Splitting vector bundles outside the stable range and homotopy theory of punctured affine spaces

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Abstract

We discuss the relationship between the $\mathbb{A}^1$-homotopy sheaves of $\mathbb{A}^n \setminus 0$ and the problem of splitting off a trivial rank 1 summand from a rank n-vector bundle. We begin by computing $\boldsymbol{\pi}_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)$, and providing a host of related computations of "non-stable" $\mathbb{A}^1$-homotopy sheaves. We then use our computation to deduce that a rank 3 vector bundle on a smooth affine 4-fold over an algebraically closed field having characteristic unequal to 2 splits off a trivial rank 1 summand if and only if its third Chern class (in Chow theory) is trivial. This result provides a positive answer to a case of a conjecture of M.P. Murthy.

Contents

1 Introduction

This paper is motivated in part by the following classical question: if X is a smooth affine variety of dimension d over a field k, under what conditions does a rank r vector bundle on X split as the direct sum of a rank $r-1$ vector bundle and a free module of rank 1 (briefly: when does a rank r vector bundle split off a rank 1 trivial summand)? The main idea of this paper, which is the third in a series after [AF12b] and [AF12a], is to apply $\mathbb{A}^1$-homotopy theory to provide some new results regarding this splitting problem.

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The answer to the question posed in the previous paragraph depends on the relationship between r and d. For example, in 1958, Serre proved that if X is a connected affine scheme of Krull dimension d, then any vector bundle on X of rank $r > d$ is the direct sum of a vector bundle of rank d and a free module [Ser58, Théorème 1]. Answering the question when $r = d$ led to a torrent of work. It follows from the results of [MS76] that if X is a smooth affine surface over an algebraically closed field, then a rank 2 bundle on X splits off a trivial rank 1 summand if and only if its second Chern class in $CH^2(X)$ is zero. In [KM82, Corollary 2.4] Murthy and Mohan Kumar showed that if X is a smooth affine threefold over an algebraically closed field, then a rank 3 bundle $\mathcal{E}$ splits off a trivial rank 1 summand if and only if $0 = c_3(\mathcal{E}) \in CH^3(X)$. This result was subsequently generalized by Murthy [Mur94, Remark 3.6 and Theorem 3.7]: he showed (in particular) that if X is a smooth affine variety of dimension d over an algebraically closed field k, and if $\mathcal{E}$ is a rank d bundle on X, then $\mathcal{E}$ splits off a trivial rank 1 summand if and only if $0 = c_d(\mathcal{E}) \in CH^d(X)$.

In [Mor12], Morel revisited the splitting problem in terms of obstruction theory in the setting of $\mathbb{A}^1$-homotopy theory. Using his classification theorem for vector bundles on smooth affine schemes, he was able to recast the splitting problem in terms precisely analogous to the classical theory of the Euler class studied, e.g., in Milnor-Stasheff [MS74]. In particular, he showed that over an arbitrary perfect field k, there is an "Euler class" obstruction to splitting a trivial rank 1 summand off a bundle with trivial determinant (see also [Fas08] and [FS09] for $d = 2, 3$). When the base field k is algebraically closed, this Euler class is precisely the top Chern class of the bundle. Rank d vector bundles on smooth affine d-folds are "at the edge of the stable range." More precisely, over an algebraically closed field k, it follows from the computations of [AF12b, AF12a] that rank d vector bundles are determined by data that is essentially K-theoretic in nature (in fact, such a vector bundle can be specified by a sequence of elements in the Chow groups of X, though these elements are necessarily not arbitrary).

Vector bundles of rank $r < d$ on general smooth affine d-folds are "outside the stable range" even if k is algebraically closed. *A priori*, one might not expect to be able to make any reasonable statements about the structure of such vector bundles. Nevertheless, Murthy wrote that he did not know an example of a vector bundle $\mathcal{E}$ of rank $d - 1$ on a smooth affine d-fold over an algebraically closed field k such that $c_{d-1}(\mathcal{E}) = 0 \in CH^{d-1}(X)$ that does not split off a trivial rank 1 summand [Mur99, p. 173]. Following a long established tradition, we reformulate this observation as a conjecture.

Conjecture 1 (Murthy's splitting conjecture)**.** *If X is a smooth affine d-fold over an algebraically closed field k and $\mathcal{E}$ is a vector bundle of rank $d - 1$ over X, then $\mathcal{E}$ splits off a trivial rank 1 summand if and only if $c_{d-1}(\mathcal{E}) = 0$ in $CH^{d-1}(X)$.*

With one exception, we were not aware of any general (algebro-geometric) results regarding splitting vector bundles outside the stable range. In [AF12b] we proved that, given an algebraically closed field k and a smooth affine threefold over k, there is a unique rank 2 vector bundle on X with given c_1 and c_2; consequently, a rank 2 vector bundle on such a variety splits off a trivial rank 1 summand if and only if c_2 is trivial. In particular, Conjecture 1 is true under the additional assumptions that k has characteristic unequal to 2 and $d = 3$. In this work, we provide a solution to Conjecture 1 under the additional assumptions that k has characteristic unequal to 2 and $d = 4$.

Theorem 2. *If X is a smooth affine 4-fold over an algebraically closed field k having characteristic unequal to 2 and if $\mathcal{E}$ is a rank 3 vector bundle on X, then $\mathcal{E}$ splits off a trivial rank 1 summand if and only if $0 = c_3(\mathcal{E}) \in CH^3(X)$.*

This result and the one mentioned in the previous paragraph were deduced by the link between the splitting problem and $\mathbb{A}^1$-homotopy theory. To explain this, write BGL_n for the classifying space for GL_n-torsors (the reader is encouraged to think of an appropriate infinite Grassmannian). Write $\mathcal{H}(k)$ for the Morel-Voevodsky $\mathbb{A}^1$-homotopy category. Given any smooth scheme X, we write $[X, BGL_n]_{\mathbb{A}^1}$ for the set of morphisms in $\mathcal{H}(k)$ from X to BGL_n. Morel showed [Mor12] that the pointed set $[X, BGL_n]_{\mathbb{A}^1}$ is canonically in bijection with the set $\mathcal{V}_n(X)$ of isomorphism classes of rank n vector bundles on X (provided X is affine).

There is a canonical morphism $BGL_{n-1} \to BGL_n$ corresponding to the inclusion map $GL_{n-1} \to GL_n$ sending an invertible matrix M to the block diagonal $n \times n$-matrix with blocks M and 1. This morphism induces a map $[X, BGL_{n-1}]_{\mathbb{A}^1} \to [X, BGL_n]$ that sends a rank $n-1$ vector bundle $\mathcal{E}$ to the rank n vector bundle $\mathcal{E} \oplus \mathcal{O}_X$. Therefore, the splitting problem is equivalent to the following lifting question: given an element of $[X, BGL_n]_{\mathbb{A}^1}$, when can it be lifted to a morphism $[X, BGL_{n-1}]_{\mathbb{A}^1}$? By standard topology, the obstructions to existence of such a lift are governed by the structure of the ($\mathbb{A}^1$-)homotopy fiber of the above map $BGL_{n-1} \to BGL_n$. Morel then explicitly identified this $\mathbb{A}^1$-homotopy fiber by proving the existence of an $\mathbb{A}^1$-fiber sequence:

$$\mathbb{A}^n \setminus 0 \longrightarrow BGL_{n-1} \longrightarrow BGL_n.$$

By obstruction theory, understanding the lifting question is then tantamount to understanding the (unstable) $\mathbb{A}^1$-homotopy theory of $\mathbb{A}^n \setminus 0$. To provide a positive answer to Murthy's question for a given integer d, the above approach requires as input sufficiently detailed information about the $d-1$st $\mathbb{A}^1$-homotopy sheaf of $\mathbb{A}^{d-1} \setminus 0$. In particular, in [AF12b] we computed $\pi_2^{\mathbb{A}^1}(\mathbb{A}^2 \setminus 0)$. In this paper, we deduce the above result from a computation of $\boldsymbol{\pi}_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)$. For completeness, we record this computation here.

Theorem 3 (See Theorem 3.7 and Proposition 4.1)**.** *If k is an infinite perfect field having characteristic unequal to 2, there is a short exact sequence of the form*

$$0 \longrightarrow \mathbf{F}_5 \longrightarrow \boldsymbol{\pi}_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0) \longrightarrow \mathbf{GW}_4^3 \longrightarrow 0,$$

where $\mathbf{GW}_4^3$ is a sheafification of a certain Karoubi U-theory group for the Nisnevich topology, and $\mathbf{F}_5$ is a quotient of the sheaf $\mathbf{T}_5$ introduced in [AF12a, Theorem 2.3] *which is itself a fiber product of the form*

$$\begin{array}{ccc} \mathbf{T}_5 & \longrightarrow & \mathbf{I}^5 \\ \downarrow & & \downarrow \\ \mathbf{S}_5 & \longrightarrow & \mathbf{K}_5^M/2, \end{array}$$

and $\mathbf{S}_5$ admits an epimorphism from $\mathbf{K}_5^M/24$. Moreover, the epimorphism $\mathbf{T}_5 \to \mathbf{F}_5$ becomes an isomorphism after 4-fold contraction.

The space $\mathbb{A}^d \setminus 0$ is a motivic sphere, and the computation above, together with the parallels in algebraic topology, hint at an extraordinarily rich structure in its unstable $\mathbb{A}^1$-homotopy sheaves. The results above exemplify how this structure is reflected in the splitting problem for projective modules. We draw the reader's attention to some tantalizing features of the above computation. The Grothendieck-Witt sheaf $\mathbf{GW}^3_4$ that appears corresponds to the part of the $\mathbb{A}^1$-homotopy sheaf detected by the "degree" homomorphism in Hermitian K-theory, though we defer a detailed explanation of this connection to a subsequent paper [AF12c]. On the other hand, the kernel of the surjective map to the Grothendieck-Witt sheaf is closely related to the motivic version of the classical J-homomorphism (see Theorem 4.17). In a sense we will make precise (see Proposition 4.1 and Corollary 4.2), the 24 that appears is the "same" factor of 24 that intervenes in the third stable homotopy group of the classical sphere spectrum (see [Hu59, Theorem 16.4]): our computation therefore mixes together topological information about the unstable homotopy groups of spheres and arithmetic information about the base-field and its finitely generated extensions!

The factor of $\mathbf{I}^5$ appearing in Theorem 3 appears to be a purely unstable phenomenon (see Corollary 4.4 and Remark 4.5); detailed analysis of this phenomenon is deferred to [AF12d]. Up to this factor, the sheaf $\boldsymbol{\pi}_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)$ is an extension of two sheaves that are of "stable" provenance (in the sense of stable $\mathbb{A}^1$-homotopy theory [Mor04a]). While we cannot yet compute the groups $\boldsymbol{\pi}_d^{\mathbb{A}^1}(\mathbb{A}^d \setminus 0)$ for $d > 3$, based on the analogy with classical unstable homotopy groups of spheres, we still expect these sheaves to exhibit similar behaviors: they should be an extension of a (subsheaf of a) Grothendieck-Witt sheaf by an appropriate Milnor K-theory sheaf modulo 24. Moreover, the phenomenon that $\boldsymbol{\pi}_d^{\mathbb{A}^1}(\mathbb{A}^d \setminus 0)$ is an extension of two "stable" pieces in the known examples, together with computations from classical unstable homotopy theory [Tod62, Mah67], hint at the existence of a meta-stable range for $\mathbb{A}^1$-homotopy sheaves of $\mathbb{A}^d \setminus 0$.

Detailed description of contents

The computation of $\boldsymbol{\pi}_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)$ involves a number of ingredients, some of which are established in greater generality than actually required for the applications to projective modules envisioned in this paper. We begin with a review of Bott periodicity in orthogonal algebraic K-theory. The motivic spectrum $\mathbf{KO}$ is known to be $(8,4)$-periodic [Hor05]; and because of this periodicity, one constructs $\mathbf{KO}$ out of 4 cohomology theories. Two of these cohomology theories are "geometrically understood", i.e., orthogonal K-theory is known to be geometrically representable by work in progress of Schlichting-Tripathi [ST12], and symplectic K-theory is known to be geometrically representable by work of [PW10a]. However, to the best of our knowledge, no one has written down "geometric" models for the other cohomology theories that appear. Section 2 produces the required geometric models.

Section 3 details various $\mathbb{A}^1$-fiber sequences attached to some classical groups and their homogeneous spaces, including GL_{2n}/Sp_{2n} and Sp_{2n}. In each case, we describe the first non-stable $\mathbb{A}^1$-homotopy sheaf and discuss the connections with corresponding calculations in classical topology. Only the computation of the first non-stable homotopy sheaf of GL_4/Sp_4 (really SL_4/Sp_4) is necessary for proof of Theorem 3. Nevertheless, this section derives some of its length from the detailed computations of the first non-stable $\mathbb{A}^1$-homotopy sheaf of Sp_{2n} and also of Sp_{2n}/GL_n, which we will use in subsequent work.

