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ELEMENTARY CONSTRUCTIONS OF EXCEPTIONAL GROUPS

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Abstract. The talk is devoted to two recent elementary representation theoretic approaches towards the construction of exceptional groups. Exposition will be as elementary as possible, and, ideally, should be accessible to anyone familiar with group theory and linear algebra, even those without any previous exposure to Lie algebras and representation theory.

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1. Introduction

Traditionally, exceptional groups are considered as an esoteric subject, and their conventional constructions require serious familiarity with representation theory of semi-simple Lie algebras and algebraic group theory. However, they naturally arise in various branches of mathematics, each time when one has to invoke classification of the finite simple groups, simple Lie groups, simple algebraic groups, or the like, each time when one mentions non-associative algebras, exceptional geometries, etc. In the last decades exceptional groups made their triumphal appearance in physics, not only in string theory, but also in most classical subjects. Each day there are more evidence, that the group E_8 is responsible for the Theory of Everything.

Moreover, a working knowledge of exceptional groups is fundamental even for a better understanding of classical groups. Jacques Tits remarked that whenever we wish to *really* understand some phenomenon for the group $\mathrm{GL}(n,K)$, we should come up with such a proof that covers also the case of E_8.

In this talk I will outline an elementary purely combinatorial approach towards construction of the exceptional groups of types E_6, E_7, E_8 and

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F_4, and calculations therein, which should be accessible to an undergraduate student. Also, I will sketch some recent applications of these methods.

Essentially, we can think about these groups, as certain groups of 27×27, 56×56, 248×248 and 27×27 matrices (sic!), more or less in the same way as we think about classical groups. With some technical tools, and a little practice, you can do in exceptional groups everything you can do in the classical ones.

Since *all* exceptional objects in mathematics are interrelated, and form *octonionic* mathematics — the one, which Vladimir Arnold forgot in his subdivision of mathematics into the real, the complex, and the quaternionic ones! — the contents of this talk is of general mathematical interest.

2. Elementary constructions of exceptional groups

In fact, first such elementary constructions were proposed by Dickson (1901), Chevalley (1948), and Freudenthal (1952), and in characteristics $\neq 2, 3$ were widely used by the Belgian and the Dutch school (Tits, Springer, Feldkamp, et al.), starting from 1950-ies. Later, they were overshadowed by the general methods of the French school, and are far less known, than they deserve.

However, recently, in the process of solution of some *really* challenging problems, this subject came to light again. This applies to the study of maximal subgroups (Aschbacher, Cohen, Cooperstein, and others), classification of forms of simple groups over a non-closed field (Rost, Garibaldi, an others), etc.

Recently, Lurje, Luzgarev and the author succeeded in removing remaining restrictions on characteristic in the minimal and adjoint representation of exceptional groups, and define them by explicit equations over $\mathbb{Z}$. We describe the following constructions:

• Explicit combinatorial description of the elementary generators $x_\alpha(\xi)$ as matrices;

• Realisation of exceptional groups as isometry groups of cubic/quartic forms, and explicit equations on matrix entries of their elements,

both of which can be fully appreciated by a 2nd year undergraduate student. In this sense, THE TALK WILL BE MUCH MORE ELEMENTARY THAN THE PRESENT EXPOSITION. I will try also to outline some further related constructions and recent applications.

3. Basic notation

Let us fix basic notation. This notation is explained in [33, 49, 34], where one can also find many further references.

• Φ is a reduced irreducible root system.

• Fix an order on Φ, let Φ^+, Φ^- and $\Pi = \{\alpha_1, \dots, \alpha_l\}$ are the sets of positive, negative and fundamental roots, respectively.

• Let $W = W(\Phi)$ be the Weyl group of Φ.

• Let $Q(\Phi)$ be the root lattice of Φ, $P(\Phi)$ be the weight lattice of Φ and P be any lattice such that $Q(\Phi) \leq P \leq P(\Phi)$.

• R is a commutative ring with 1;

• $G = G_P(\Phi, R)$ is the Chevalley group of type (Φ, P) over R.

In most cases P does not play essential role and we simply write $G = G(\Phi, R)$ for any Chevalley group of type Φ over R. However, when the answer depends on P we usually write $G_{\text{sc}}(\Phi, R)$ for the simply connected group, for which $P = P(\Phi)$ and $G_{\text{ad}}(\Phi, R)$ for the adjoint group, for which $P = Q(\Phi)$.

• $T = T(\Phi, R)$ is a split maximal torus of G.

• $x_\alpha(\xi)$, where $\alpha \in \Phi$, $\xi \in R$, denote root unipotents G elementary with respect to T.

