathrm{Nis}}(Gr_{n,\infty}, \mathbf{K}_n^Q) \longrightarrow H^{n+1}_{\mathrm{Nis}}(Gr_{n,\infty}, \mathbf{S}_{n+1})$$

The image of $o_{n,n} \in H^n_{\mathrm{Nis}}(Gr_{n,\infty}, \mathbf{K}_n^Q) = \mathbb{Z}$ is a multiple of c_n.

Remark 6.7. In [Pet59, Lemma 4.5], an explicit relationship is given between c_n and $o_{n,n}$. Using this and compatibility of our constructions with complex realization, one deduces a more precise relationship between $o_{n,n}$ and c_n.

For $n \leq 3$, the lifting classes can be defined analogously, and we can be even more explicit. For $n = 1$, $BGL_1 \cong B\mathbf{G}_m = K(\mathbf{K}_1^M, 1)$ and $o_{1,1} \in H^1(B\mathbf{G}_m, \mathbf{K}_1^M)$. For $n = 2$, we can give the lifting class a slightly different description. Instead of considering the identity map $BGL_2 \to BGL_2$, we consider the identity map $BSL_2 \to BSL_2$, which is $\mathbf{G}_m$-equivariant. In that case, we can identify the lifting class canonically as $\mathbf{G}_m$-equivariant cohomology class since $BSL_2^{(2)} = K(\mathbf{K}_2^{\mathrm{MW}}, 2)$. More precisely, the lifting class is a canonical element $o_{2,2} \in H^2_{\mathbf{G}_m}(BSL_2, \mathbf{K}_2^{\mathrm{MW}}) = H^2(BGL_2, \mathbf{K}_2^{\mathrm{MW}}(\det \xi))$: here the action of $\boldsymbol{\pi}_1(BGL_n) = \mathbf{G}_m$ on $K(\mathbf{K}_2^{MW}, 2)$ is indicated by the notation (and depends on fixing a determinant line bundle). However, the image of this lifting class in $H^2(BGL_2, \mathbf{K}_2^M)$ induced by the epimorphism $\mathbf{K}_2^{\mathrm{MW}}(\det \xi) \to \mathbf{K}_2^M$ is independent of these choices.

Proposition 6.8. *If $n = 1$ or $n = 2$, then $o_{n,n} = c_n$.*

Proof. If $n = 1$, then $BGL_1 \cong B\mathbf{G}_m = K(\mathbf{K}_1^Q, 1)$ as mentioned above, so there is nothing to check. If $n = 2$, this statement is the content of the last paragraph of [Mor12, Remark 7.22]. $\square$

The action of $\mathbf{G}_m = \mathbf{K}_1^Q$ on $\mathbf{K}_n^Q$

As we observed above, the sheaf $\boldsymbol{\pi}_1^{\mathbb{A}^1}(BGL_n) = \mathbf{G}_m$ acts on $\boldsymbol{\pi}_i^{\mathbb{A}^1}(BGL_n)$ for any $n \geq 1$. When $i < n$, the sheaves $\boldsymbol{\pi}_i^{\mathbb{A}^1}(BGL_n)$ are in the stable range, and the action of $\mathbf{G}_m$ on these sheaves coincides with the action of $\mathbf{G}_m$ on $\boldsymbol{\pi}_i^{\mathbb{A}^1}(BGL_\infty) = \mathbf{K}_i^Q$.

Lemma 6.9. *For any $i > 0$, the action of $\mathbf{G}_m$ on $\mathbf{K}_i^Q$ induced by the identifications $\boldsymbol{\pi}_1^{\mathbb{A}^1}(BGL_\infty) = \mathbf{G}_m$ and $\boldsymbol{\pi}_i^{\mathbb{A}^1}(BGL_\infty)$ is trivial.*

Proof. The action in question is determined by a morphism of sheaves $\mathbf{G}_m \to \underline{\mathrm{Hom}}(\mathbf{K}^Q_i, \mathbf{K}^Q_i)$. Since both sheaves are strongly $\mathbb{A}^1$-invariant, so it suffices to prove the induced maps on sections over fields is trivial. If L is a field, identifying $K^Q_i(L)$ as $\pi_i(BGL_\infty(L))$, the result follows from the definition of the plus construction. □

Classification of rank 2 bundles

Henceforth, we assume that k is algebraically closed and has characteristic unequal to 2. In that case, we can assume $X(k)$ is non-empty, and we will fix a base k-point. Assume X is a smooth affine k-scheme. Here is the structure to which proofs of all results below will conform. If X is of small dimension, by means of the $\mathbb{A}^1$-Postnikov tower, and Theorems 2.9, 3.20, or 3.9, we can describe $\mathbb{A}^1$-homotopy classes of *pointed* maps $[(X,x), BGL_i]_{\mathbb{A}^1}$. The set of isomorphism classes of vector bundles of rank n on X is described by the set of *unpointed* homotopy classes of maps (see Theorem 2.3). To set of unpointed homotopy classes of maps $[X, BGL_i]_{\mathbb{A}^1}$ can be obtained by factoring out the the conjugation action of $\boldsymbol{\pi}^{\mathbb{A}^1}_1(BGL_n)(k) = \mathbf{G}_m(k)$ on $[(X,x), BGL_i]_{\mathbb{A}^1}$.