Section 4 is devoted to analyzing the computation of $\boldsymbol{\pi}_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)$ in greater detail. The results of this section are not used in the remainder of the paper, but we feel they are integral because they illuminate some of the more mysterious aspects of the computation and shape our expectations about the structure of $\boldsymbol{\pi}_d^{\mathbb{A}^1}(\mathbb{A}^d \setminus 0)$. The fiber sequences of Section 3 identify $\boldsymbol{\pi}_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)$ as an extension of a Grothendieck-Witt sheaf by $\mathbf{F}_5$. The main goal of this section is to understand the origins of the $\mathbf{F}_5$ factor. In a sense we make more precise, the factor of $\mathbf{F}_5$ is "generated" by a map we call δ. We then define an algebro-geometric version of Hopf map ν and study its properties, and use this to show that δ is stably non-trivial. We believe, but are unable to prove, that δ actually coincides with an appropriate suspension of ν.

Finally, Section 5 is devoted to analyzing the problem of splitting a trivial rank 1 summand off a vector bundle by means of the techniques of obstruction theory. We give a detailed treatment of the primary obstruction to splitting, which complements Morel's discussion of the Euler class. For rank $d-1$ vector bundles on a smooth affine d-fold, we formulate a general cohomological vanishing conjecture that implies Murthy's splitting conjecture. Theorem 2 is then proven by establishing the vanishing theorem alluded to above in the case $d = 4$. This calculation depends on the explicit form of the computation of $\boldsymbol{\pi}_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)$ given in Section 3.

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2 Grothendieck-Witt sheaves and geometric Bott periodicity

In this section, we begin by reviewing some notation regarding $\mathbb{A}^1$-homotopy theory. We then discuss some aspects of $\mathbb{A}^1$-representability of Grothendieck-Witt theory including some results about Bott periodicity at the space level. Throughout, we assume k is a field having characteristic unequal to 2. The results should be familiar, though the proofs are, in a sense, backwards: they are deduced from known representability statements for various flavors of K-theory. We refer to [Sch10] and [Sch12] as general references for higher Grothendieck-Witt theory of schemes; [Hor05] for a discussion in the context of $\mathbb{A}^1$-homotopy theory, and [AF12b, §4] for a discussion in the context of the present work.

Classifying spaces

As usual, if G is a Nisnevich simplicial sheaf of groups, we write $B_{\bullet}G$ for a fibrant model of the the usual simplicial classifying space of G (see [MV99, §4.1]). If G is a linear algebraic group, then by [MV99, §4 Proposition 1.15], the space $B_{\bullet}G$ classifies Nisnevich locally trivial G-torsors in $\mathcal{H}_s^{\mathrm{Nis}}(k)$. In particular, if $P \to X$ is a Nisnevich locally trivial G-torsor over a smooth scheme X, there is a (well-defined up to simplicial homotopy) morphism $X \to B_{\bullet}G$ such that P is the pullback of the universal G-torsor over $B_{\bullet}G$.

Grassmannians and Stiefel manifolds

If V is a finite dimensional k-vector space of dimension n, we write $Gr_m(V)$ for the Grassmannian parameterizing m-dimensional subspaces of V. Upon fixing a k-point of $Gr_m(V)$, there is an isomorphism $Gr_m(V) \cong GL_n/P_m$ where P_m is a parabolic subgroup of GL_n with Levi factor $GL_{n-m} \times GL_m$. (Since we will always work with based spaces, it will be convenient to have a base-point fixed from the beginning.) An inclusion $V \hookrightarrow V'$ determines a morphism $Gr_m(V) \to Gr_m(V')$ and we write Gr_m for $\operatorname{colim}_n Gr_m(k^{\oplus n})$, where the transition morphisms are induced by the inclusions $k^{\oplus n} \to k^{\oplus n+1}$ as the first n-factors.

Consider the inclusion $GL_m \hookrightarrow GL_{m+1}$ obtained by sending an invertible $m \times m$ matrix M to the block $(m+1) \times (m+1)$-matrix

$$\begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}.$$

Taking the product of this morphism with the identity map, this inclusion yields a map $GL_{n-m} \times GL_m \to GL_{n-m} \times GL_{m+1}$, that can be extended to a morphism of parabolic subgroups of GL_n and therefore to a morphism $Gr_{m,n} \to Gr_{m+1,n+1}$. These morphisms are compatible with the transition morphisms corresponding to increasing n and yield morphisms $Gr_m \to Gr_{m+1}$ upon taking the colimit, and we write Gr for $\operatorname{colim}_m Gr_m$. Finally, we write KGL for the space $\mathbb{Z} \times Gr$. The importance of the space KGL is that it represents algebraic K-theory in $\mathcal{H}(k)$ by [MV99, §4 Theorem 3.13].

Write H for the trivial symplectic k-vector space $(k^{\oplus 2}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$. Write $H^{\oplus n}$ for the n-fold orthogonal direct sum of H with itself. The quaternionic Grassmannian $HGr_m(H^{\oplus n})$ is the open subscheme of $Gr(2m, H^{\oplus n})$ parameterizing subspaces to which the symplectic form restricts non-degenerately. Upon choosing a base-point $HGr_m(H^{\oplus n})$ becomes isomorphic to $Sp_{2n}/(Sp_{2(n-m)} \times Sp_{2m})$. Any inclusion $H^{\oplus n} \to H^{\oplus n'}$ determines a morphism $HGr_m(H^{\oplus n}) \to HGr_m(H^{\oplus n'})$ and we write HGr_m for $\operatorname{colim}_n HGr_m(H^{\oplus n})$ (for the morphism induced by the inclusion as the first n summands).

The inclusions $H^{\oplus n} \to H^{\oplus n+1}$ yield morphisms $Sp_{2n} \to Sp_{2n+2}$ and there are corresponding morphisms $HGr_{m,n} \to HGr_{m+1,n+1}$. As above, using these maps, one defines a morphism of spaces $HGr_m \to HGr_{m+1}$, and we write HGr for $\operatorname{colim}_m HGr_m$ and KSp for the space $\mathbb{Z} \times HGr$. The importance of the space KSp is that, if k is a field having characteristic unequal to 2, it represents symplectic K-theory in $\mathcal{H}(k)$ by [PW10a, Theorem 8.2].

The forgetful map

The inclusion $f_{m,n} : HGr_{m,n} \to Gr_{2m,2n}$ is compatible with the various transition maps relating (quaternionic) Grassmannians for different values of m and n. As a consequence, there are induced morphisms $f_m : HGr_m \to Gr_{2m}$ and $f : HGr \to Gr$ that arise by taking the various colimits. Taking the product with the multiplication by $2 : \mathbb{Z} \to \mathbb{Z}$, we obtain a map $f : KSp \to KGL$ that we call the forgetful map.

Remark 2.1. The inclusion $Sp_{2n} \hookrightarrow GL_{2n}$ yields a morphism $B_\bullet Sp_{2n} \to B_\bullet GL_{2n}$ and by taking the colimit with respect to n (for appropriate inclusions) one obtains $B_\bullet Sp \to B_\bullet GL$. We will, presently, compare this morphism to the one studied in the previous paragraph. Given a smooth scheme X and a simplicial homotopy class of maps $X \to B_\bullet Sp_{2n}$, the composite map to $B_\bullet GL_{2n}$ by means of the above inclusion yields a vector bundle of rank $2n$. The multiplication by 2 appearing in the last line of the previous paragraph encodes the fact that the rank of the vector bundle underlying a symplectic bundle is even.

We want to identify the $\mathbb{A}^1$-homotopy fibers of the maps f_m and the resulting maps obtained by taking the relevant colimits. To this end, we will replace the map $HGr_m \to Gr_{2m}$ by an $\mathbb{A}^1$-homotopy equivalent map whose homotopy fiber (almost) coincides with the point-set fiber.

First, we construct a candidate for the homotopy fiber: consider the homogeneous space GL_{2n}/Sp_{2n}. There are induced morphisms of homogeneous spaces $GL_{2n}/Sp_{2n} \to GL_{2n+2}/Sp_{2n+2}$ and we set

$$GL/Sp := \operatorname{colim}_n GL_{2n}/Sp_{2n}$$

We now relate the $\mathbb{A}^1$-homotopy type of GL/Sp to the Grassmannians above.

Let us describe the candidate replacement for HGr_m. Let $V_{m,n}$ be the variety parameterizing m-dimensional subspaces of an n-dimensional k-vector space equipped with a basis, i.e., the Stiefel variety of m-frames in an n-dimensional k-vector space. The canonical morphism $V_{m,n} \to Gr_{m,n}$ that forgets the basis is a GL_m-torsor. The inclusion $Sp_{2m} \hookrightarrow GL_{2m}$ then determines a morphism $V_{2m,2n}/Sp_{2m} \to V_{2m,2n}/GL_{2m}$ of quotients (the quotient $V_{2m,2n}/Sp_{2m}$ exists as a smooth scheme), which is precisely the projection map in the following contracted product:

$$V_{2m,2n} \times^{GL_{2m}} GL_{2m}/Sp_{2m} \longrightarrow V_{2m,2n}/GL_{2m}.$$

As the associated fiber bundle of a GL_m-torsor, this sequence is an $\mathbb{A}^1$-fiber sequence by [Wen11, Proposition 5.2]; in particular, the canonical map from the actual fiber to the $\mathbb{A}^1$-homotopy fiber is an $\mathbb{A}^1$-weak equivalence.

There are maps $V_{2m,2n} \to V_{2m,2(n+1)}$ and the collection of such spaces and maps yields an admissible gadget (over $\operatorname{Spec} k$) in the sense of [MV99, §4 Definition 2.1]. These transition maps yield morphisms

$$V_{2m,2n}/Sp_{2m} \longrightarrow V_{2m,2(n+1)}/Sp_{2m}.$$

We set $HGr'_m := \operatorname{colim}_n V_{2m,2n}/Sp_{2m}$ with respect to these morphisms, and we write $f'_m : HGr'_m \to Gr_{2m}$ for the colimit of the morphisms from the previous paragraph. The goal of the next few paragraphs is to identify the $\mathbb{A}^1$-homotopy type of HGr'_m.

The Sp_{2m}-torsor $V_{2m,2n} \to V_{2m,2n}/Sp_{2m}$ is classified by a morphism (well defined up to simplicial homotopy)

$$V_{2m,2n}/Sp_{2m} \longrightarrow B_\bullet Sp_{2m},$$

and the GL_{2m}-torsor $V_{2m,2n}/GL_{2m}$ is classified by a morphism $Gr_{2m} \to B_\bullet GL_{2m}$.

These classifying maps are compatible with the transition maps relating Stiefel manifolds for different values of m and n and yield a morphism

$$\pi'_m : HGr'_m \longrightarrow B_\bullet Sp_{2m}.$$

On the other hand, the quotient $Sp_{2n}/Sp_{2(n-m)} \to HGr_{m,n}$ is also an Sp_{2m}-torsor and is therefore also classified by a map $HGr_{m,n} \to B_\bullet Sp_{2m}$. Taking the colimit with respect to n yields a classifying morphism (well defined up to simplicial homotopy)

$$\pi_m : HGr_m \longrightarrow B_\bullet Sp_{2m}.$$

The basic geometric fact about these classifying morphisms is summarized in the following statement.

Lemma 2.2. *The morphisms π_m and π'_m are $\mathbb{A}^1$-weak equivalences. If $i_m : Sp_{2m} \to GL_{2m}$ is the obvious inclusion map, and $i'_m : B_\bullet Sp_{2m} \to B_\bullet GL_{2m}$ is the induced morphism then the following diagram is homotopy commutative:*

$$\begin{tikzcd} HGr'_m \arrow[r, "\pi'_m"] \arrow[d, "f'_m"] & B_\bullet Sp_{2m} \arrow[d, "i_{m'}"] & HGr_m \arrow[l, "\pi_m"'] \arrow[d, "f_m"] \\ Gr_{2m} \arrow[r] & B_\bullet GL_{2m} & Gr_{2m}, \arrow[l] \end{tikzcd}$$

where the bottom horizontal morphisms are the classifying maps for the universal GL_{2m}-torsors $V_{2m} \to Gr_{2m}$.

Proof. That the map π'_m is an $\mathbb{A}^1$-weak equivalence follows from the results of [MV99, §4 Proposition 2.3, Lemma 2.5 and Proposition 2.6], which shows that $\operatorname{colim}_n V_{2m,2n}$ is $\mathbb{A}^1$-contractible.

Panin and Walter show [PW10a] that π_m is an $\mathbb{A}^1$-weak equivalence by showing that the spaces $Sp_{2n}/Sp_{2(n-m)}$ and the obvious inclusion maps form an acceptable gadget in the sense of [PW10a, Definition 8.3]; this allows one to conclude that $\operatorname{colim}_n Sp_{2n}/Sp_{2(n-m)}$ is $\mathbb{A}^1$-contractible and then identify HGr_m as the quotient of an $\mathbb{A}^1$-contractible space by a free action of Sp_{2m}.

The homotopy commutativity follows by unwinding the definitions of the various maps. ☐

Our next goal is to identify the $\mathbb{A}^1$-homotopy fiber of the map $HGr'_m \to Gr_{2m}$. Because the maps $GL_{2m}/Sp_{2m} \to \operatorname{hofib}(V_{2m,2n}/Sp_{2m} \to V_{2m,2n}/GL_{2m})$ are all $\mathbb{A}^1$-weak equivalences, we conclude that the induced map

$$GL_{2m}/Sp_{2m} \longrightarrow \operatorname{colim}_n \operatorname{hofib}(V_{2m,2n}/Sp_{2m} \to V_{2m,2n}/GL_{2m})$$

is also an $\mathbb{A}^1$-weak equivalence. Note that since we are taking a filtered colimit here the obvious map from the homotopy colimit to the colimit is an $\mathbb{A}^1$-weak equivalence [MV99, §2 Corollary 1.21]; we use this observation repeatedly below to identify the space level colimit as a model for the homotopy colimit.