• $E(\Phi, R)$ is the [absolute] elementary subgroup of $G(\Phi, R)$, generated by all root unipotents $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$.

4. Chevalley groups versus elementary subgroups

Many authors not familiar with algebraic groups or algebraic K-theory do not distinguish Chevalley groups and their elementary subgroups. Actually, these groups are defined dually.

• Chevalley groups $G(\Phi, R)$ are [the groups of R-points of] algebraic groups. In other words, $G(\Phi, R)$ is defined as

$$G(\Phi, R) = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], R),$$

where $\mathbb{Z}[G]$ is the affine algebra of G. By definition $G(\Phi, R)$ consists of solutions in R of certain algebraic equations.

• As opposed to that, *elementary* Chevalley groups $E(\Phi, R)$ are generated by elementary generators

$$E(\Phi, R) = \big\langle x_\alpha(\xi),\ \alpha \in \Phi,\ \xi \in R \big\rangle.$$

When $R = K$ is a field, one knows relations among these elementary generators, so that $E(\Phi, R)$ can be defined by generators and relations. However, in general, the elementary generators are described by their action in certain representations.

By the very construction of these groups $E(\Phi, R) \leq G(\Phi, R)$ but in general $E(\Phi, R)$ can be strictly smaller than $G(\Phi, R)$ even for fields.

5. Classical groups

Actually, classical Chevalley groups can be easily constructed as groups of isometries of bilinear/quadratic forms. Recall identification of Chevalley groups and elementary Chevalley groups for the classical cases. The second column of the following table lists traditional notation of classical groups, according to types: A_l the special linear group, B_l the odd orthogonal group, C_l the symplectic group, and D_l the even orthogonal group. These groups are defined by algebraic equations. Orthogonal groups are not simply connected, the corresponding simply connected groups are the spin groups. Orthogonal groups [and spin groups] in this table are the *split* orthogonal groups. The last column lists the names of their elementary subgroups.

Φ	$G(\Phi, R)$	$E(\Phi, R)$
A_l	$\mathrm{SL}(l+1, R)$	$E(l+1, R)$
B_l	$\mathrm{Spin}(2l+1, R)$	$\mathrm{Epin}(2l+1, R)$
	$\mathrm{SO}(2l+1, R)$	$\mathrm{EO}(2l+1, R)$
C_l	$\mathrm{Sp}(2l, R)$	$\mathrm{Ep}(2l, R)$
D_l	$\mathrm{Spin}(2l, R)$	$\mathrm{Epin}(2l, R)$
	$\mathrm{SO}(2l, R)$	$\mathrm{EO}(2l, R)$

6. Minimal modules

Our constructions rely on combinatorics and geometry of representation theory.

- $\varpi_1, \ldots, \varpi_l$ are the fundamental weights;
- $P_{++}(\Phi)$ is the cone of dominant integral weights. Every weight $\omega \in P_{++}(\Phi)$ is a non-negative integral linear combination of $\varpi_1, \ldots, \varpi_l$.
- Let us fix a dominant weight $\omega \in P_{++}(\Phi)$ and let $V = V(\omega)$ be the Weyl module of the group G with the highest weight ω.
- The corresponding representation $G \longrightarrow \mathrm{GL}(V)$ will be denoted by $\pi = \pi(\omega)$.
- By $\Lambda(\pi) = \Lambda(\omega)$ one denotes the set of weights of the representation π *with multiplicities*.
- One can choose in V an *admissible base* v^λ, $\lambda \in \Lambda(\omega)$, consisting of weight vectors (i.e. v^λ is in fact a vector of weight λ, when one considers λ as a weight in the usual sense, *without multiplicities*), such that the action of the root unipotents $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$, is descrived by the matrices, whose entries are polynomials in ξ with integer coefficients.

We are mostly interested in *minimal modules*:

• *microweight representations*, such that $\Lambda(\omega) = W(\Phi)\omega$ is a single Weyl orbit;

• *short root representations*, such that $\Lambda(\omega)$ is the union of the set Φ_s of short roots, and $|\Phi_s \cap \Pi|$ copies of zero weight 0_α, $\alpha \in \Phi_s \cap \Pi$. For simply laced systems these are precisely the adjoint representations.

In the sequel we envisage elements of the Chevalley group $G(\Phi, R)$ as matrices $g = (g_{\lambda\mu})$, $\lambda, \mu \in \Lambda$, with respect to the base v^λ, $\lambda \in \Lambda$. In other words, the columns of this matrix are the coordinate columns of the vectors $gv^\mu = \pi(g)v^\mu$, $\mu \in \Lambda$, with respect to the base v^λ, $\lambda \in \Lambda$. The μ-th column of the matrix g is denoted by $g_{*\mu}$, while the λ-th row of this matrix is denoted by $g_{\lambda*}$.