Theorem 6.10. *If k is an algebraically closed field having characteristic unequal to 2, and X is a smooth affine 3-fold, the map sending a vector bundle of rank 2 to its Chern classes determines a bijection between the pointed set of isomorphism classes of rank 2 vector bundles on X and $CH^1(X) \times CH^2(X)$.*

Proof. As observed in Corollary 6.6, the primary obstruction to existence of a rank 2 vector bundle on X vanishes. Fix a class $\xi \in Pic(X)$. Since the primary obstruction vanishes, the first lifting class is an element of $H^2(X, \boldsymbol{\pi}^{\mathbb{A}^1}_2(BGL_2))$, which by the descriptions of homotopy sheaves (and in the notation) given above is isomorphic to $H^2_{\mathrm{Nis}}(X, \mathbf{K}^{\mathrm{MW}}_2(\xi))$. The short exact sequence of sheaves on X of the form

$$0 \longrightarrow \mathbf{I}^3(\xi) \longrightarrow \mathbf{K}^{\mathrm{MW}}_2(\xi) \longrightarrow \mathbf{K}^M_2 \longrightarrow 0.$$

Taking cohomology of this short exact sequence gives rise to the sequence

$$\longrightarrow H^2(X, \mathbf{I}^3(\xi)) \longrightarrow H^2(X, \mathbf{K}^{\mathrm{MW}}_2(\xi)) \longrightarrow H^2(X, \mathbf{K}^M_2) \longrightarrow H^3(X, \mathbf{I}^3(\xi)) \longrightarrow \cdots .$$

Since k is algebraically closed, $H^2(X, \mathbf{I}^3(\xi))$ and $H^3(X, \mathbf{I}^3(\xi))$ vanish by Proposition 5.8, so the morphism in the middle is an isomorphism. By Bloch's formula, $H^2(X, \mathbf{K}^M_2) = CH^2(X)$. As a consequence, the primary lifting class is uniquely determined by an element of $CH^2(X)$. Moreover, since $BGL^{(2)}_2 = K_{\mathbf{G}_m}(\mathbf{K}^{MW}_2, 2)$, the discussion above shows that the lifting class is exactly an element of $CH^2(X)$.

The secondary lift is an element of $H^3(X, \boldsymbol{\pi}^{\mathbb{A}^1}_3(BGL_2))$. We know that $\boldsymbol{\pi}^{\mathbb{A}^1}_3(BGL_2)$ is an extension of $\mathbf{K}^{Sp}_3$ by $\mathbf{S}''_4$, and the latter is an extension of $\mathbf{S}'_4$ by $\mathbf{I}^5(\xi)$; we also know that the former is a quotient of $\mathbf{K}^M_4/12$. By Corollary 5.7, the sheaf $\mathbf{I}^5(\xi)$ is trivial and therefore the sheaves $\mathbf{S}''_4$ and $\mathbf{S}'_4$ are isomorphic. For reasons of cohomological dimension, there is a surjective map $H^3_{\mathrm{Nis}}(X, \mathbf{K}^M_4/12) \to H^3_{\mathrm{Nis}}(X, \mathbf{S}'_4)$. The first group vanishes by Proposition 5.10 and therefore $H^3_{\mathrm{Nis}}(X, \mathbf{S}'_4) = 0$. Thus, the long exact sequence in cohomology gives rise to an isomorphism $H^3_{\mathrm{Nis}}(X, \boldsymbol{\pi}^{\mathbb{A}^1}_3(BGL_2)) \to H^3_{\mathrm{Nis}}(X, \mathbf{K}^{Sp}_3)$. By Theorem 4.11, $H^3_{\mathrm{Nis}}(X, \mathbf{K}^{Sp}_3)$ is a quotient of $CH^3(X)/2$. Since X is affine, $CH^3(X)$ is uniquely divisible by [Sri89] and therefore $CH^3(X)/2$ is trivial.

Given the above data, we have built a pointed $\mathbb{A}^1$-homotopy class of maps $X \to BGL_2$. The action of $\mathbf{G}_m$ on $\boldsymbol{\pi}_i^{\mathbb{A}^1}(BGL_2)$ is induced by change of base-points. For $i = 2$, this action is not trivial, but we only care about the induced action of $\mathbf{G}_m(k)$ on $H^2_{\mathrm{Nis}}(X, \mathbf{K}_2^{\mathrm{MW}}(\xi))$. However, since k is algebraically closed, as explained above, the action factors through an action on $H^2_{\mathrm{Nis}}(X, \mathbf{K}_2^M)$. The morphism $\mathbf{G}_m \to \mathbf{K}_2^M$ is induced by the stabilization, and Lemma 6.9 shows the action is trivial in the stable setting. Since $H^3(X, \boldsymbol{\pi}_3^{\mathbb{A}^1}(BGL_2))$ is trivial, the action of $\mathbf{G}_m(k)$ on this group is also trivial. Thus, pointed and unpointed homotopy classes of maps coincide. Finally, we identify the lifting classes with the Chern classes by means of Proposition 6.8. The map $(c_1, c_2) : \mathscr{V}_2(X) \to CH^1(X) \times CH^2(X)$ is therefore a bijection: it is injective since the arguments above show that a vector bundle is uniquely determined by its lifting classes, alias Chern classes, and it is surjective since, given the data of Chern classes, we can build a map $X \to BGL_2^{(2)}$ by the obstruction theory arguments above, and such a map extends uniquely to a vector bundle on X. □

Classification of rank 3 bundles

Theorem 6.11. *If k is an algebraically closed field having characteristic unequal to 2, and X is a smooth affine 3-fold, then the map sending a rank 3 vector bundle to its Chern classes determines a bijection between the pointed set of isomorphism classes of rank 3 bundles on X and the set $CH^1(X) \times CH^2(X) \times CH^3(X)$.*