Now, we "commute the filtered homotopy colimit past the homotopy fiber" (in the sequel, we will simply use this phrase to stand for the argument of the next few lines). More precisely, there is a canonical morphism

$$\operatorname{colim}_n \operatorname{hofib}(V_{2m,2n}/Sp_{2m} \to V_{2m,2n}/GL_{2m}) \longrightarrow \operatorname{hofib}(HGr'_m \to V_{2m,2n}/GL_{2m}),$$

that we claim is an $\mathbb{A}^1$-weak equivalence. Since $\mathbb{A}^1$-fibrant replacement is functorial, we can replace all the maps $V_{2m,2n}/Sp_{2m} \to V_{2m,2n}/GL_{2m}$ by $\mathbb{A}^1$-fibrations of $\mathbb{A}^1$-fibrant spaces. In that case, the set-theoretic fiber of each such map coincides with the $\mathbb{A}^1$-homotopy theoretic fiber. To check that the above morphism is an $\mathbb{A}^1$-weak equivalence, we can check this stalkwise. Each stalk is a fibrant (Kan) simplicial set, and in the category of fibrant simplicial sets, we can commute filtered colimits past fiber products (this follows from the corresponding fact in the category of sets [ML98, IX.2 Theorem 1]), i.e., after $\mathbb{A}^1$-fibrant replacement, the above map is stalkwise an $\mathbb{A}^1$-weak equivalence. Combining these observations, we have established the following result.

Lemma 2.3. *The morphism $GL_{2m}/Sp_{2m} \to \mathrm{hofib}(HGr'_m \to Gr_{2m})$ is an $\mathbb{A}^1$-weak equivalence.*

Using the $\mathbb{A}^1$-weak equivalence from $HGr_m \to \mathrm{colim}_n V_{2m,2n}/Sp_{2m}$, we get an $\mathbb{A}^1$-fiber sequence of the form

$$GL_{2m}/Sp_{2m} \longrightarrow HGr_m \longrightarrow Gr_m.$$

Now, we take the colimit with respect to m. In particular, using the same argument commuting the filtered homotopy colimits past the homotopy fiber, we deduce that $GL/Sp = \mathrm{colim}_m GL_{2m}/Sp_{2m}$ is precisely the $\mathbb{A}^1$-homotopy fiber of the map $HGr \to Gr$. Since the fiber over the base-point only depends on the connected component of the identity, we can summarize the discussion so far with the following statement.

Lemma 2.4. *The map $GL/Sp \to \mathrm{hofib}(KSp \to KGL)$ constructed above is an $\mathbb{A}^1$-weak equivalence.*

Remark 2.5. As we explain at the beginning of Section 3, it is possible to define directly a ($\mathbb{A}^1$-homotopy associative) multiplication map

$$GL_{2n}/Sp_{2n} \times GL_{2m}/Sp_{2m} \longrightarrow GL_{2(n+m)}/Sp_{2(n+m)}$$

that is compatible with stabilization (up to $\mathbb{A}^1$-homotopy). Thus, one obtains a multiplication on GL/Sp by taking an appropriate colimit. Presumably this multiplication underlies an infinite loop space structure on GL/Sp making the above result into a homotopy fiber sequence of (simplicial) infinite loop spaces.

The hyperbolic morphism

Consider the inclusion $\gamma_m : GL_m \to Sp_{2m}$ given by sending an invertible $m \times m$-matrix M to the matrix

$$\begin{pmatrix} M & 0 \\ 0 & M^{T^{-1}} \end{pmatrix},$$

which is symplectic with respect to the standard symplectic form. If $B_\bullet GL_m$ is a simplicially fibrant model of the simplicial classifying space for GL_m, then the morphism γ_m induces a morphism

$$h_m : B_\bullet GL_m \longrightarrow B_\bullet Sp_{2m}.$$

The importance of this map is summarized in the following result.

Proposition 2.6. *For any smooth scheme X, the morphism*

$$[X, B_\bullet GL_m]_s \to [X, B_\bullet Sp_{2m}]_s$$

induced by h_m is precisely the map sending a rank n vector bundle $\mathcal{E}$ on X to the symplectic bundle $\mathcal{E} \oplus \mathcal{E}^\vee$ equipped with the standard symplectic form.

Proof. It suffices to check this in the universal case. In that case, if W is the standard $2n$-dimensional representation of Sp_{2n}, and V is the standard n-dimensional representation of GL_n, then we have $Res^{Sp_{2n}}_{GL_n}(W) \cong V \oplus V^\vee$. Translating this into statements about associated vector bundles: the pullback of the vector bundle obtained by twisting the universal Sp_{2m}-torsor via W along the map $B_\bullet GL_m \to B_\bullet Sp_{2m}$ is the direct sum of the universal vector bundle on $B_\bullet GL_m$ and its dual. $\square$

Our goal is to identify the $\mathbb{A}^1$-homotopy fiber of h_m and, to this end, we begin by constructing a candidate for the $\mathbb{A}^1$-homotopy fiber. The inclusion $GL_m \hookrightarrow GL_{m+1}$ and the inclusion $Sp_{2m} \hookrightarrow Sp_{2m+2}$ studied above are compatible with γ_m and together yield a morphism

$$Sp_{2m}/GL_m \longrightarrow Sp_{2m+2}/GL_{m+1}.$$

We set

$$Sp/GL := \operatorname{colim}_m Sp_{2m}/GL_m$$

with respect to the above morphisms.

Next, we construct a geometric model for the above hyperbolic map where the $\mathbb{A}^1$-homotopy fiber is easier to understand. The space $Sp_{2n}/Sp_{2(n-m)}$ is an Sp_{2m}-torsor over $HGr(m,n)$. The inclusion $GL_m \hookrightarrow Sp_{2m}$ studied above yields a GL_m-action on $Sp_{2n}/Sp_{2(n-m)}$, and a quotient of $Sp_{2n}/Sp_{2(n-m)}$ by this action exists as a smooth scheme. The inclusion $GL_m \hookrightarrow GL_{m+1}$ yields a commutative square of the form

$$\begin{tikzcd}
Sp_{2n}/(Sp_{2(n-m)} \times GL_m) \arrow[r] \arrow[d] & HGr_{m,n} \arrow[d] \\
Sp_{2n+2}/(Sp_{2(n-m)} \times GL_{m+1}) \arrow[r] & HGr_{m+1,n+1}.
\end{tikzcd}$$

We then set $Gr'_m := \operatorname{colim}_m Sp_{2n}/(Sp_{2(n-m)} \times GL_m)$, and there is an induced morphism $h'_m : Gr'_m \to HGr_m$.

Since the quotient morphism $Sp_{2n}/Sp_{2(n-m)} \to Sp_{2n}/(Sp_{2(n-m)} \times GL_m)$ is a GL_m-torsor, there is a (well-defined up to simplicial homotopy) morphism

$$Sp_{2n}/(Sp_{2(n-m)} \times GL_m) \longrightarrow B_\bullet GL_m,$$

classifying this GL_m-torsor. There is an induced morphism

$$\psi_m : Gr'_m \longrightarrow B_\bullet GL_m.$$

We already saw that the classifying map $\pi'_m : HGr_m \to B_\bullet Sp_m$ is an $\mathbb{A}^1$-weak equivalence. Analogously, we have the following result.

Lemma 2.7. *The morphism ψ_m is an $\mathbb{A}^1$-weak equivalence, and if τ is the morphism of simplicial classifying spaces induced by the group homomorphism $GL_m \to Sp_{2m}$ described in the beginning of this section, the following diagram is homotopy commutative:*

$$\begin{tikzcd}
Gr'_m \arrow[r, "\psi_m"] \arrow[d, "h'_m"'] & B_\bullet GL_m \arrow[d, "h_m"] \\
HGr_m \arrow[r, "\pi'_m"'] & B_\bullet Sp_{2m},
\end{tikzcd}$$

where h'_m is the map on quotients induced by h_m.

Proof. The spaces $Sp_{2n}/Sp_{2(n-m)}$ form an acceptable gadget in the sense of [PW10a, Definition 8.3]. In particular, $\operatorname{colim}_n Sp_{2n}/Sp_{2(n-m)}$ is $\mathbb{A}^1$-contractible. If we consider the Čech simplicial object associated with ψ_m, the proof follows in the same way as the proof of Lemma 2.2 above. The homotopy commutativity is clear from the construction. □

Now, the map $Sp_{2n}/(Sp_{2(n-m)} \times GL_m) \to HGr(m,n)$ is the projection morphism of the following associated fiber space:

$$Sp_{2m}/GL_m \longrightarrow Sp_{2n}/Sp_{2(n-m)} \times^{Sp_{2m}} Sp_{2m}/GL_m \longrightarrow HGr_{m,n}$$

and so provides an $\mathbb{A}^1$-fiber sequence by [Wen11, Proposition 5.2]. In particular, the map

$$Sp_{2m}/GL_m \longrightarrow \operatorname{hofib}(Sp_{2n}/(Sp_{2(n-m)} \times GL_m) \to HGr(m,n))$$

is an $\mathbb{A}^1$-weak equivalence for any m. As in the previous section, since a filtered colimit of $\mathbb{A}^1$-weak equivalences is again an $\mathbb{A}^1$-weak equivalence, the map

$$Sp_{2m}/GL_m \longrightarrow colim_n \operatorname{hofib}(Sp_{2n}/(Sp_{2(n-m)} \times GL_m) \to HGr(m,n)),$$

induced by taking colimits is an $\mathbb{A}^1$-weak equivalence. Again, commuting the filtered homotopy colimit past the homotopy fiber we obtain an $\mathbb{A}^1$-weak equivalence

$$\begin{aligned}&\operatorname{colim}_n \operatorname{hofib}(Sp_{2n}/(Sp_{2(n-m)} \times GL_m) \to HGr_{m,n}) \cong\\ &\operatorname{hofib}(Gr'_m \to HGr_m).\end{aligned}$$

Thus, combining the observations above, and taking the colimit with respect to n, we have deduced the following result.

Lemma 2.8. *The morphism above yields an $\mathbb{A}^1$-weak equivalence*

$$Sp_{2m}/GL_m \longrightarrow \operatorname{hofib}(\operatorname{colim}_n Sp_{2n}/(Sp_{2(n-m)} \times GL_m) \to HGr_m).$$

Combining the two lemmas above, yields an $\mathbb{A}^1$-fiber sequence of the form

$$Sp_{2m}/GL_m \longrightarrow Gr_m \longrightarrow HGr_m$$

We can then take the colimit with respect to m and, using the fact that $\operatorname{colim}_m Sp_{2m}/GL_m = Sp/GL$, and once more commuting the filtered homotopy colimits past the homotopy fiber, we deduce the following result.

Lemma 2.9. *The map $Sp/GL \to \operatorname{hofib}(KGL \to KSp)$ constructed above is an $\mathbb{A}^1$-weak equivalence.*

Algebraic avatars of Bott periodicity

Let $f : KSp \to KGL$ and $h : KGL \to KSp$ be the morphisms constructed above. We already know that KGL represents algebraic K-theory by [MV99, §4 Theorem 3.13] and that KSp represents symplectic K-theory by [PW10a, Theorem 8.2]. Following the conventions of higher Grothendieck-Witt groups, we write $[\Sigma_s^i X_+, KSp]_{\mathbb{A}^1} = GW_i^2(X)$. In that case, for any smooth k-scheme X, the maps f and h yield morphisms

$$f_* : GW_i^2(X) \longrightarrow K_i(X) \quad \text{ and } h_* : K_i(X) \longrightarrow GW_i^2(X),$$

which are functorial in X. Presently, our goal is to identify these morphisms.

Proposition 2.10. *The morphisms f_* and h_* coincide with the forgetful and hyperbolic morphisms on Grothendieck-Witt groups.*

Proof. To establish this fact, we need to show that f_* and h_* induce maps of cohomology theories; we will do this by showing that f and h arise from morphisms of Γ-spaces. To this end, recall that the morphism $GL_n \times GL_m \to GL_{n+m}$ given by block-sum of matrices yields a morphism of classifying spaces $B_\bullet GL_n \times B_\bullet GL_m \to B_\bullet GL_{n+m}$, and that these morphisms can be collected together into a monoid $\coprod_{n\geq 0} B_\bullet GL_n$. Similar considerations for the symplectic group yield a monoid structure on $\coprod_{n\geq 0} B_\bullet Sp_{2n}$. The construction explained in [Seg74, §2; p. 299] shows that each of these monoids is part of a Γ-space in the category of simplicial sheaves. The point is that because of the explicit nature of the Γ-space construction, the sequences of group homomorphisms $i_m : Sp_{2m} \to GL_{2m}$ and $\gamma_m : GL_m \to Sp_{2m}$ yield morphisms of Γ-spaces corresponding to the above monoids, and therefore to morphisms of the corresponding cohomology theories.