7. Weight diagrams

Weight diagrams were introduced by Dynkin and Vinberg in the 1950-ies, but never made their way to the published works of the Moscow school. Later they appeared in combinatorial context, and in various problems of the theory of algebraic groups and Lie algebras. Later still, in the work of Matsumoto and Stein [20, 28], they became a working tool in the study of exceptional groups over rings and their K-theory. From the present viewpoint, they are a special case of the crystal graphs of Lusztig and Kashiwara.

We show that the weight diagrams in fact encode all information about Lie algebras, and the correspodning groups: signs of structure constants, action of elementary generators, defining equations, multilinear invariants, ..., and a lot more.

The *weight diagram* of a microweight/short root/adjoint representation π is a colour graph constructed as follows:

• It vertices correspond to $\Lambda(\pi)$;

• Two non-zero weights $\lambda, \mu \in \Lambda(\pi)$ are joined by an arrow from μ to λ, of colour i, if $\lambda - \mu = \alpha_i$;

• The zero weight 0_{α_i} is joined to $\pm\alpha_i$, by arrows of colour i.

One can find weight digrams of all minimal and adjoint modules in [26]. The ones relevant for our constructions of exceptional groups are weight diagrams (E_6, ϖ_1), (E_7, ϖ_7), (E_6, ϖ_2), (E_7, ϖ_1), (E_8, ϖ_8), which can be found also in [33, 34, 37].

A *weight graph* of a representation is defined similarly, only that now colours correspond to *all* positive roots, rather than the just fundamental ones. Below, the distance $d(\lambda, \mu)$ between tow weight $\lambda, \mu \in \Lambda(\pi)$, always refers to their distance in the weight *graph*. In particular, $d(\lambda, \mu) = 1$ means that $\lambda - \mu$ is a root. Similarly, $d(\lambda, \mu) = 2$ means that $\lambda - \mu$ is the sum of two roots (but $\lambda \neq \mu$ and $\lambda - \mu \notin \Phi$).

8. Construction of $E(\Phi, R)$

For simplicity, let us describe the technically easier case of microweight representations. In this case, weight diagrams describe the action of root unipotents $x_\alpha(\xi)$ *up to sign*. Namely, let $\alpha = \sum m_i\alpha_i$ be the linear expansion of α with respect to the fundamental roots. Then it suffices to find all pairs (λ, μ) of weights $\lambda, \mu \in \Lambda(\pi)$, joined [in positive direction] by a path with m_1 arrows of colour α_1,...,m_l arrows of colour α_l, in any order. Then, by the formula $x_\alpha(\xi)v^\lambda = v^\lambda + c_{\lambda\alpha}\xi v^{\lambda+\alpha}$, action of $x_\alpha(\xi)$ adds v^λ to or or subtracts it from v^μ.

Thus, the only remaining problem is to determine the signs of structure constants $c_{\lambda\alpha}$. Various such algorithms were known for quite some time, see references in [36, 37].

However, as observed in [34, 37], this can be done directly from the weight diagram. The following result is a special case of results by Lusztig and Kashiwara on crystal bases, but in [34] and [37] I gave two entirely elementary proofs, on based on the theory of Lie algebras, and another one purely combinatorial, based on the results of [21].

Theorem 1. *Let (V, π) be a microweight representation. Then there exists an admissible base v^λ, $\lambda \in \Lambda(\pi)$, such that $c_{\lambda\alpha} = +1$ for all fundamental and negative fundamental roots α.*

Now we can describe an easy purely combinatorial algortihm to calculate $c_{\lambda\alpha}$ inductively. Define the *canonical string* of a root $\alpha \in \Phi^+$ as follows. The canonical string of a fundamental root α_i is i. If the height of α is ≥ 2 and α_i is the smallest fundamental root such that $\alpha - \alpha_i \in \Phi$, then the canonical string of α is obtained from the canonical string of $\alpha - \alpha_i$ by appending i on the left. For example, the canonical strings of the maximal roots of E_6 and E_7 are 24315423456 and 13425431654234567, respectively.

Now to calculate $c_{\lambda\alpha}$ we proceed as follows. Let $i_1 \dots i_h$, be the canonical string of α. We search for a path in the *negative* direction starting at $\lambda_1 = \mu = \lambda + \alpha$ which has labels $i_1, \dots, i_h$ in the same order. Such a path does not necessarily exist. If there is a bond labeled i_1 hanging on λ_1 in the negative direction, we set $\lambda_2 = \lambda_1 - \alpha_{i_1}$, otherwise we say that i_1 is *nasty* for λ and set $\lambda_2 = \lambda_1$. We proceed like this until we get to the end of the canonical string. Let $n = n(\alpha, \lambda)$ be the number of labels in the canonical string of α nasty for λ.