Proof. Again, we fix a class $\xi : X \to BGL_3^{(1)} = B\mathbf{G}_m$, which corresponds to a class in $CH^1(X)$. As in the proof of Theorem 6.10, the obstruction classes vanish, so it suffices to understand the relevant lifts. The next lift is an element of $H^2(X, \boldsymbol{\pi}_2^{\mathbb{A}^1}(BGL_3)) = H^2(X, \mathbf{K}_2^Q) = CH^2(X)$. The subsequent lift is an element of $H^3(X, \boldsymbol{\pi}_3^{\mathbb{A}^1}(BGL_3))$. However, we know that $\boldsymbol{\pi}_3^{\mathbb{A}^1}(BGL_3)$ is an extension of $\mathbf{K}_3^Q$ by $\mathbf{S}_4$, where $\mathbf{S}_4$ is a quotient of $\mathbf{K}_4^M/6$. We know that $H^3(X, \mathbf{K}_4^M/6)$ vanishes by Proposition 5.10, and for reasons of cohomological dimension, it follows that $H^3(X, \mathbf{S}_4) = 0$. Therefore, there is an isomorphism $H^3(X, \boldsymbol{\pi}_3^{\mathbb{A}^1}(BGL_3)) \xrightarrow{\sim} H^3(X, \mathbf{K}_3^Q) = CH^3(X)$.

Thus, the set of pointed $\mathbb{A}^1$-homotopy classes of maps is in bijection with the set in the statement. We claim that the induced action of $\mathbf{G}_m(k)$ on $CH^i(X)$ is trivial for $i = 1, 2, 3$. To see this, observe that for each i, since k is algebraically closed, the action of $\mathbf{G}_m(k)$ on $H^i(X, \boldsymbol{\pi}_i^{\mathbb{A}^1}(BGL_3))$ factors through $H^i(X, \mathbf{K}_i^Q)$ and this factorization is induced by the stabilization map $\boldsymbol{\pi}_i^{\mathbb{A}^1}(BGL_3) \to \boldsymbol{\pi}_i^{\mathbb{A}^1}(BGL_{3+j})$. Thus, it suffices to check that the action is trivial in the stable setting, in which case the triviality follows from 6.9.

The identification of the first two lifting classes with Chern classes follows from Proposition 6.8. In-so-far as the third Chern class is concerned, we proceed as follows. We know $o_{3,3}$ is a multiple of c_3 in the universal setting and since the lifting class on X is a pull-back of $o_{3,3}$ by means of an $\mathbb{A}^1$-homotopy class of maps $f : X \to BGL_3$ we deduce that $f^*o_{3,3}$ is a *fixed* multiple of $f^*(c_3)$. Since X is a smooth affine 3-fold and k is algebraically closed, we know that $CH^3(X)$ is uniquely divisible (again, see [Sri89]). Therefore, by dividing by the fixed multiple if necessary, we can replace the third lifting class by c_3. Given these observations, the proof can be concluded as in the rank 2 case: the map $(c_1, c_2, c_3) : \mathscr{V}_3(X) \to CH^1(X) \times CH^2(X) \times CH^3(X)$ is injective since the Chern classes uniquely determine the lifting classes, and surjectivity follows since, given the data of Chern classes, obstruction theory tells us how to build a pointed map $X \to BGL_3$ with prescribed lifting classes. □

Remark 6.12. Suppose k is algebraically closed and has characteristic unequal to 2. Assume $d \geq 3$ is an odd integer. With more care in the description of the lifting classes, it should be possible to obtain classification results for rank d vector bundles on smooth affine k-folds of dimension d. In particular, if there is no $(d-1)!$-torsion in $CH^*(X)$, it should be the case that vector bundles of rank d are determined by their Chern classes.

Isomorphism vs. stable isomorphism

As a corollary of the above results, we get the following statement which is a strengthening of [Fas11, Theorem 5.4].

Corollary 6.13. *Let X be a smooth affine 3-fold and let E and E' be two vector bundles over X. Then E and E' are stably isomorphic if and only if they are isomorphic.*

Proof. We have to prove that stably isomorphic vector bundles are indeed isomorphic. If E has rank one, this is obvious. If E has rank ≥ 2, then since E and E' are stably isomorphic, they have the same Chern classes. When E has rank 2, the required isomorphism follows from Theorem 6.10. When E has rank 3, the required isomorphism follows from Theorem 6.11. When E has rank $n \geq 4$, then the resulting homotopy sheaves are already in the stable range. Since E and E' are stably isomorphic, the composite of the classifying maps $X \to BGL_n$ become isomorphic when composed with the stabilization morphism $BGL_n \to BGL_\infty$. Pick a base-point of X arbitrarily. The same obstruction theory arguments as above show that $[(X,x), BGL_n]_{\mathbb{A}^1} \longrightarrow [(X,x), BGL_{\infty,\infty}]$ is a bijection for X of homotopy dimension ≤ 3. This follows inductively from the observation that the $\mathbb{A}^1$-homotopy fiber of the morphism $BGL_n \to BGL_{n+1}$ is GL_{n+1}/GL_n, which is $\mathbb{A}^1$-$(n-1)$-connected by Theorem 2.6. $\square$

A On the sheaf $\mathbf{S}_n$

Recall that in Section 3 we defined the sheaf $\mathbf{S}_n$ as the cokernel of a morphism of sheaves $\mathbf{K}_n^Q \to \mathbf{K}_n^M$. Lemma 3.8 showed that, assume our base field F was infinite, upon taking sections over an extension field L/F, the morphism of the previous sentence coincides with the (functorial in L) homomorphism $K_n^Q(L) \to K_n^M(L)$ defined by Suslin (see 3.1). Since both sheaves are strictly $\mathbb{A}^1$-invariant, it followed from this observation that there was an epimorphism

$$\mathbf{K}_n^M/(n-1)! \to \mathbf{S}_n.$$

To prove this epimorphism is an isomorphism, it is necessary and sufficient to check that this is the case on sections over finitely generated extensions of F, i.e., the above morphism is an isomorphism if and only if, for every finitely generated extension L/F, Suslin's homomorphism $K_n^Q(L) \to K_n^M(L)$ has image precisely equal to $(n-1)!K_n^M(L)$; the question of whether this is so is what we will refer to as Suslin's question (in degree n).