Morel and Voevodsky explain that the group completion $\mathbf{R}\Omega_s^1 B(\coprod_{n\geq 0} B_\bullet GL_n)$ represents algebraic K-theory [MV99, §4 Proposition 3.9] in the simplicial homotopy category and they construct an $\mathbb{A}^1$-weak equivalence

$$\mathbb{Z} \times B_\bullet GL_\infty \longrightarrow \mathbf{R}\Omega_s^1 B(\coprod_{n\geq 0} B_\bullet GL_n).$$

As Hornbostel remarks [Hor05, Remark 3.8], the analogous proof with $B_\bullet GL_n$ replaced by $B_\bullet Sp_{2n}$ (replace references to Thomason's results in [MV99] by the corresponding results due to Schlichting in [Sch10] and use the fact that Sp_{2n} is a special group, i.e., that all Sp_{2n}-torsors over smooth schemes are Zariski locally trivial) provides a corresponding model of symplectic K-theory. $\square$

Theorem 2.11. *If F is a field having characteristic unequal to 2, and if X is a smooth F-scheme, then for any integer $i \geq 0$, there are canonical isomorphisms*

$$[\Sigma_s^i X_+, Sp/GL]_{\mathbb{A}^1} \xrightarrow{\sim} GW_i^1(X), \text{ and}$$
$$[\Sigma_s^i X_+, GL/Sp]_{\mathbb{A}^1} \xrightarrow{\sim} GW_{i+1}^3(X).$$

Proof. We have an $\mathbb{A}^1$-homotopy fiber sequence of the form

$$GL/Sp \longrightarrow KSp \xrightarrow{f} KGL.$$

For any smooth scheme X, there is an associated long exact sequence obtained by applying $[X_+,]_{\mathbb{A}^1}$. Since in Proposition 2.10 we have identified the map $f_* : [\Sigma^i_s X_+, KSp]_{\mathbb{A}^1} \to [\Sigma^i_s X_+, KGL]_{\mathbb{A}^1}$ as the forgetful map on symplectic K-theory, the result follows by comparison with Schlichting's Bott sequence [Sch12, Theorem 8.11] via [Sch12, Theorem 9.3]. The result for the hyperbolic functor is analogous. □

For any $i, j \in \mathbb{N}$, let $\mathbf{GW}^j_i$ be the Nisnevich sheaf associated to the presheaf $U \mapsto GW^j_i(U)$; we refer to these sheaves as Grothendieck-Witt sheaves. In view of the above theorem, the proof of the following corollary is a straightforward consequence of sheafification.

Corollary 2.12. *For any integer $i \geq 0$, we have canonical isomorphisms*

$$\begin{aligned} \boldsymbol{\pi}^{\mathbb{A}^1}_i(Sp/GL) &\xrightarrow{\sim} \mathbf{GW}^1_i \\ \boldsymbol{\pi}^{\mathbb{A}^1}_i(GL/Sp) &\xrightarrow{\sim} \mathbf{GW}^3_{i+1}. \end{aligned}$$

Contracted sheaves

Let $\mathcal{G}$ be a Nisnevich sheaf of abelian groups. Recall from that $\mathcal{G}$ is called strictly $\mathbb{A}^1$-invariant if the map $H^i_{\mathrm{Nis}}(X, \mathcal{G}) \to H^i_{\mathrm{Nis}}(X \times \mathbb{A}^1, \mathcal{G})$ induced by the projection is an isomorphism for any $i \in \mathbb{N}$ [Mor12, Definition 7].

For any smooth k-scheme U, the unit map $\operatorname{Spec} k \to \mathbf{G}_m$ yields a morphism $U \to U \times \mathbf{G}_m$. The sheaf $\mathcal{G}_{-1}$ is then defined by

$$\mathcal{G}_{-1}(U) = \ker(\mathcal{G}(\mathbf{G}_m \times U) \to \mathcal{G}(U)).$$

We can then inductively define $\mathcal{G}_{-n} := (\mathcal{G}_{-n+1})_{-1}$ for any integer $n \geq 1$; we call $\mathcal{G}_{-n}$ the n-th contraction of G. It turns out that contraction is an exact functor (see, e.g., [Mor12, Lemma 7.33]). If $(\mathcal{X}, x)$ is a pointed $\mathbb{A}^1$-connected space, the $\mathbb{A}^1$-homotopy sheaves of $\mathbf{G}_m$-loop spaces of $\mathcal{X}$ are related to those of $\mathcal{X}$ by the following result of Morel.

Theorem 2.13 ([Mor12, Theorem 6.13])**.** *If $(\mathcal{X}, x)$ is a pointed $\mathbb{A}^1$-connected space, then for every pair of integers $i, j \geq 1$*

$$\boldsymbol{\pi}^{\mathbb{A}^1}_{i,j}(\mathcal{X}) := \boldsymbol{\pi}^{\mathbb{A}^1}_i(\mathbf{R}\Omega^j_{\mathbf{G}_m}\mathcal{X}) = \boldsymbol{\pi}^{\mathbb{A}^1}_i(\mathcal{X})_{-j}.$$

In the sequel, we will need computations of contractions of various strictly $\mathbb{A}^1$-invariant sheaves; the results we use are summarized in the following statement.

Proposition 2.14. *For any integers $i, n \geq 0$ and any $j \in \mathbb{Z}$, we have*

i) $(\mathbf{K}^Q_i)_{-n} = \mathbf{K}^Q_{i-n}$.

ii) $(\mathbf{K}^M_i)_{-n} = \mathbf{K}^M_{i-n}$.

iii) $(\mathbf{K}^{MW}_i)_{-n} = \mathbf{K}^{MW}_{i-n}$.

iv) $(\mathbf{I}^i)_{-n} = \mathbf{I}^{i-n}$.

v) $(\mathbf{GW}^j_i)_{-n} = \mathbf{GW}^{j-n}_{i-n}$.

Proof. The identifications in (i) and (ii) follow from [AF12b, Lemma 5.3], while that in (v) is [AF12b, Proposition 5.4]. The identification in (iii) is a direct consequence of [Mor12, Theorem 2.24]. For (iv), first observe that there is an exact sequence of strictly $\mathbb{A}^1$-invariant sheaves

$$0 \longrightarrow \mathbf{I}^i \longrightarrow \mathbf{K}^{MW}_{i-1} \longrightarrow \mathbf{K}^{M}_{i-1} \longrightarrow 0$$

for any $i \in \mathbb{Z}$ [Mor04b, Corollaire 5.4]. The identification of (iv) is then an immediate consequence of exactness of the contraction construction. □

3 Some homotopy sheaves of classical groups and symmetric spaces

The goal of this section is, after establishing an appropriate stable range, to compute the first non-stable $\mathbb{A}^1$-homotopy sheaves of GL_{2n}/Sp_{2n}, Sp_{2n} and Sp_{2n}/GL_n. The computation of the first non-stable homotopy sheaf of GL_{2n}/Sp_{2n} will, in particular, give the computation of $\boldsymbol{\pi}_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)$ mentioned in the introduction. However, the more general computation has other applications, e.g., to obstructions to existence of algebraic symplectic structures on smooth varieties. We will also discuss compatibility of our computations with complex realization. The topological results corresponding to our computations are classical and can be found, e.g., in [Har63] and [MT64].

Geometry of GL_{2n}/Sp_{2n}

Let W_{2n} be the $2n$-dimensional standard k-rational representation of GL_{2n}, and consider the vector space $(\wedge^2 W_{2n})^\vee$ of anti-symmetric k-bilinear forms on W_{2n}. The GL_{2n}-representation carried by $(\wedge^2 W_{2n})^\vee$ yields an action of GL_{2n} on $\mathbb{A}((\wedge^2 W_{2n})^\vee)$. There is an open subscheme $A_{2n} \subset \mathbb{A}((\wedge^2 W_{2n})^\vee)$ consisting of non-degenerate anti-symmetric k-bilinear forms on W_{2n}, and this subscheme is stable under GL_{2n}.

If we fix a non-degenerate anti-symmetric form on W_{2n}, then the corresponding k-point of A_{2n} has stabilizer isomorphic to Sp_{2n} and this choice yields an identification $A_{2n} \cong GL_{2n}/Sp_{2n}$. In discussing $\mathbb{A}^1$-homotopy theory of A_{2n}, we will always fix a base-point and, for that reason, we prefer to refer to GL_{2n}/Sp_{2n} instead. The determinant map $GL_{2n}/Sp_{2n} \to \mathbf{G}_m$ induces a morphism

$$Pf : A_{2n} \longrightarrow \mathbf{G}_m$$

such if we pick a basis of W_{2n}, then A_{2n} can be identified with the space of anti-symmetric $2n \times 2n$-matrices, and Pf sends a $2n \times 2n$ anti-symmetric matrix to its Pfaffian. We set

$$X_n := Pf^{-1}(1)$$

The scheme X_n is smooth, and a choice of base-point provides an isomorphism $X_n \cong SL_{2n}/Sp_{2n}$.

There is a morphism

$$A_{2n} \times A_{2m} \longrightarrow A_{2(n+m)},$$

which corresponds in coordinates to block-sum of anti-symmetric matrices (this "multiplication" is the one referred to in Remark 2.5). If H is the standard hyperbolic matrix described at the beginning of the previous section, then block sum with H determines a stabilization morphism

$$A_{2n} \longrightarrow A_{2(n+1)}.$$

Since $Pf(H) = 1$, it follows that block sum with H yields a stabilization morphism

$$X_n \longrightarrow X_{n+1}.$$

We set $SL/Sp := \operatorname{colim}_n X_n$, where the morphisms defining the colimit are as just specified. One goal of what follows is to understand the $\mathbb{A}^1$-homotopy fiber of the stabilization morphisms.

The first non-stable $\mathbb{A}^1$-homotopy sheaf of GL_{2n}/Sp_{2n}

The goal of this section is to compute the "low-degree" homotopy sheaves of GL_{2n}/Sp_{2n}. We will establish a stable range for these homotopy sheaves, and then describe the first non-stable $\mathbb{A}^1$-homotopy sheaf. To establish these results, we will perform a series of reductions. First, let us understand the $\mathbb{A}^1$-connected components of GL_{2n}/Sp_{2n}.

Lemma 3.1. *The inclusions $Sp_{2n} \hookrightarrow SL_{2n} \longrightarrow GL_{2n}$ yield a split $\mathbb{A}^1$-fiber sequence of the form*

$$SL_{2n}/Sp_{2n} \longrightarrow GL_{2n}/Sp_{2n} \longrightarrow \mathbf{G}_m;$$

the splitting is given by $t \mapsto diag(t, 1 \ldots, 1)$. In particular, the first morphism is the inclusion of the $\mathbb{A}^1$-connected component of the base-point.

Proof. Since $GL_{2n}/Sp_{2n} = GL_{2n} \times^{SL_{2n}} SL_{2n}/Sp_{2n}$ the sequence

$$SL_{2n}/Sp_{2n} \longrightarrow GL_{2n}/Sp_{2n} \longrightarrow \mathbf{G}_m$$

is a $\mathbb{A}^1$-fibre sequence by [Wen11, Proposition 5.1]. Now the $\mathbb{A}^1$-fiber sequence associated with the classifying morphism

$$SL_{2n} \longrightarrow SL_{2n}/Sp_{2n} \longrightarrow BSp_{2n}$$

demonstrates that the space SL_{2n}/Sp_{2n} is $\mathbb{A}^1$-connected. Since the morphism

$$GL_{2n}/Sp_{2n} \longrightarrow \mathbf{G}_m$$

splits, it follows that $\boldsymbol{\pi}_0^{\mathbb{A}^1}(GL_{2n}/Sp_{2n}) = \boldsymbol{\pi}_0^{\mathbb{A}^1}(\mathbf{G}_m) = \mathbf{G}_m$. This proves the Lemma. ☐

Corollary 3.2. *There is a canonical isomorphism $\boldsymbol{\pi}_0^{\mathbb{A}^1}(GL/Sp) \cong \mathbf{G}_m$, and the induced morphism $SL/Sp \to GL/Sp$ is the inclusion of the $\mathbb{A}^1$-connected component of the base-point.*

Proof. The results above show that the determinant yields an isomorphism $\boldsymbol{\pi}_0^{\mathbb{A}^1}(GL_{2n}/Sp_{2n}) \cong \mathbf{G}_m$ for every $n > 0$; this is obviously compatible with the inclusion maps. For the second statement, since the following diagram commutes

$$\begin{tikzcd} SL_{2n}/Sp_{2n} \arrow[r] \arrow[d] & GL_{2n}/Sp_{2n} \arrow[d] \\ SL_{2n+2}/Sp_{2n+2} \arrow[r] & GL_{2n+2}/Sp_{2n+2}, \end{tikzcd}$$

the result follows by taking the colimit. ☐

The practical consequence of the above statements is that the $\mathbb{A}^1$-homotopy theory of the map $GL_{2n}/Sp_{2n} \to GL/Sp$ is reduced to studying the map $SL_{2n}/Sp_{2n} \to SL/Sp$.

In [AF12b, Proposition 3.11], we observed the existence of a fiber square of the form

$$
\begin{tikzcd}
Sp_{2n} \arrow[r] \arrow[d] & Sp_{2n+2} \arrow[d] \\
SL_{2n+1} \arrow[r] & SL_{2n+2}.
\end{tikzcd}
\tag{3.1}
$$

Since Sp_{2n+2} acts transitively on SL_{2n+2}/SL_{2n+1}, and the stabilizer of the identity coset is Sp_{2n}, we conclude the existence of an isomorphism of schemes $SL_{2n+2}/SL_{2n+1} \cong Sp_{2n+2}/Sp_{2n}$. Analogously, we can deduce the following result.