Theorem 2. *Let v^λ be an admissible base satisfying conclusion of Theorem* 1. *Then for all $\alpha \in \Phi^+$ and all $\lambda \in L(\pi)$ such that $\lambda + \alpha \in \Lambda(\pi)$ one has $c_{\lambda\alpha} = (-1)^{n(\alpha,\lambda)}$.*

Clearly, Theorems 1 and 2 allow to completely restore the elementary generators of the group $E(\Phi, R)$ in a microweight representation $V =$

$V(\omega)$ directly from the graph of that representation. The resulting tables for (E_6, ϖ_1) and (E_7, ϖ_7) are reproduced in [47] and [44], respectively.

For adjoint representations, similar explicit formulas are classically known, and in [37] we describe, how to read the signs directly from the weight diagrams purely combinatorially, in the same style, as above.

Apart from that, weight diagrams encode, in a very compact manner wealth of information on the groups $E(\Phi, R)$ and $G(\Phi, R)$, including, for instance, explicit quadratic equations defining the highest weight orbit of these representations, see [38, 16, 17].

In fact, the *extended* Chevalley group $\overline{G}(\Phi, R)$ can be characterised as the largest subgroup of $\mathrm{GL}(V)$ consisting entirely of matrices, whose first columns satisfy this system of quadrics. In absolutely irreducible representations this group coincides with the normaliser of $G(\Phi, R)$ in $\mathrm{GL}(V)$. Within $\overline{G}(\Phi, R)$ is then described by another algebraic equation.

9. Construction of $G(\Phi, R)$

At the end of the preceding section we hinted to an algebraic description of the normaliser of the Chevalley group $G(\mathrm{E}_6, R)$ in $\mathrm{GL}(27, R)$. However, it does not answer the question, when does an *individual* matrix $g \in \mathrm{GL}(27, R)$ belong to $\overline{G}(\mathrm{E}_6, R)$?

Below Θ_0 denotes the set of *unordered* triads $\{\lambda, \mu, \nu\}$, where $d(\lambda, \mu) = (\lambda, \nu) = d(\mu, \nu) = 2$. To fix signs, we pick up the standard [*ordered*] triad

$$(\lambda_0, \mu_0, \nu_0) = \left({234321 \atop 2}, {012221 \atop 1}, {000001 \atop 0} \right).$$

Clearly, a triad is completely determined by any two of its elements. In other words, for any two weights λ, μ at distance 2 in the weight graph, there exists a *unique* weight $\nu = \lambda \circ \mu$ such that $(\lambda, \mu, \lambda \circ \mu)$ forms a triad. Thus, $|\Theta_0| = 45$.

Now the the cubic form Q on $V(\varpi_1)$ is defined as follows. For a vector $x = \sum x_\lambda v^\lambda$, set

$$Q(x) = \sum \mathrm{sign}(w) x_\lambda x_\mu x_\nu,$$

where the sum is taken over $\{\lambda, \mu, \nu\} \in \Theta_0$, while $w \in W(\mathrm{E}_6)$ is such that $w(\lambda_0, \mu_0, \nu_0) = (\lambda, \mu, \nu)$.

Further, let us introduce the following notation for the *polarisation* of a partial derivative of the cubic form Q:

$$f_\lambda(x, y) = F(e_\lambda, x, y) = \sum \mathrm{sign}(w) x_\mu y_{\lambda \circ \mu},$$

where F denotes the complete polarisation of Q. Here, the sum is taken over all weights ν such that $d(\lambda, \mu) = 2$, and $w \in W(\mathrm{E}_6)$ is chosen in such a way, that $w(\lambda_0, \mu_0, \nu_0) = (\lambda, \mu, \lambda \circ \mu)$. Clearly, this sign depends only on λ and μ themselves, $\mathrm{sign}(w) = (-1)^{h(\lambda, \mu, \lambda \circ \mu)}$.

Now, we are all set to state the main result of [43]. Its proof relies on the main results of Michael Aschbacher [2] for fields, on a Lie algebra calculation, and a lot of explicit fiddling with multilinear forms and matrix entries. But the final result is as explicit and elementary as one could expect, and can be taken as a *definition* of E_6.