In what follows, we will study Suslin's homomorphism in greater detail by considering the canonical homomorphism $K_n^M(L) \to K_n^Q(L)$. Under the identification of Milnor K-theory with an appropriate motivic cohomology group, this latter homomorphism can be thought of as an edge map in the motivic spectral sequence, as was observed originally (to our knowledge) by T. Geisser and M.

Levine. One can give explicit conditions under which Suslin's homomorphism has image precisely equal to $(n-1)!K_n^M$, but these hypotheses are rather cumbersome except for small values of n. We describe an "absolute rigidity conjecture" for a certain motivic cohomology group, apparently posed by Suslin, that, as Sasha Merkurjev explained to us, guarantees that $\mathbf{S}_4 \cong \mathbf{K}_4^M/6$.

The motivic spectral sequence

Recall that there is a spectral sequence with $E_2^{p,q} = H^{p-q}(\operatorname{Spec} L, \mathbb{Z}(-q))$ that converges to $K^Q_{-p-q}(L)$ [Sus03, FS02, Lev08]. Here, since the complexes $\mathbb{Z}(-q)$ are trivial for $-q < 0$, and since $H^{p-q}(\operatorname{Spec} L, \mathbb{Z}(-q))$ vanishes if $p-q > -q$, it follows that this is a third quadrant spectral sequence. The complex $\mathbb{Z}$ has cohomology in degree 0 only, and the complex $\mathbb{Z}(1)$ has cohomology only in degrees 0 and 1. Since the motivic cohomology groups $H^n(\operatorname{Spec} L, \mathbb{Z}(n)) = K_n^M(L)$, it follows that $E_2^{0,-n} = K_n^M(L)$.

Because of the Voevodsky-Rost solution [Voe03, Voe11] to the Bloch-Kato conjecture, which implies the Beilinson-Lichtenbaum conjecture by work of Suslin-Voevodsky, much more is known about the groups that appear in the E_2-page of the motivic spectral sequence.

Lemma A.1. *If $p-q \leq 0$, then $E_2^{p,q}$ is uniquely divisible (except for $p = q = 0$).*

Proof. The coefficient sequence $\mathbb{Z} \hookrightarrow \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ induces morphisms of motivic cohomology

$$H^{i-1,j}(X, \mathbb{Q}/\mathbb{Z}) \longrightarrow H^{i,j}(X, \mathbb{Z}) \longrightarrow H^{i,j}(X, \mathbb{Q}) \longrightarrow H^{i,j}(X, \mathbb{Q}/\mathbb{Z})$$

Now, $\mathbb{Q}/Z$ is isomorphic to the product of its p-primary components. The p-primary components is the direct limit of $\mathbb{Z}/p^n$. Now, if $i < j$, the Bloch-Kato conjecture says that motivic cohomology with $\mathbb{Z}/p^n$ coefficients is isomorphic to étale cohomology. If $i < 0$, then the corresponding étale cohomology group vanishes since étale cohomology in negative degrees is trivial. Applying these observations, we deduce that $E^{p,q}$ is uniquely divisible for $p - q < 0$ and torsion free for $p - q \leq 0$. It remains to prove that when $p - q = 0$ that $E^{p,q}$ is divisible.

The universal coefficient theorem gives a commutative diagram of the form

$$\begin{array}{ccccc}
0 & \longrightarrow & H^{0,q}(L, \mathbb{Z})/\ell & \longrightarrow & H^{0,q}(L, \mathbb{Z}/\ell) \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^{0,q}(L, \mathbb{Z}_\ell)/\ell & \longrightarrow & H^{0,q}(L, \mathbb{Z}/\ell)
\end{array}$$

where the horizontal maps are injective and the right vertical map is an equality.

We may assume without loss of generality that L is finitely generated over the prime field [MVW06, Lemma 3.9]. Now, the Beilinson-Lichtenbaum conjecture implies that the map $\mathbb{Z}_\ell(i) \to B_\ell(i)$ is a quasi-isomorphism (the latter is a limit of truncations of μ_ℓ restricted to the Nisnevich site). In particular, since L has only finitely many roots of unity, it follows that $H^0(L, B_\ell(q)) = 0$. As a consequence, $H^{0,q}(L, \mathbb{Z}_\ell) = 0$. It follows that $H^{0,q}(L, \mathbb{Z})$ is ℓ-divisible, and since ℓ was arbitrary, it follows that $H^{0,q}(L, \mathbb{Z})$ is divisible. $\square$

Thus, the E_2-page of the spectral sequence then takes the form

$$\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & \mathbb{Z} \\
0 & 0 & 0 & 0 & 0 & K_1^M(L) \\
E^{-5,-2} & E^{-4,-2} & E^{-3,-2} & E^{-2,-2} & E^{-1,-2} & K_2^M(L) \\
E^{-5,-3} & E^{-4,-3} & E^{-3,-3} & E^{-2,-3} & E^{-1,-3} & K_3^M(L) \\
E^{-5,-4} & E^{-4,-4} & E^{-3,-4} & E^{-2,-4} & E^{-1,-4} & K_4^M(L)
\end{array}$$

where all the terms $E_2^{p,q}$ on and above the line $p = q$ are uniquely divisible.

The next result is a consequence of the multiplicative structure of the motivic spectral sequence, which is established in [FS02, Theorem 15.5]; the result was originally established in [GL00, Proposition 3.3].