Lemma 3.3. *For any integer $n \geq 1$, there is a canonical isomorphism of schemes $SL_{2n+1}/Sp_{2n} \cong X_{n+1}$.*

Lemma 3.4. *The sequences of closed immersion group homomorphisms*

i) $Sp_{2n} \hookrightarrow SL_{2n} \hookrightarrow SL_{2n+1}$,

ii) $Sp_{2n} \hookrightarrow SL_{2n+1} \hookrightarrow SL_{2n+2}$, *and*

iii) $Sp_{2n} \hookrightarrow Sp_{2n+2} \hookrightarrow SL_{2n+2}$

yield $\mathbb{A}^1$-fiber sequences of the form

i) $SL_{2n}/Sp_{2n} \to SL_{2n+1}/Sp_{2n} \to SL_{2n+1}/SL_{2n}$,

ii) $SL_{2n+1}/Sp_{2n} \to SL_{2n+2}/Sp_{2n} \to SL_{2n+2}/SL_{2n+1}$, *and*

iii) $Sp_{2n+2}/Sp_{2n} \to SL_{2n+2}/Sp_{2n} \to SL_{2n+2}/Sp_{2n+2}$.

Proof. In each case, these fiber sequences are the associated fiber bundles to Zariski locally trivial SL_n or Sp_{2n}-torsors for appropriate values of n; we then apply [Wen11, Proposition 5.2]. □

The following result provides a description of the connectivity of the $\mathbb{A}^1$-homotopy fiber of the stabilization map $X_n \to X_{n+1}$, together with some complements.

Proposition 3.5. *For any $n \geq 1$, there is an $\mathbb{A}^1$-fiber sequence of the form*

$$X_n \longrightarrow X_{n+1} \longrightarrow SL_{2n+1}/SL_{2n}.$$

In particular, $X_n \to X_{n+1}$ is $(2n-2)$-$\mathbb{A}^1$-connected, $X_n \to SL/Sp$ is $(2n-2)$-$\mathbb{A}^1$-connected, and there is an exact sequence of the form

$$\boldsymbol{\pi}_{2n}^{\mathbb{A}^1}(GL/Sp) \longrightarrow \mathbf{K}_{2n+1}^{MW} \longrightarrow \boldsymbol{\pi}_{2n-1}^{\mathbb{A}^1}(X_n) \longrightarrow \boldsymbol{\pi}_{2n-1}^{\mathbb{A}^1}(GL/Sp) \longrightarrow 0.$$

Proof. For the first statement, combine Lemmas 3.4(i) and 3.3. The second statement follows immediately from the first since SL_{2n+1}/SL_{2n} is $\mathbb{A}^1$-weak equivalent to $\mathbb{A}^{2n+1}\setminus 0$ which is $(2n-1)$-$\mathbb{A}^1$-connected. The third statement follows from the second by induction on n, and the fourth statement follows from the previous three together with Corollary 3.2 by looking at the long exact sequence in homotopy sheaves attached to the stated $\mathbb{A}^1$-fiber sequence and using the fact that $\boldsymbol{\pi}^{\mathbb{A}^1}_{2n}(SL_{2n+1}/SL_{2n}) \cong \mathbf{K}^{MW}_{2n+1}$ by [Mor12, Theorem 5.40]. ☐

By means of Corollary 2.12, the exact sequence in Proposition 3.5 takes the form

$$\mathbf{GW}^3_{2n+1} \xrightarrow{\chi_{2n+1}} \mathbf{K}^{MW}_{2n+1} \longrightarrow \boldsymbol{\pi}^{\mathbb{A}^1}_{2n-1}(X_n) \longrightarrow \mathbf{GW}^3_{2n} \longrightarrow 0,$$

and we set

$$\mathbf{F}_{2n+1} := \mathrm{coker}(\mathbf{GW}^3_{2n+1} \xrightarrow{\chi_{2n+1}} \mathbf{K}^{MW}_{2n+1}).$$

Our goal in what follows is to describe $\mathbf{F}_{2n+1}$ (here $\mathbf{F}$ stands for “forgetful”).

Since the morphism $\mathbf{GW}^3_{2n+1} \to \mathbf{K}^{MW}_{2n+1}$ is induced by the morphism $X_{n+1} \to SL_{2n+1}/SL_{2n}$ by means of the identification $X_{n+1} \cong SL_{2n+1}/Sp_{2n}$, we can consider the composite morphism

$$SL_{2n+1} \to X_{n+1} \to SL_{2n+1}/SL_{2n}.$$

This composite is precisely the projection morphism of the SL_{2n}-torsor $SL_{2n+1} \to SL_{2n+1}/SL_{2n}$. As a consequence, the morphism $\psi_{2n+1} : \boldsymbol{\pi}^{\mathbb{A}^1}_{2n}(SL_{2n+1}) \to \mathbf{K}^{MW}_{2n+1}$ factors through a morphism $\mathbf{K}^Q_{2n+1} \to 2\mathbf{K}^M_{2n+1} \subset \mathbf{K}^{MW}_{2n+1}$ by [AF12b, Lemma 3.2] and the image of this homomorphism contains $(2n)!\mathbf{K}^M_{2n+1}$ by [AF12b, Lemma 3.8].

Since the morphism $\boldsymbol{\pi}^{\mathbb{A}^1}_{2n}(SL_{2n+1}) \to \boldsymbol{\pi}^{\mathbb{A}^1}_{2n}(X_{n+1})$ can be factored through stabilization to a morphism $\mathbf{K}^Q_{2n+1} \to \mathbf{GW}^3_{2n+1}$, the results of the previous section identify this homomorphism with the hyperbolic homomorphism $H_{3,2n+1}$, and we deduce that the following diagram commutes

$$\begin{tikzcd} \mathbf{K}^Q_{2n+1} \arrow[r, "H_{3,2n+1}"] \arrow[d, "\psi_{2n+1}"'] & \mathbf{GW}^3_{2n+1} \arrow[d, "\chi_{2n+1}"] \\ \mathbf{Im}(\psi_{2n+1}) \arrow[r] & \mathbf{K}^{MW}_{2n+1}. \end{tikzcd} \tag{3.2}$$

In particular, since $\mathbf{T}_{2n+1} = \mathrm{coker}(\psi_{2n+1})$ by definition (see [AF12a, Theorem 2.3] for a more detailed discussion of this sheaf), we obtain an epimorphism $\mathbf{T}_{2n+1} \to \mathbf{F}_{2n+1}$.

Theorem 3.6. *The canonical morphism $GL_{2n}/Sp_{2n} \to GL/Sp$ is $(2n-2)$-$\mathbb{A}^1$-connected, and there is a short exact sequence of the form*

$$0 \longrightarrow \mathbf{F}_{2n+1} \longrightarrow \boldsymbol{\pi}^{\mathbb{A}^1}_{2n-1}(GL_{2n}/Sp_{2n}) \longrightarrow \boldsymbol{\pi}^{\mathbb{A}^1}_{2n-1}(GL/Sp) \longrightarrow 0,$$

where $\mathbf{F}_{2n+1}$ is a quotient of $\mathbf{T}_{2n+1}$.

Taking $n = 2$ and using the fact (from Lemma 3.3) that $X_2 \cong SL_3/SL_2$, which is $\mathbb{A}^1$-weakly equivalent to $\mathbb{A}^3 \setminus 0$, Theorem 3.6 yields the following result.

Theorem 3.7. *There is a short exact sequence of the form*

$$0 \longrightarrow \mathbf{F}_5 \longrightarrow \boldsymbol{\pi}_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0) \longrightarrow \mathbf{GW}_4^3 \longrightarrow 0,$$

where $\mathbf{F}_5$ is a quotient of $\mathbf{T}_5$.

The computation of Theorem 3.7 can be refined to provide more detailed information about $\mathbf{F}_5$: the next result shows that the epimorphism $\mathbf{T}_5 \longrightarrow \mathbf{F}_5$ becomes an isomorphism after repeated contraction.

Lemma 3.8. *The epimorphism $\mathbf{T}_5 \to \mathbf{F}_5$ induces an isomorphism $(\mathbf{T}_5)_{-4} \to (\mathbf{F}_5)_{-4}$ and there is a cartesian square of the form:*

$$\begin{tikzcd}
(\mathbf{T}_5)_{-4} \arrow[r] \arrow[d] & \mathbf{I} \arrow[d] \\
\mathbf{K}_1^M/24 \arrow[r] & \bar{\mathbf{I}}.
\end{tikzcd}$$

Proof. Fix $n = 2$, and consider Diagram (3.2) above. Contracting this diagram 4-times and using Proposition 2.14, yields a cartesian square

$$\begin{tikzcd}
\mathbf{K}_1^Q \arrow[r, "H_{3,1}"] \arrow[d, "(\psi_5)_{-4}"'] & \mathbf{GW}_1^3 \arrow[d, "\chi_{2n+1}"] \\
\mathbf{Im}(\psi_5)_{-4} \arrow[r] & \mathbf{K}_1^{MW}.
\end{tikzcd}$$

In order to show that $(\mathbf{T}_5)_{-4} \to (\mathbf{F}_5)_{-4}$ is an isomorphism, it therefore suffices to prove that $H_{3,1}$ is surjective. This surjectivity statement follows from [FRS12, Lemma 2.3]. The fact that $(\mathbf{T}_5)_{-4}$ sits in the Cartesian square

$$\begin{tikzcd}
(\mathbf{T}_5)_{-4} \arrow[r] \arrow[d] & \mathbf{I} \arrow[d] \\
\mathbf{K}_1^M/24 \arrow[r] & \bar{\mathbf{I}}.
\end{tikzcd}$$

is a direct consequence of [AF12a, Theorem 2.3 and Lemma 2.9]. $\square$

Lemma 3.9. *The space X_m is $\mathbb{A}^1$-simply connected for arbitrary m.*

Proof. When $m = 1$, the space SL_2/Sp_2 is a single point, so we can assume that $m \geq 2$. In that case, we know that $X_m \to X_{m+1}$ is a $(2m-2)$-$\mathbb{A}^1$-connected by Theorem 3.6. As a consequence, it suffices to prove that SL/Sp is $\mathbb{A}^1$-1-connected. Since $\boldsymbol{\pi}_1^{\mathbb{A}^1}(SL/Sp) \cong \mathbf{GW}_2^3$ by Corollary 2.12, it suffices to observe that if F is a field, then $\mathbf{GW}_2^3(F)$ is trivial by, e.g., [FRS12, Lemma 2.2]. In what amounts to the same thing, one can also consider the portion of the long exact sequence

$$\boldsymbol{\pi}_1^{\mathbb{A}^1}(Sp) \to \boldsymbol{\pi}_1^{\mathbb{A}^1}(SL) \longrightarrow \boldsymbol{\pi}_1^{\mathbb{A}^1}(SL/Sp) \longrightarrow 0$$

in $\mathbb{A}^1$-homotopy sheaves. There are identifications $\boldsymbol{\pi}_1^{\mathbb{A}^1}(Sp) = \mathbf{K}_2^{MW}$ and $\boldsymbol{\pi}_1^{\mathbb{A}^1}(SL) = \mathbf{K}_2^M$, and the map $\mathbf{K}_2^{MW} \to \mathbf{K}_2^M$ is the natural epimorphism. $\square$

Remark 3.10. In classical topology, the first few non-stable homotopy groups of $X_n(\mathbb{C})$ were computed in [Har63]. In particular, if n is even, then $\pi_{4n}(X_n(\mathbb{C})) = \mathbb{Z}/(2n!)$, while if n is odd, then $\pi_{4n}(X_n(\mathbb{C})) = \mathbb{Z}/((2n!)/2)$. Complex realization gives a morphism

$$\boldsymbol{\pi}^{\mathbb{A}^1}_{2n-1,2n+1}(X_n) \longrightarrow \pi_{4n}(X_n(\mathbb{C})).$$

The group $\boldsymbol{\pi}^{\mathbb{A}^1}_{2n-1,2n+1}(X_n)$ can be computed by contracting the result of Theorem 3.6 $(2n+1)$ times. Since $(\mathbf{GW}^3_{2n})_{-2n-1} = \mathbf{W}^{1-2n}$ is trivial by [BW02], it follows that $\boldsymbol{\pi}^{\mathbb{A}^1}_{2n-1,2n+1}(X_n) = (\mathbf{F}_{2n+1})_{-2n-1}$, and furthermore there is an epimorphism from $(\mathbf{T}_{2n+1})_{-2n-1}$ onto this group. The latter contraction was discussed in detail in [AF12a, Theorem 4.5], where it was established that $(\mathbf{T}_{2n+1})_{-2n-1} = \mathbb{Z}/(2n!)$. The classical computation suggests that, when n is even, the homomorphism $\mathbf{F}_{2n+1} \to \mathbf{T}_{2n+1}$ is an isomorphism, while if n is odd then it has a non-trivial kernel.