Theorem 3. *For a matrix $g \in \mathrm{GL}(27, R)$ to belong to $\overline{G}(\mathrm{E}_6, R))$ it is necessary and sufficient that its entries satisfy the following equations.*

• **Equations on a pair of adjacent columns.** *For all $\lambda, \mu, \nu \in \Lambda$ such that $d(\mu, \nu) \leq 1$, one has*

$$f_\lambda(g_{*\mu}, g_{*\nu}) = 0.$$

• **Equations on two pairs of non-adjacent columns.** *For all $\lambda, \mu, \nu, \rho, \sigma, \tau \in \Lambda$ such that $d(\mu, \nu) = d(\sigma, \tau) = 2$, one has*

$$(-1)^{h(\mu\circ\nu,\mu,\nu)} g'_{\mu\circ\nu,\lambda} f_\rho(g_{*\sigma}, g_{*\tau}) = (-1)^{h(\sigma\circ\tau,\sigma,\tau)} g'_{\sigma\circ\tau,\rho} f_\lambda(g_{*\mu}, g_{*\nu}).$$

In his Thesis, as part of description of overgroups of $G(\mathrm{F}_4, R)$ in $G(\mathrm{E}_6, R)$, Alexander Luzgarev characterised $G(\mathrm{F}_4, R)$ by further explicit equations in $G(\mathrm{E}_6, R)$, see [14].

Recently, Alexander Luzgarev and myself [45] have made final touches to the proof of a similar result for [simply connected] Chevalley groups of type E_7. In this case explicit equations on entries of an element $g \in G_{\mathrm{sc}}(\mathrm{E}_7, R) \leq \mathrm{GL}(56, R)$ can be stated in terms of [some] bilinear and trilinear forms related to the *four-linear* invariants of $G_{\mathrm{sc}}(\mathrm{E}_7, R)$.

The remarkable new observation by Jacob Lurie and Luzgarev [13, 15], which allowed to completely remove all traces of the condition $2 \in R^*$, is that one has to simultaneously consider 4 such four-linear invariants, and these invariants themselves are not symmetric (but the space they generate is!). Recall, that Michael Aschbacher and Bruce Cooperstein [3, 4] used condition $2 \in R^*$ in a very crucial way.

We cannot reproduce these invariants here, since they consist of 19768 monomials, see the construction in [15] and [44] but it would only take a few more pages to reproduce their partial derivatives, and state an explicit analogue of Theorem 3, and this is done in [45, 17].

Quite recently, Alexander Luzgarev and myself obtained similar results also for adjoint representations, and thus, in particular, described $G(\mathrm{E}_8, R)$ by explicit equations of the same sort as in Theorem 3 above.

10. Final remarks

As described in [19], these elementary constructions can be used to provide extremely efficient ways to calculate in exceptional groups.

• A first such working geometric approach was DECOMPOSITION OF UNIPOTENTS, developped jointly by Alexei Stepanov, myself and Eugene Plotkin [33, 49, 31, 34].

• For computer implementations, see [36, 47, 44].

• Recently, together with Mikhail Gavrilovich, Sergei Nikolenko, and Alexander Luzgarev, we proposed another generation of geometric methods, THE PROOF FROM THE BOOK $\approx$ A_2-proof, which uses minimum information about the group [41, 42, 48, 46].

Together with the works by Douglas Costa and Gordon Keller [5], we put the last dots in the proof of structure theorems for exceptional groups, initiated by Eiichi Abe, Kazuo Suzuki, Giovanni Taddei, Leonid Vaserstein, and others. Combined with the papers based on localisation papers, by Anthony Bak, Roozbeh Hazrat, Victor Petrov, Alexei Stepanov, the author, and Zuhong Zhang (see [1, 6, 7, 8, 9, 10, 11, 32, 29, 30] and references there), this allowed us to establish structure theorems over *arbitrary* commutative rings, for all groups of rank ≥ 2, and many further remarkable generalisations and refinements of these theorems.

• Quite recently, building on the geometric methods, I have shown that they are even much more flexible and powerful, and that we can *easily* calculate with several columns and rows of a matrix from an exceptional group in a minimal representation, see, in particular, [35, 39, 40].

• Finally, let us mention a very broad ongoing project currently carried through by Anastasia Stavrova, Victor Petrov, Alexander Luzgarev and Ekaterina Kulikova, whose ultimate goal is to generalise these results and methods to all isotropic reductive groups [22, 23, 18, 12, 27].

The bibliography lists some works which treat exceptional groups in the spirit especially close to the the present talk, and some recent works by myself, my students, and collaborators, where one can find thorough description of the background, constructions of exceptional groups, various approaches to the proofs of structure theorems, and many further applications.

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