Lemma A.2 (Geisser-Levine)**.** *The edge homomorphism $K_n^M(L) \to K_n^Q(L)$ is the homomorphism induced by the natural isomorphism $K_1^M(L) \xrightarrow{\sim} K_1^Q(L)$ and compatibility with products.*

Suslin's question in degree 4 and motivic cohomology

We now consider Suslin's question when $n = 4$. In that case, Suslin's question has a positive answer, i.e., the map $K_4^M(L)/6 \to \mathbf{S}_4(L)$ is an isomorphism, if and only if the reduction modulo 6 map $K_4^Q(L)/6 \to K_4^M(L)/6$ is the trivial map. Using the motivic spectral sequence, we will factor this map through a different motivic cohomology group.

To unburden the already suffering notation, we drop the superscript Q used to denote Quillen K-theory. The description of the motivic spectral sequence above implies that the filtration on K_4 has 3 non-trivial steps: $F^{-2}K_4(L)/F^{-1}K_4(L) = E_\infty^{-2,-2}$, $F^{-1}K_4(L)/F^0K_4(L) = E_\infty^{-1,-3}$, and $F^0K_4(L)/F^1K_4(L) \cong E_\infty^{0,-4}$.

From the above picture it is clear that $K_4^M(L)$ surjects onto $E_\infty^{0,-4}$. To analyze the remaining terms we will need some additional information about the differentials in the motivic spectral sequence. Classically, it is known that the differentials in the Atiyah-Hirzebruch spectral sequence are torsion, and Soulé established a motivic analog of this fact. More precisely, Soulé constructed an action of Adams operations on the motivic spectral sequence that is compatible with the differentials (see [GS99, §7.1] for one construction).

Lemma A.3 (Soulé [Sou]; see, e.g., [PW01, Lemma 1.3])**.** *For any given r, there exists an integer M, independent of p and q such that $M \cdot d^r_{p,q} = 0$.*

The only differential incident on $E_2^{-1,-3}$ comes from $E_2^{-3,-2}$, and all outgoing differentials are trivial. By Lemma A.1 $E_2^{-3,-2}$ is a uniquely divisible group, so by Lemma A.3 the differential incident on $E_2^{-1,-3}$ is trivial. All higher differentials are trivial, and as a consequence we deduce that $H^{2,3}(L,\mathbb{Z}) = E_2^{-1,-3} = E_\infty^{-1,-3}$. Similarly, the group $E_2^{-2,-2}$ is uniquely divisible, again by Lemma A.1. There are no non-trivial incoming differentials, and the outgoing differential is trivial by Lemma A.3. All higher differentials are trivial and therefore $E_2^{-2,-2} = E_\infty^{-2,-2}$.

The image of the homomorphism $K_4^M(L) \to K_4(L)$ is precisely $F^0K_4(L)$, which is a subgroup of $F^{-1}K_4(L)$. The composite map $K_4^M(L) \to K_4(L) \to K_4^M(L)$ is trivial when reduced modulo 6 by Suslin's theorem. Therefore, the map $K_4(L)/6 \to K_4^M(L)/6$ induced by reducing Suslin's homomorphism modulo 6 factors through $(F^{-2}K_4(L)/F^0K_4(L))/6$.

Now, there is a short exact sequence of the form

$$0 \longrightarrow H^{2,3}(L,\mathbb{Z}) \longrightarrow F^{-2}K_4(L)/F^0K_4(L) \longrightarrow H^{0,2}(L,\mathbb{Z}) \longrightarrow 0.$$

Since the group $H^{0,2}(L,\mathbb{Z})$ is uniquely divisible, reducing modulo 6 yields an isomorphism

$$H^{2,3}(L,\mathbb{Z})/6 \xrightarrow{\sim} F^{-2}K_4(L)/F^0K_4(L)/6.$$

and so the morphism $K_4(L)/6 \to K_4^M(L)/6$ factors through a morphism

$$H^{2,3}(L,\mathbb{Z})/6 \longrightarrow K_4^M(L)/6.$$

Therefore, the homomorphism $K_4(L)/6 \to K_4^M(L)/6$ is trivial if and only if the factored map $H^{2,3}(L)/6 \to K_4^M(L)/6$ is trivial.

Suslin's question in degree 4: number fields

We now show that Suslin's question has a positive answer for "small" fields.

Lemma A.4. *If F is a number field, the canonical map $K_4(F)/6 \to K_4^M(F)/6$ is trivial. In particular, the factorized map $H^{2,3}(F)/6 \to K_4^M(F)/6$ is trivial.*

Proof. If F is a number field, then write r for the number of real embeddings of F. The maps $F \to \mathbb{R}$ induced by the various real embeddings yield a morphism

$$K_i^M(F) \longrightarrow \prod_r K_i^M(\mathbb{R}).$$

Now, by results of Bass-Tate, we know that for $i \geq 3$, that $K_i^M(F)$ is finitely generated and that the above homomorphism induces an isomorphism $K_4^M(F) \cong (\mathbb{Z}/2)^r$ [BT73, Theorem II.2.1]. In particular, the reduction modulo 6 map determines an isomorphism

$$K_i^M(F)/6 \xrightarrow{\sim} \prod_{i=1}^{r} K_i^M(\mathbb{R})/6 \cong (\mathbb{Z}/2)^r.$$

On the other hand, we know that $K_4(\mathbb{R})$ is uniquely divisible [Wei12, Chapter VI.3] so $K_4(\mathbb{R})/6$ is the trivial group. By functoriality of Suslin's homomorphism, it follows that the map $K_4(F)/6 \to K_4^M(F)/6$ factors through $\prod_{i=1}^r K_4(\mathbb{R})/6 = 0$. The second statement is an immediate consequence of the first by the discussion above. $\square$

Suslin's question in degree 4 and absolute rigidity

If L is a field, let L_c denote the algebraic closure of the prime field in L (the field L_c is sometimes called the *field of absolute constants of* L). If we fix a base field F, we will say that a (covariant) functor $\mathcal{F}$ on the category of finitely generated extensions L/F is *absolutely rigid* if the map $\mathcal{F}(L_c) \to \mathcal{F}(L)$ is an isomorphism. Sasha Merkurjev attributed the following question about absolute rigidity to Suslin, though it has not appeared in print; for additional context, the reader may consult, e.g., [Sus87a, Conjecture 5.4], where related questions are posed in the context of the study of K_3.