Remark 3.11. The above results provide a non-trivial obstruction to existence of an algebraic symplectic structure on a smooth algebraic variety of dimension $2d$. Indeed, if Y is a smooth algebraic variety of dimension $2d$ with trivial (co)tangent bundle, we can fix such a trivialization and therefore obtain an $\mathbb{A}^1$-homotopy class of maps $Y \to BSL_{2d}$ classifying this structure. The existence of an algebraic symplectic structure yields a reduction of the structure group for the tangent bundle from SL_{2d} to Sp_{2d}, i.e., a lift of the given map $Y \to BSL_{2d}$ to a map $Y \to BSp_{2d}$. Whether such a lift exists is governed by the homotopy fiber of the map, which is precisely X_d. We know that $\boldsymbol{\pi}^{\mathbb{A}^1}_i(X_d) = \mathbf{GW}^3_{i+1}$ for $i \leq 2d-2$, and the results above compute $\boldsymbol{\pi}^{\mathbb{A}^1}_{2d-1}(X_d)$. Since $\boldsymbol{\pi}^{\mathbb{A}^1}_1(X_d) = 0$ for any $d \geq 1$, the inductively defined obstructions to existence of an algebraic symplectic structure lie in the (untwisted) groups $H^{i+1}(X, \boldsymbol{\pi}^{\mathbb{A}^1}_i(X_d))$. The sheaves $\boldsymbol{\pi}^{\mathbb{A}^1}_i(X_d)$ are stable for $i \leq 2d-2$, in which case they coincide with $\mathbf{GW}^3_{i+1}$ as we observed above. Thus, we obtain an inductively defined sequence of elements of $H^{i+1}(Y, \mathbf{GW}^3_{i+1})$ for $i \leq 2d-2$. If all of these obstructions vanish, then there is a final obstruction in $H^{2d}(Y, \boldsymbol{\pi}_{2d-1}(X_d))$. We will revisit this interpretation in subsequent work.

The first non-stable $\mathbb{A}^1$-homotopy sheaf of Sp_{2n}

In this section, we describe the first non-stable $\mathbb{A}^1$-homotopy sheaf of Sp_{2n} (the stable range was identified by Wendt; see [Wen11, AF12b]). There are two approaches to identifying this sheaf: either we can generalize the approach to the computation of $\boldsymbol{\pi}^{\mathbb{A}^1}_2(Sp_2)$ provided in [AF12b, Theorem 3.20], or we can use the results of the previous section regarding the first non-stable homotopy sheaf of X_m; we follow the first approach and describe the second approach in slightly more detail later.

To begin, observe that the long exact sequence in $\mathbb{A}^1$-homotopy sheaves associated with the $\mathbb{A}^1$-fiber sequence arising from the Sp_{2n}-torsor $Sp_{2n+2} \to Sp_{2n+2}/Sp_{2n}$ yields an exact sequence of the form

$$\boldsymbol{\pi}^{\mathbb{A}^1}_{2n+1}(Sp_{2n+2}) \stackrel{\psi_{2n+2}}{\longrightarrow} \boldsymbol{\pi}^{\mathbb{A}^1}_{2n+1}(Sp_{2n+2}/Sp_{2n}) \longrightarrow \boldsymbol{\pi}^{\mathbb{A}^1}_{2n}(Sp_{2n}) \longrightarrow \boldsymbol{\pi}^{\mathbb{A}^1}_{2n}(Sp_{2n+2}) \longrightarrow 0.$$

The sheaves involving Sp_{2n+2} are already in the stable range, and since there is an $\mathbb{A}^1$-weak equivalence $Sp_{2n+2}/Sp_{2n} \longrightarrow \mathbb{A}^{2n+2} \setminus 0$, we obtain an exact sequence of the form

$$\mathbf{K}^{Sp}_{2n+2} \stackrel{\psi_{2n+2}}{\longrightarrow} \mathbf{K}^{MW}_{2n+2} \longrightarrow \boldsymbol{\pi}^{\mathbb{A}^1}_{2n}(Sp_{2n}) \longrightarrow \mathbf{K}^{Sp}_{2n+1} \longrightarrow 0.$$

If we set
$$\mathbf{S}''_{2n+2} := \mathrm{coker}(\mathbf{K}^{Sp}_{2n+2} \stackrel{\psi_{2n+2}}{\longrightarrow} \mathbf{K}^{MW}_{2n+2}),$$
then our goal is to describe $\mathbf{S}''_{2n+2}$ explicitly.

To this end, we again use the morphism of fiber sequences associated with the fiber square 3.1 (see [AF12b, Corollary 3.12]), to obtain a commutative diagram of long exact sequences in $\mathbb{A}^1$-homotopy sheaves of the form:

$$\begin{tikzcd}
\mathbf{K}^{Sp}_{2n+2} \arrow[r, "\psi_{2n+2}"] \arrow[d] & \boldsymbol{\pi}^{\mathbb{A}^1}_{2n+1}(\mathbb{A}^{2n+2}\setminus 0) \arrow[r] \arrow[d, equal] & \boldsymbol{\pi}^{\mathbb{A}^1}_{2n}(Sp_{2n}) \arrow[r] \arrow[d] & \mathbf{K}^{Sp}_{2n+1} \arrow[d] \\
\boldsymbol{\pi}^{\mathbb{A}^1}_{2n+1}(SL_{2n+2}) \arrow[r] & \boldsymbol{\pi}^{\mathbb{A}^1}_{2n+1}(\mathbb{A}^{2n+2}\setminus 0) \arrow[r] & \boldsymbol{\pi}^{\mathbb{A}^1}_{2n}(SL_{2n+1}) \arrow[r] & \mathbf{K}^{Q}_{2n+1}.
\end{tikzcd}$$

Now, the sheaf $\boldsymbol{\pi}^{\mathbb{A}^1}_{2n+1}(SL_{2n+2})$ was computed in [AF12a, Theorem 2.3]. In particular, we observed in [AF12a, Lemma 2.2] that the homomorphism $\boldsymbol{\pi}^{\mathbb{A}^1}_{2n+1}(SL_{2n+2}) \to \mathbf{K}^{MW}_{2n+2}$ factors through the stabilization homomorphism $\boldsymbol{\pi}^{\mathbb{A}^1}_{2n+1}(SL_{2n+2}) \to \mathbf{K}^{Q}_{2n+2}$, and through $\mathbf{K}^{Q}_{2n+2} \to 2\mathbf{K}^{M}_{2n+2} \subset \mathbf{K}^{MW}_{2n+2}$.

Since the above diagram indicates that ψ_{2n+2} factors through $\boldsymbol{\pi}^{\mathbb{A}^1}_{2n+1}(SL_{2n+2})$, it follows that the homomorphism ψ_{2n+2} factors through $\mathbf{K}^{Q}_{2n+2}$. Moreover, since the stabilization homomorphism in question is induced by the composite $Sp_{2n+2} \to SL_{2n+2}$, it follows that the induced homomorphism $\mathbf{K}^{Sp}_{2n+2} \to \mathbf{K}^{Q}_{2n+2}$ is precisely the forgetful homomorphism.

Let ψ'_{2n+2} be the composite of the forgetful morphism $\mathbf{K}^{Sp}_{2n+2} \to \mathbf{K}^{Q}_{2n+2}$ and the morphism $\mathbf{K}^{Q}_{2n+2} \to 2\mathbf{K}^{M}_{2n+2}$ described above, and set

$$\mathbf{S}'_{2n+2} := \mathrm{coker}(\mathbf{K}^{Sp}_{2n+2} \stackrel{\psi'_{2n+2}}{\longrightarrow} 2\mathbf{K}^{M}_{2n+2}).$$

The discussion above also yields an epimorphism $\mathbf{S}''_{2n+2} \to \mathbf{S}'_{2n+2}$, but the kernel of this morphism can be understood more precisely using the techniques of the proof of [AF12a, Theorem 2.3]

We have the fiber product presentation (see [AF12a, §2] for a discussion):

$$\begin{tikzcd}
\mathbf{K}^{MW}_{*} \arrow[r] \arrow[d] & \mathbf{I}^{*} \arrow[d] \\
\mathbf{K}^{M}_{*} \arrow[r] & \mathbf{K}^{M}_{*}/2.
\end{tikzcd}$$

The composite morphism $\mathbf{K}^{Q}_{2n+2} \to \mathbf{K}^{MW}_{2n+2} \to \mathbf{I}^{2n+2}$ is trivial, while the composite morphism $\mathbf{K}^{Q}_{2n+2} \to \mathbf{K}^{M}_{2n+2}$ is precisely the one described above (it has image contained in $2\mathbf{K}^{M}_{2n+2}$). We therefore obtain an induced morphism $\mathbf{S}'_{2n+2} \to \mathbf{K}^{M}_{2n+2}/2$. On the other hand, the morphism $\mathbf{K}^{Sp}_{2n+2} \to \mathbf{K}^{MW}_{2n+2}$ induces a morphism to $\mathbf{I}^{2n+2}$. Since this morphism factors through $\mathbf{K}^{Q}_{2n+2}$, it follows that this composite map is trivial, and we obtain a morphism $\mathbf{S}''_{2n+2} \to \mathbf{I}^{2n+2}$. The next result is an immediate consequence of this discussion.

Proposition 3.12. *There is a fiber product diagram of the form*

$$\begin{tikzcd} \mathbf{S}''_{2n+2} \arrow[r] \arrow[d] & \mathbf{I}^{2n+2} \arrow[d] \\ \mathbf{S}'_{2n+2} \arrow[r] & \mathbf{K}^M_{2n+2}/2, \end{tikzcd}$$

where the morphisms in the fiber product are those defined above.

Finally, we understand the order in the torsion of the sheaf $\mathbf{S}'_{2n+2}$.

Lemma 3.13. *If the base field k is assumed to have characteristic unequal to 2, there is an epimorphism $\mathbf{K}^M_{2n+2}/(2(2n+1)!) \to \mathbf{S}'_{2n+2}$.*

Proof. This is proven in a fashion identical to [AF12b, Lemma 3.19]. It suffices to understand what happens on the "symbolic part" of $K^{Sp}_{2n+2}(F)$ for any finitely generated field extension $k \subset F$. To understand this, recall that we have the hyperbolic morphism $K^Q_{2n+2}(F) \to K^{Sp}_{2n+2}(F)$ and the natural homomorphism $K^M_{2n+2}(F) \to K^Q_{2n+2}(F)$. The composite map

$$K^M_{2n+2}(F) \to K^Q_{2n+2}(F) \to K^M_{2n+2}(F)$$

is multiplication by $-(2n+1)!$ by Suslin's result [Sus84, Corollary 4.4], and the composite map

$$K^M_{2n+2}(F) \longrightarrow K^Q_{2n+2}(F) \longrightarrow K^{Sp}_{2n+2}(F) \longrightarrow K^Q_{2n+2}(F) \longrightarrow K^M_{2n+2}(F)$$

is multiplication by $-2(2n+1)!$ from the above fact combined with [AF12b, Lemma 4.3]. $\square$

For convenient reference, we summarize the above results in the following statement.

Theorem 3.14. *There is an exact sequence of the form*

$$0 \longrightarrow \mathbf{S}''_{2n+2} \longrightarrow \boldsymbol{\pi}^{\mathbb{A}^1}_{2n}(Sp_{2n}) \longrightarrow \mathbf{K}^{Sp}_{2n+1} \longrightarrow 0,$$

and a fiber product diagram

$$\begin{tikzcd} \mathbf{S}''_{2n+2} \arrow[r] \arrow[d] & \mathbf{I}^{2n+2} \arrow[d] \\ \mathbf{S}'_{2n+2} \arrow[r] & \mathbf{K}^M_{2n+2}/2 \end{tikzcd}$$

where $\mathbf{S}''_{2n+2}$ and $\mathbf{S}'_{2n+2}$ are defined above, and $\mathbf{S}'_{2n+2}$ is a quotient of $\mathbf{K}^M_{2n+2}/(2(2n+1)!)$.

As in the previous section, more precise statements regarding the structure of the sheaf $\mathbf{S}''_{2n+2}$ can be made after sufficiently many contractions. The following results show that the structure of the sheaf $\mathbf{S}''_{2n+2}$ depends on the parity of n.

Lemma 3.15. *If n is an even integer, then the morphism of sheaves*

$$\mathbf{K}^M_{2n+2}/(2(2n+1)!) \longrightarrow \mathbf{S}'_{2n+2}$$

induces an isomorphism $\mathbf{K}^M_2/((2n+1)!) \to (\mathbf{S}'_{2n+2})_{-2n}$. Moreover, there is a cartesian square of the form

$$\begin{tikzcd} (\mathbf{S}''_{2n+2})_{-2n} \arrow[r] \arrow[d] & \mathbf{I}^2 \arrow[d] \\ \mathbf{K}^M_2/((2n+1)!) \arrow[r] & \mathbf{K}^M_2/2. \end{tikzcd}$$

Proof. Recall that the sheaf $\mathbf{S}''_{2n+2}$ is the cokernel of the composite map

$$\mathbf{K}^{Sp}_{2n+2} = \mathbf{GW}^2_{2n+2} \xrightarrow{F_{2,2n+2}} \mathbf{K}^Q_{2n+2} \longrightarrow 2\mathbf{K}^M_{2n+2}$$

where $F_{2,2n+2}$ is the forgetful homomorphism. Contracting $2n$ times and using Proposition 2.14, we obtain a composite

$$\mathbf{GW}^{2-2n}_2 \xrightarrow{F_{2-2n,2}} \mathbf{K}^Q_2 \longrightarrow 2\mathbf{K}^M_2$$

whose cokernel is $(\mathbf{S}''_{2n+2})_{-2n}$. We know from [AF12a, Lemma 2.9] that the cokernel of the morphism $\mathbf{K}^Q_2 \to 2\mathbf{K}^M_2$ is precisely $\mathbf{K}^M_2/((2n+1)!)$ and it suffices to show that $F_{2-2n,2}$ is onto to conclude.