Question A.5 (Suslin)**.** *If F is a fixed base-field, is the functor $L \mapsto H^{2,3}(L, \mathbb{Z})$ on the category of finitely generated extensions L/F absolutely rigid?*

Since we have established that Suslin's question in degree 4 has a positive answer in the case of a number field, a positive answer to the absolute rigidity conjecture implies a positive answer to Suslin's question in degree 4 in general. More precisely, the next result is an immediate corollary of functoriality of the homomorphism $H^{2,3}(L, \mathbb{Z})/6 \to K_4^M(L)/6$ with respect to L and the computations we recalled above for finite fields or number fields.

Corollary A.6. *If F is a fixed base-field (assumed to have characteristic 0) and the functor $L \mapsto H^{2,3}(L, \mathbb{Z})$ is absolutely rigid, then for any field L, Suslin's homomorphism $K_4(L) \to K_4^M(L)$ has image precisely $6K_4^M(L)$.*

Remark A.7. By the universal coefficient sequence, there is a short exact sequence of the form

$$0 \longrightarrow H^{2,3}(L, \mathbb{Z})/6 \longrightarrow H^{2,3}(L, \mathbb{Z}/6) \longrightarrow H^{3,3}(L, \mathbb{Z})_6 \longrightarrow 0.$$

The group $H^{2,3}(L, \mathbb{Z}/6)$ is, by the Beilinson-Lichtenbaum conjecture (really, we just need weight 3), isomorphic to $H^2_{\text{ét}}(L, \mu_6^{\otimes 3})$. If L contains sixth roots of unity, then this group can be identified with the 6-torsion in the Brauer group of L and is therefore non-trivial in general. Note also that $H^{3,3}(L, \mathbb{Z})$ is $K_3^M(L)$. Furthermore, the map on the right hand side is induced by the integral Bockstein homomorphism $H^{2,3}(L, \mathbb{Z}/6) \to H^{3,3}(L, \mathbb{Z})$.

Remark A.8. Finally, we close with one comment about Suslin's question about $\mathbf{S}_n$. Suppose L is any field having characteristic coprime to $n!$, then the Bloch-Kato conjecture gives an isomorphism

$$K^M_{n+1}/n!(L) \xrightarrow{\sim} H^{n+1}_{\text{ét}}(L, \mu_{n!}^{\otimes n+1}).$$

Thus, if L has étale 2- and 3-cohomological dimension $< n+1$, then it follows that $K^M_{n+1}/n!(L)$ is trivial. Thus, $\mathbf{S}_{n+1}(L)$ is trivial under these hypotheses.

References

[AD08] A. Asok and B. Doran. Vector bundles on contractible smooth schemes. *Duke Math. J.*, 143(3):513–530, 2008. 3

[AD09] A. Asok and B. Doran. $\mathbb{A}^1$-homotopy groups, excision, and solvable quotients. *Adv. Math.*, 221(4):1144–1190, 2009. 12

[AF12a] A. Asok and J. Fasel. Algebraic vector bundles on spheres. 2012. *Preprint*, available at http://arxiv.org/abs/1204.4538. 2, 6

[AF12b] A. Asok and J. Fasel. Splitting vector bundles outside the stable range and homotopy theory of punctured affine spaces. 2012. *Preprint* available at http://arxiv.org/abs/1209.5631. 2

[AP71] J. Kr. Arason and A. Pfister. Beweis des Krullschen Durchschnittsatzes für den Wittring. *Invent. Math.*, 12:173–176, 1971. 38

[Bal05] P. Balmer. Witt groups. In *Handbook of K-theory. Vol. 1, 2*, pages 539–576. Springer, Berlin, 2005. 28, 34

[Bas64] H. Bass. K-theory and stable algebra. *Inst. Hautes Études Sci. Publ. Math.*, (22):5–60, 1964. 2, 3

[Bas74] H. Bass. Clifford algebras and spinor norms over a commutative ring. *Amer. J. Math.*, 96:156–206, 1974. 32

[Bha03] S. M. Bhatwadekar. A cancellation theorem for projective modules over affine algebras over C_1-fields. *J. Pure Appl. Algebra*, 183(1-3):17–26, 2003. 2

[Blo86] S. Bloch. Algebraic cycles and higher K-theory. *Adv. in Math.*, 61(3):267–304, 1986. 6

[BO74] S. Bloch and A. Ogus. Gersten's conjecture and the homology of schemes. *Ann. Sci. École Norm. Sup. (4)*, 7:181–201 (1975), 1974. 39

[Bro03] P. Brosnan. Steenrod operations in Chow theory. *Trans. Amer. Math. Soc.*, 355(5):1869–1903 (electronic), 2003. 33

[BS62] H. Bass and S. Schanuel. The homotopy theory of projective modules. *Bull. Amer. Math. Soc.*, 68:425–428, 1962. 2