Since n is even, we can identify $\mathbf{GW}^{2-2n}_2 = \mathbf{GW}^2_2$ and the forgetful map $F_{2,2}$ is the natural morphism $\mathbf{GW}^2_2 = \mathbf{K}^{MW}_2 \to \mathbf{K}^M_2$, which is surjective by construction. $\square$

Lemma 3.16. *If n is an odd integer, then the morphism of sheaves*

$$\mathbf{K}^M_{2n+2}/(2(2n+1)!) \to \mathbf{S}'_{2n+2}$$

induces an isomorphism $\mathbf{K}^M_1/(2(2n+1)!) \to (\mathbf{S}'_{2n+2})_{-2n-1}$. Moreover, there is a cartesian square of the form

$$\begin{tikzcd} (\mathbf{S}''_{2n+2})_{-2n-1} \arrow[r] \arrow[d] & \mathbf{I} \arrow[d] \\ \mathbf{K}^M_1/((2n+1)!) \arrow[r] & \mathbf{K}^M_1/2. \end{tikzcd}$$

Proof. Arguing as in the previous lemma, we obtain a composite morphism

$$\mathbf{GW}^{1-2n}_1 \xrightarrow{F_{1-2n,1}} \mathbf{K}^Q_1 \longrightarrow 2\mathbf{K}^M_1$$

whose cokernel is $(\mathbf{S}''_{2n+2})_{-2n-1}$. The cokernel of $\mathbf{K}^Q_1 \to 2\mathbf{K}^M_1$ is $\mathbf{K}^M_1/((2n+1)!)$ by [AF12a, Lemma 2.9] and it remains to show that the image of $\mathbf{GW}^{1-2n}_1 \to \mathbf{K}^Q_1$ is $2\mathbf{K}^Q_1$. Since n is odd, we can identify $\mathbf{GW}^{1-2n}_1 = \mathbf{GW}^3_1$. Combining [FRS12, Lemma 2.3] and [AF12b, Lemma 4.3] yields the required statement regarding the image. $\square$

Remark 3.17. The complex realization of Sp_{2n} is the group $Sp_{2n}(\mathbb{C})$, which is homotopy equivalent to the compact symplectic group Sp_{2n}. On the other hand, the real realization of Sp_{2n} is the real Lie group $Sp_{2n}(\mathbb{R})$, which is homotopy equivalent to its maximal compact subgroup $U(n)$. It is known that $\pi_{4n+2}(Sp_{2n})$ is $\mathbb{Z}/(2n+1)!$ if n is even and $\mathbb{Z}/(2(2n+1)!)$ if n is odd [Har63]. There is a canonical morphism from $\boldsymbol{\pi}^{\mathbb{A}^1}_{2n,2n+2}(Sp_{2n})(\mathbb{C}) \to \pi_{4n+2}(Sp_{2n})$. In view of the two lemmas above, this morphism is an isomorphism for arbitrary n since, in each case, one can explicitly lift a generator.

The first non-stable $\mathbb{A}^1$-homotopy sheaf of Sp_{2n}/GL_n

As above, we can study the $\mathbb{A}^1$-connectivity of the morphism $Sp_{2n}/GL_n \to Sp/GL$. The next result is established in exactly the same way as Lemma 3.4.

Lemma 3.18. *The sequences of closed immersion group homomorphisms*

i) $GL_n \hookrightarrow Sp_{2n} \hookrightarrow Sp_{2n+2}$, *and*

ii) $GL_n \hookrightarrow GL_{n+1} \hookrightarrow Sp_{2n+2}$

yield $\mathbb{A}^1$-fiber sequences of the form

i) $Sp_{2n}/GL_n \to Sp_{2n+2}/GL_n \to Sp_{2n+2}/Sp_{2n}$, *and*

ii) $GL_{n+1}/GL_n \to Sp_{2n+2}/GL_n \to Sp_{2n+2}/GL_{n+1}$.

Using the above fiber sequences, we can study the first non-stable $\mathbb{A}^1$-homotopy sheaf of Sp_{2n}/GL_n. To state the result, we first make two definitions.

Theorem 3.19. *The morphism $Sp_{2n}/GL_n \to Sp_{2n+2}/GL_{n+1}$ is $(n-1)$-$\mathbb{A}^1$-connected. For any integer $n \geq 2$, there is a short exact sequence of the form*

$$0 \longrightarrow \mathbf{V}_{n+1} \longrightarrow \boldsymbol{\pi}^{\mathbb{A}^1}_n(Sp_{2n}/GL_n) \longrightarrow \mathbf{GW}^1_n \longrightarrow 0,$$

where if n is even, then $\mathbf{V}_{n+1}$ is the cokernel of the morphism $\mathbf{GW}^1_{n+1} \to \mathbf{K}^{MW}_{n+1}$, while if n is odd, then $\mathbf{V}_{n+1}$ is the cokernel of a morphism $\mathbf{GW}^1_{n+1} \to \mathbf{K}^M_{n+1}$, and, in each case, the morphism in question factors through the forgetful morphism $\mathbf{GW}^1_{n+1} \longrightarrow \mathbf{K}^Q_{n+1}$.

Proof. For the first statement, we factor the inclusion $Sp_{2n}/GL_n \to Sp_{2n+2}/GL_{n+1}$ through Sp_{2n+2}/GL_n. From Lemma 3.18, we see that $Sp_{2n}/GL_n \to Sp_{2n+2}/GL_n$ is $(2n-1)$-$\mathbb{A}^1$-connected (since Sp_{2n+2}/Sp_{2n} is $\mathbb{A}^1$-weak equivalent to $\mathbb{A}^{2n+2} \setminus 0$), and that $Sp_{2n+2}/GL_n \to Sp_{2n+2}/GL_{n+1}$ is $(n-1)$-$\mathbb{A}^1$-connected.

For the second statement, the fiber sequences in Lemma 3.18 yield isomorphisms

$$\boldsymbol{\pi}^{\mathbb{A}^1}_{n+1}(Sp_{2(n+1)}/GL_{n+1}) \xrightarrow{\sim} \boldsymbol{\pi}^{\mathbb{A}^1}_{n+1}(Sp_{2(n+2)}/GL_{n+1}), \text{ and}$$
$$\boldsymbol{\pi}^{\mathbb{A}^1}_{n}(Sp_{2n}/GL_{n}) \xrightarrow{\sim} \boldsymbol{\pi}^{\mathbb{A}^1}_{n}(Sp_{2(n+1)}/GL_{n}).$$

Applying these identifications in the long exact sequence in $\mathbb{A}^1$-homotopy sheaves associated with the second fiber sequence in Lemma 3.18, and combining this with the long exact sequence in $\mathbb{A}^1$-homotopy sheaves associated with the first fiber sequence in the same lemma with n replaced by $n+1$ yields the following diagram

$$\begin{tikzcd}
\boldsymbol{\pi}_{n+1}^{\mathbb{A}^1}(GL_{n+2}/GL_{n+1}) \arrow[d] & & & \\
\boldsymbol{\pi}_{n+1}^{\mathbb{A}^1}(Sp_{2n+4}/GL_{n+1}) \arrow[r] \arrow[d] & \boldsymbol{\pi}_{n}^{\mathbb{A}^1}(GL_{n+1}/GL_{n}) \arrow[r] & \boldsymbol{\pi}_{n}^{\mathbb{A}^1}(Sp_{2n+2}/GL_{n}) \arrow[r] & \boldsymbol{\pi}_{n}^{\mathbb{A}^1}(Sp_{2n+2}/GL_{n+1}) \arrow[r] & 0 \\
\boldsymbol{\pi}_{n+1}^{\mathbb{A}^1}(Sp_{2n+4}/GL_{n+2}) \arrow[d] & & & \\
0 & & &
\end{tikzcd}$$

By the first statement there are isomorphisms of the form $\boldsymbol{\pi}_{n}^{\mathbb{A}^1}(Sp_{2n+2}/GL_{n+1}) \cong \boldsymbol{\pi}_{n}^{\mathbb{A}^1}(Sp/GL)$ and also $\boldsymbol{\pi}_{n+1}^{\mathbb{A}^1}(Sp_{2n+4}/GL_{n+2}) \cong \boldsymbol{\pi}_{n+1}^{\mathbb{A}^1}(Sp/GL)$, and these homotopy sheaves were identified with $\mathbf{GW}_n^1$ and $\mathbf{GW}_{n+1}^1$ by Corollary 2.12.

The map

$$\boldsymbol{\pi}_{n+1}^{\mathbb{A}^1}(Sp_{2n+4}/GL_{n+1}) \to \boldsymbol{\pi}_{n}^{\mathbb{A}^1}(GL_{n+1}/GL_{n})$$

factors through $\boldsymbol{\pi}_{n+1}^{\mathbb{A}^1}(GL_{n+1})$ by definition and we have a commutative diagram

$$\begin{tikzcd}
\boldsymbol{\pi}_{n+1}^{\mathbb{A}^1}(Sp_{2n+4}/GL_{n+1}) \arrow[r] \arrow[d] & \boldsymbol{\pi}_{n}^{\mathbb{A}^1}(GL_{n+1}) \arrow[d] \\
\boldsymbol{\pi}_{n+1}^{\mathbb{A}^1}(Sp_{2n+4}/GL_{n+2}) \arrow[r] & \boldsymbol{\pi}_{n}^{\mathbb{A}^1}(GL_{n+2}).
\end{tikzcd}$$

Under the identifications $\boldsymbol{\pi}_{n+1}^{\mathbb{A}^1}(Sp_{2n+4}/GL_{n+2}) = \mathbf{GW}_{n+1}^1$ and $\boldsymbol{\pi}_{n}^{\mathbb{A}^1}(GL_{n+2}) = \mathbf{K}_{n+1}^Q$, the bottom map is the forgetful homomorphism. The composite

$$\boldsymbol{\pi}_{n+1}^{\mathbb{A}^1}(GL_{n+2}/GL_{n+1}) \to \boldsymbol{\pi}_{n+1}^{\mathbb{A}^1}(Sp_{2n+4}/GL_{n+1}) \to \boldsymbol{\pi}_{n}^{\mathbb{A}^1}(GL_{n+1})$$

is precisely the connecting homomorphism in the long exact sequence of homotopy groups induced by the fiber sequence

$$GL_{n+1} \to GL_{n+2} \to GL_{n+2}/GL_{n+1},$$

and therefore we get a commutative diagram with exact columns

$$\begin{tikzcd}
\boldsymbol{\pi}^{\mathbb{A}^1}_{n+1}(GL_{n+2}/GL_{n+1}) \arrow[r, equal] \arrow[d] & \boldsymbol{\pi}^{\mathbb{A}^1}_{n+1}(GL_{n+2}/GL_{n+1}) \arrow[d] \arrow[dr] & \\
\boldsymbol{\pi}^{\mathbb{A}^1}_{n+1}(Sp_{2n+4}/GL_{n+1}) \arrow[r] \arrow[d] & \boldsymbol{\pi}^{\mathbb{A}^1}_{n}(GL_{n+1}) \arrow[r] \arrow[d] & \boldsymbol{\pi}^{\mathbb{A}^1}_{n}(GL_{n+1}/GL_n) \\
\mathbf{GW}^1_{n+1} \arrow[r] \arrow[d] & \mathbf{K}^Q_{n+1} \arrow[d] & \\
0 & 0 &
\end{tikzcd}$$

We can understand the two right-hand columns as in [AF12b, proof of Lemma 3.2] (if n is odd) and [AF12a, proof of Lemma 2.2] (if n is even). In particular, the diagonal map

$$\boldsymbol{\pi}^{\mathbb{A}^1}_{n+1}(GL_{n+2}/GL_{n+1}) \to \boldsymbol{\pi}^{\mathbb{A}^1}_{n}(GL_{n+1}/GL_n)$$

is multiplication by η if n is odd and trivial if n is even. Consequently, the morphism $\mathbf{GW}^1_{n+1} \to \mathbf{K}^{MW}_{n+1}$ described above factors as

$$\mathbf{GW}^1_{n+1} \longrightarrow \mathbf{K}^Q_{n+1} \longrightarrow \mathbf{K}^M_{n+1}$$

if n is odd, and

$$\mathbf{GW}^1_{n+1} \longrightarrow \mathbf{K}^Q_{n+1} \longrightarrow \mathbf{K}^{MW}_{n+1}$$

if n is even; in each case, the first morphism is the forgetful morphism. The second morphism for n odd was described in [AF12b, §3], and for n even was described in [AF12a, §2]. $\square$

Remark 3.20. Suppose that n is even. Then we can study the map

$$\mathbf{GW}^1_{n+1} \longrightarrow \mathbf{K}^Q_{n+1} \longrightarrow \mathbf{K}^{MW}_{n+1}$$

as follows. We precompose with the hyperbolic map $\mathbf{K}^Q_{n+1} \to \mathbf{GW}^1_{n+1}$ and we observe that the composite

$$\mathbf{K}^Q_{n+1} \to \mathbf{GW}^1_{n+1} \to \mathbf{K}^Q_{n+1}$$

is multiplication by 2 on the symbolic part of $\mathbf{K}^Q_{n+1}$ ([AF12b, Lemma 4.3]). Thus we get an epimorphism of sheaves $\mathbf{K}^M_{n+1}/2(n!) \times_{\mathbf{K}^M_{n+1}/2} \mathbf{I}^{n+1} \to \mathbf{V}_{n+1}$.