[BT73] H. Bass and J. Tate. The Milnor ring of a global field. In *Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic (Proc. Conf., Seattle, Wash., Battelle Memorial Inst., 1972)*, pages 349–446. Lecture Notes in Math., Vol. 342. Springer, Berlin, 1973. 49

[BW02] P. Balmer and C. Walter. A Gersten-Witt spectral sequence for regular schemes. *Ann. Sci. École Norm. Sup. (4)*, 35(1):127–152, 2002. 34, 35

[DK12] A. M. Dhorajia and M. K. Keshari. A note on cancellation of projective modules. *J. Pure Appl. Algebra*, 216(1):126–129, 2012. 3

[DW59] A. Dold and H. Whitney. Classification of oriented sphere bundles over a 4-complex. *Ann. of Math. (2)*, 69:667–677, 1959. 3

[Fas08] J. Fasel. Groupes de Chow-Witt. *Mém. Soc. Math. Fr. (N.S.)*, (113):viii+197, 2008. 31

[Fas09] J. Fasel. The projective bundle theorem for I^j-cohomology. Preprint, 2009. 38, 39

[Fas11] J. Fasel. Stably free modules over smooth affine threefolds. *Duke Math. J.*, 156(1):33–49, 2011. 2, 46

[Fas12] J. Fasel. A degree map on unimodular rows. *J. Ramanujan Math. Soc.*, 27(1):23–42, 2012. 14

[FRS12] J. Fasel, R. A. Rao, and R. G. Swan. On stably free modules over affine algebras. *Publ. Math. Inst. Hautes Études Sci.*, 116(1):223–243, 2012. 3

[FS02] E. M. Friedlander and A. Suslin. The spectral sequence relating algebraic K-theory to motivic cohomology. *Ann. Sci. École Norm. Sup. (4)*, 35(6):773–875, 2002. 47, 48

[FS08] J. Fasel and V. Srinivas. A vanishing theorem for oriented intersection multiplicities. *Math. Res. Lett.*, 15(3):447–458, 2008. 32

[FS09] J. Fasel and V. Srinivas. Chow-Witt groups and Grothendieck-Witt groups of regular schemes. *Adv. Math.*, 221(1):302–329, 2009. 6, 29, 30, 32

[Gil02] S. Gille. On Witt groups with support. *Math. Ann.*, 322(1):103–137, 2002. 31

[Gil07a] S. Gille. The general dévissage theorem for Witt groups of schemes. *Arch. Math. (Basel)*, 88(4):333–343, 2007. 30, 31

[Gil07b] S. Gille. A graded Gersten-Witt complex for schemes with a dualizing complex and the Chow group. *J. Pure Appl. Algebra*, 208(2):391–419, 2007. 38

[GJ09] P. G. Goerss and J. F. Jardine. *Simplicial homotopy theory*. Modern Birkhäuser Classics. Birkhäuser Verlag, Basel, 2009. Reprint of the 1999 edition [MR1711612]. 40

[GL00] T. Geisser and M. Levine. The K-theory of fields in characteristic p. *Invent. Math.*, 139(3):459–493, 2000. 48

[GS99] H. Gillet and C. Soulé. Filtrations on higher algebraic K-theory. In *Algebraic K-theory (Seattle, WA, 1997)*, volume 67 of *Proc. Sympos. Pure Math.*, pages 89–148. Amer. Math. Soc., Providence, RI, 1999. 48

[Gui89] D. Guin. Homologie du groupe linéaire et K-théorie de Milnor des anneaux. *J. Algebra*, 123(1):27–59, 1989. 19

[Hor02] J. Hornbostel. Constructions and dévissage in Hermitian K-theory. *K-Theory*, 26(2):139–170, 2002. 28

[Hor05] J. Hornbostel. $\mathbb{A}^1$-representability of hermitian K-theory and Witt groups. *Topology*, 44:661–687, 2005. 11

[Hov99] M. Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999. 7, 8

[HT10] K. Hutchinson and L. Tao. Homology stability for the special linear group of a field and Milnor-Witt K-theory. *Doc. Math.*, (Extra volume: Andrei A. Suslin sixtieth birthday):267–315, 2010. 26

[Kar73] M. Karoubi. Périodicité de la K-théorie hermitienne. In *Algebraic K-theory, III: Hermitian K-theory and geometric applications (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 301–411. Lecture Notes in Math., Vol. 343. Springer, Berlin, 1973. 28

[Kar80] M. Karoubi. Le théorème fondamental de la K-théorie hermitienne. *Ann. of Math. (2)*, 112(2):259–282, 1980. 28

[KM82] N. M. Kumar and M. P. Murthy. Algebraic cycles and vector bundles over affine three-folds. *Ann. of Math. (2)*, 116(3):579–591, 1982. 2

[Lev08] M. Levine. The homotopy coniveau tower. *J. Topol.*, 1(1):217–267, 2008. 47

[Lin82] H. Lindel. On the Bass-Quillen conjecture concerning projective modules over polynomial rings. *Invent. Math.*, 65(2):319–323, 1981/82. 3

[Mat69] H. Matsumoto. Sur les sous-groupes arithmétiques des groupes semi-simples déployés. *Ann. Sci. École Norm. Sup. (4)*, 2:1–62, 1969. 4

[Mil80] J. S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980. 39

[MK85] N. Mohan Kumar. Stably free modules. *Amer. J. Math.*, 107(6):1439–1444 (1986), 1985. 2

[Mor04] F. Morel. Sur les puissances de l'idéal fondamental de l'anneau de Witt. *Comment. Math. Helv.*, 79(4):689–703, 2004. 25, 38

[Mor05] F. Morel. The stable $\mathbb{A}^1$-connectivity theorems. *K-Theory*, 35(1-2):1–68, 2005. 15

[Mor06] F. Morel. $\mathbb{A}^1$-algebraic topology. In *International Congress of Mathematicians. Vol. II*, pages 1035–1059. Eur. Math. Soc., Zürich, 2006. 4

[Mor12] F. Morel. *$\mathbb{A}^1$-algebraic topology over a field*, volume 2052 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2012. 3, 4, 5, 6, 8, 9, 10, 12, 13, 15, 16, 32, 37, 40, 42, 43

[Mos11] L.-F. Moser. $\mathbb{A}^1$-locality results for linear algebraic groups [draft]. 2011. *Preprint*. 3, 8, 16

[MV99] F. Morel and V. Voevodsky. $\mathbb{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, (90):45–143 (2001), 1999. 3, 7, 9, 10, 11, 12, 16, 17, 20, 40, 41, 42

[MVW06] C. Mazza, V. Voevodsky, and C. Weibel. *Lecture notes on motivic cohomology*, volume 2 of *Clay Mathematics Monographs*. American Mathematical Society, Providence, RI, 2006. 47

[NS89] Yu. P. Nesterenko and A. A. Suslin. Homology of the general linear group over a local ring, and Milnor's K-theory. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53(1):121–146, 1989. 19

[OVV07] D. Orlov, A. Vishik, and V. Voevodsky. An exact sequence for $K_*^M/2$ with applications to quadratic forms. *Ann. of Math. (2)*, 165(1):1–13, 2007. 38, 39

[Pet59] F. P. Peterson. Some remarks on Chern classes. *Ann. of Math. (2)*, 69:414–420, 1959. 43

[PW01] C. Pedrini and C. Weibel. The higher K-theory of a complex surface. *Compositio Math.*, 129(3):239–271, 2001. 48

[PW10] I. Panin and C. Walter. On the motivic commutative ring spectrum $\mathbf{BO}$. 2010. available at `http://www.math.uiuc.edu/K-theory/0978/`. 11, 28

[Sch10a] M. Schlichting. Hermitian K-theory of exact categories. *J. K-Theory*, 5(1):105–165, 2010. 27, 28, 29

[Sch10b] M. Schlichting. The Mayer-Vietoris principle for Grothendieck-Witt groups of schemes. *Invent. Math.*, 179(2):349–433, 2010. 27, 29, 31

[Sch11] M. Schlichting. Hermitian K-theory, derived equivalences and Karoubi's fundamental theorem. Preprint available at `http://www.warwick.ac.uk/~masiap/research/prelim.html`, 2011. 28

[Ser94] J.-P. Serre. *Cohomologie galoisienne*, volume 5 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, fifth edition, 1994. 38

[SGA70] *Schémas en groupes. I: Propriétés générales des schémas en groupes*. Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 151. Springer-Verlag, Berlin, 1970. 20

[Sou] C. Soulé. Letter to Bloch and Lichtenbaum. Feb. 8, 1995. 48

[Sri89] V. Srinivas. Torsion 0-cycles on affine varieties in characteristic p. *J. Algebra*, 120(2):428–432, 1989. 44, 45

[Ste99] N. Steenrod. *The topology of fibre bundles*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1999. Reprint of the 1957 edition, Princeton Paperbacks. 3

[Sus77] A. A. Suslin. A cancellation theorem for projective modules over algebras. *Dokl. Akad. Nauk SSSR*, 236(4):808–811, 1977. 2

[Sus79] A. A. Suslin. The cancellation problem for projective modules and related topics. In *Ring theory (Proc. Conf., Univ. Waterloo, Waterloo, 1978)*, volume 734 of *Lecture Notes in Math.*, pages 323–338. Springer, Berlin, 1979. 2

[Sus82] A. A. Suslin. Cancellation for affine varieties. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 114:187–195, 222, 1982. Modules and algebraic groups. 2

[Sus84] A. A. Suslin. Homology of GL_n, characteristic classes and Milnor K-theory. In *Algebraic K-theory, number theory, geometry and analysis (Bielefeld, 1982)*, volume 1046 of *Lecture Notes in Math.*, pages 357–375. Springer, Berlin, 1984. 5, 15, 19, 26, 27

[Sus87a] A. A. Suslin. Algebraic K-theory of fields. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pages 222–244, Providence, RI, 1987. Amer. Math. Soc. 49

[Sus87b] A. A. Suslin. Torsion in K_2 of fields. *K-Theory*, 1(1):5–29, 1987. 4

[Sus03] A. Suslin. On the Grayson spectral sequence. *Tr. Mat. Inst. Steklova*, 241(Teor. Chisel, Algebra i Algebr. Geom.):218–253, 2003. 47

[Voe03] V. Voevodsky. Motivic cohomology with $\mathbf{Z}/2$-coefficients. *Publ. Math. Inst. Hautes Études Sci.*, (98):59–104, 2003. 47

[Voe11] V. Voevodsky. On motivic cohomology with $\mathbf{Z}/l$-coefficients. *Ann. of Math. (2)*, 174(1):401–438, 2011. 47

[Wal03] C. Walter. Grothendieck-Witt groups of triangulated categories. Preprint available at `www.math.uiuc.edu/K-theory/0643/`, 2003. 28

[Wei12] C. Weibel. *The K-book: An introduction to algebraic K-theory*. 2012. *Preprint* available at `http://www.math.rutgers.edu/~weibel/Kbook.html`. 49

[Wen10] M. Wendt. $\mathbb{A}^1$-homotopy of Chevalley groups. *J. K-Theory*, 5(2):245–287, 2010. 4

[Wen11] M. Wendt. Rationally trivial torsors in $\mathbb{A}^1$-homotopy theory. *J. K-Theory*, 7(3):541–572, 2011. 5, 8, 11, 12, 21, 23