Arguing as in the previous section, we can show that the sheaf $(\mathbf{V}_{n+1})_{-n}$ depends on the class of n modulo 4. If $n \equiv 2 \pmod 4$, the epimorphism of sheaves

$$\mathbf{K}^M_1/2(n!) \times_{\mathbf{K}^M_1/2} \mathbf{I} \to (\mathbf{V}_{n+1})_{-n}$$

is in fact an isomorphism. It induces an isomorphism $\mathbf{K}^M_1/(n!) \times_{\mathbf{K}^M_1/2} \mathbf{I} \to (\mathbf{V}_{n+1})_{-n}$ if $n \equiv 0 \pmod 4$.

If n is odd, things are even simpler. Indeed, we claim that the map $\mathbf{K}^M_{n+1} \to \mathbf{V}_{n+1}$ yields an isomorphism $\mathbf{K}^M_1 \to (\mathbf{V}_{n+1})_{-n}$ provided $n \neq 1$. If $n \equiv 3 \pmod 4$, then this follows from the fact that the n-th contraction of the forgetful homomorphism $\mathbf{GW}^1_{1+n} \to \mathbf{K}^Q_{n+1}$ is trivial because $\mathbf{GW}^2_1$ is trivial. If $n \equiv 1 \pmod 4$, the n-th contraction of the forgetful homomorphism $\mathbf{GW}^1_{1+n} \to \mathbf{K}^Q_{n+1}$ is the forgetful homomorphism $\mathbf{GW}^0_1 \to \mathbf{K}^Q_1$ whose image is the constant subsheaf ± 1 by [FRS12, Lemma 2.4]. The contraction of the map $\mathbf{K}^Q_1 \to \mathbf{K}^M_1$ is multiplication by $n!$ and thus the composite $\mathbf{GW}^0_1 \to \mathbf{K}^Q_1 \to \mathbf{K}^M_1$ is trivial if $n \neq 1$.

In case $n = 1$, the arguments above prove that the map $\mathbf{K}^M_2 \to \mathbf{V}_2$ induces an isomorphism $\mathbf{K}^M_1/(\pm 1) \to (\mathbf{V}_2)_{-1}$.

4 The Hopf map ν and $\pi_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)$

In the previous section, we described the sheaf $\boldsymbol{\pi}_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)$ as an extension of the Grothendieck-Witt sheaf $\mathbf{GW}^3_4$ by a sheaf we called $\mathbf{F}_5$. The goal here is to provide a better understanding of the "topological" origin of the sheaf $\mathbf{F}_5$ and the factor of 24 that appears in the Milnor K-theory sheaf that contributes to $\mathbf{F}_5$. We will see that the 24 appearing in the description of $\mathbf{F}_5$ is the "same" as the 24 the appears in the third stable homotopy group of spheres. We place the word "topological" in quotes because the initial computations we make are purely algebraic. To begin, we study what happens under real and complex realization. Finally, we give a purely algebraic proof of the stable non-triviality of ν (see Theorem 4.17). One consequence of this is that $\mathbb{P}^1$-suspensions of ν contribute to $\boldsymbol{\pi}_n^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0)$ for $n \geq 4$ as well.

Contracted homotopy sheaves

Precomposing with elements of $\boldsymbol{\pi}^{\mathbb{A}^1}_{3,5}(\Sigma^3_s \mathbf{G}_m^{\wedge 5})$ gives $\boldsymbol{\pi}^{\mathbb{A}^1}_{3,5}(\mathcal{X})$ a $K^{MW}_0(k)$-module structure for any pointed space $\mathcal{X}$, and this module structure is covariantly functorial in $\mathcal{X}$ by construction. In particular $\boldsymbol{\pi}^{\mathbb{A}^1}_{3,5}(\mathbb{A}^3 \setminus 0)$ admits the structure of a $K^{MW}_0(k)$-module.

The suspension morphism yields a map

$$\Sigma^3_s \mathbf{G}_m^{\wedge 5} \longrightarrow \Omega^1_s \Sigma^4_s \mathbf{G}_m^{\wedge 5} \cong \Omega^1_s SL_5/SL_4.$$

We saw above that the connecting morphism in the $\mathbb{A}^1$-fiber sequence $X_2 \to X_3 \to SL_5/SL_4$ was a morphism $\delta : \Omega^1_s SL_5/SL_4 \to X_2 = SL_4/Sp_4 \cong \mathbb{A}^3 \setminus 0$. Abusing notation, we will write

$$\delta : \Sigma^3_s \mathbf{G}_m^{\wedge 5} \longrightarrow \Omega^1_s SL_5/SL_4 \longrightarrow X_2$$

for the composite map.

By Morel's $\mathbb{A}^1$-Freudenthal suspension theorem, since the space $\Sigma^3_s \mathbf{G}_m^{\wedge 5}$ is 2-$\mathbb{A}^1$-connected (by Morel's unstable $\mathbb{A}^1$-connectivity theorem), the suspension morphism induces an isomorphism upon applying $\boldsymbol{\pi}_{3,j}$ for any $j \geq 0$. In particular, the morphism

$$\mathbf{K}^{MW}_0 = \boldsymbol{\pi}^{\mathbb{A}^1}_{3,5}(\Sigma^3_s \mathbf{G}_m^{\wedge 5}) \longrightarrow \boldsymbol{\pi}^{\mathbb{A}^1}_{3,5}(\Omega^1_s SL_5/SL_4)$$

is an isomorphism. Using this notation, we can now describe the $K^{MW}_0(k)$-module structure of $\boldsymbol{\pi}^{\mathbb{A}^1}_{3,5}(\mathbb{A}^3 \setminus 0)(k)$.

Proposition 4.1. *There is a canonical isomorphism*

$$\boldsymbol{\pi}_{3,5}^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0) \cong \mathbb{Z}/24 \times_{\mathbb{Z}/2} \mathbf{W},$$

and $\boldsymbol{\pi}_{3,5}^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)(k)$ *is generated as a* $K_0^{MW}(k)$*-module by* δ.

Proof. By Theorem 3.7, we know that $\boldsymbol{\pi}_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)$ is an extension of $\mathbf{GW}_4^3$ by $\mathbf{F}_5$. By [Mor12, Theorem 6.13], we know that $\boldsymbol{\pi}_{3,5}^{\mathbb{A}^1}(\mathbb{A}^5 \setminus 0) \cong \boldsymbol{\pi}_3^{\mathbb{A}^1}(\mathbb{A}^5 \setminus 0)_{-5}$. Since $\boldsymbol{\pi}_{3,5}^{\mathbb{A}^1}(X_3) = (\mathbf{GW}_4^3)_{-5} = 0$, it follows from the long exact sequence in $\mathbb{A}^1$-homotopy sheaves associated with the fiber sequence $X_2 \to X_3 \to \mathbb{A}^5 \setminus 0$ that the morphism

$$\mathbf{K}_0^{MW} = \boldsymbol{\pi}_{3,5}^{\mathbb{A}^1}(\Omega_s^1 SL_5/SL_4) \longrightarrow \boldsymbol{\pi}_{3,5}^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)$$

is an epimorphism. In other words, $\boldsymbol{\pi}_{3,5}^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)$ is generated as a $K_0^{MW}(k)$-module by the connecting homomorphism δ.

By exactness of contractions, and the fact that $(\mathbf{GW}_4^3)_{-5} = 0$, it follows that

$$\boldsymbol{\pi}_{3,5}^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0) \cong (\mathbf{F}_5)_{-5}.$$

The result follows then from Lemma 3.8. □

Complex realization

If $k = \mathbb{C}$, then we can apply the complex realization functor to the $\mathbb{A}^1$-fiber sequence

$$X_2 \longrightarrow X_3 \longrightarrow SL_5/SL_4$$

to obtain (after shifting) the topological fiber sequence

$$\Omega^1 S^9 \longrightarrow S^5 \longrightarrow SU(6)/Sp(6).$$

By precomposing the map $\Omega^1 S^9 \to S^5$ by the suspension map $S^8 \to \Omega^1\Sigma^1 S^8$, we obtain a map $S^8 \to S^5$, which we want to identify. The long exact sequence in homotopy groups of the above fiber sequence yields:

$$\pi_8(\Omega^1 S^9) \longrightarrow \pi_8(S^5) \longrightarrow \pi_8(SU(6)/Sp(6)) \longrightarrow 0.$$

We know that $\pi_8(SU(6)/Sp(6)) \cong \pi_8(SU/Sp) = \pi_8(U/Sp)$, and by Bott periodicity, we know that $\pi_8(U/Sp) = \pi_{10}(O) = \pi_2(O) = 0$. In other words, the portion of the long exact sequence displayed above collapses to the surjection

$$\mathbb{Z} \longrightarrow \pi_8(S^5) \longrightarrow 0.$$

One knows that $\pi_8(S^5)$ is $\mathbb{Z}/24$ generated by the suspension of the Hopf map ν. Since the composite map $S^8 \to S^5$ mentioned above corresponds to the image of $1 \in \mathbb{Z}$, it follows that the composite map is ν.

Corollary 4.2. *Under complex realization, the homomorphism*

$$\boldsymbol{\pi}_{3,5}^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)(\mathbb{C}) \to \pi_8(S^5) = \mathbb{Z}/24$$

is an isomorphism.

Proof. By Proposition 4.1, we know that $\boldsymbol{\pi}_{3,5}^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0) \cong \mathbb{Z}/24 \times_{\mathbb{Z}/2} \mathbf{W}(k)$. As a consequence, complex realization yields a homomorphism $\mathbb{Z}/24 \longrightarrow \mathbb{Z}/24$. Moreover, we saw before the statement that the generator of (the topological) $\mathbb{Z}/24$ is precisely the connecting homomorphism in the fibration associated with the complex points of $X_2 \to X_3 \to \mathbb{A}^5 \setminus 0$. Since this connecting homomorphism is algebraically defined, and δ is a generator of $\boldsymbol{\pi}_{3,5}^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)(\mathbb{C})$, it follows that complex realization maps the algebraic generator to the topological generator and is therefore an isomorphism. $\square$

Remark 4.3. We interpret this result as saying that the 24 that appears in $\boldsymbol{\pi}_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)$ is the "same" as that appearing in the third stable homotopy of the spheres.

Real realization

We can compute the homotopy groups of the real points as well to study real realization. We view this computation as providing an explanation for the appearance of $\mathbf{W}$ in $\boldsymbol{\pi}_{3,5}^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)$. Up to $\mathbb{A}^1$-homotopy, the fiber sequence $X_2 \to X_3 \to SL_5/SL_4$ yields the sequence

$$\mathbb{A}^3 \setminus 0 \longrightarrow SL_6/Sp_6 \longrightarrow \mathbb{A}^5 \setminus 0,$$

which upon taking real points gives the topological fiber sequence

$$S^2 \longrightarrow SO(6)/U(3) \longrightarrow S^4.$$

A computation using Bott periodicity shows that $\pi_3(SO(6)/U(3)) = \pi_3(O/U) = \pi_4(O) = 0$. As a consequence, the map $\pi_3(\Omega^1 S^4) \to \pi_3(S^2)$ in the long exact sequence is an isomorphism. Thus the composite map $S^3 \to \Omega^1 S^4 \to S^2$ is precisely the classical Hopf map η.

Corollary 4.4. *For any $j \geq 0$, real realization defines a surjective map $\boldsymbol{\pi}_{3,j}^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)(\mathbb{R}) \to \pi_3(S^2) = \mathbb{Z}$.*

Proof. We can compute $\boldsymbol{\pi}_{3,j}^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)$ by contracting $\boldsymbol{\pi}_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)$ j-times. To prove surjectivity, we will consider only $\mathbf{F}_5$, i.e., the kernel of the map $\boldsymbol{\pi}_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0) \to \mathbf{GW}_4^3$. By definition, we know that $\mathbf{F}_5$ admits an epimorphism from $\mathbf{T}_5$, which is a fiber product of $\mathbf{S}_5$ and $\mathbf{I}^5$ over $\mathbf{K}_5^M/2$. Moreover, the map $\mathbf{T}_5 \to \mathbf{F}_5$ is injective on $\mathbf{I}^5$. Contracting repeatedly and using the fact that $\mathbf{I}^i(\mathbb{R}) = \mathbb{Z}$ for any $i \leq 5$ (by convention $\mathbf{I}^i = \mathbf{W}$ for $i \leq 0$), we see that $\boldsymbol{\pi}_{3,j}^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)(\mathbb{R})$ is non-trivial. The real realization of the connecting homomorphism lifts the generator of $\pi_3(S^2)$ by the discussion preceding the statement. $\square$

Remark 4.5. The above computation shows that the factor of $\mathbf{I}^5$ is an avatar of the topological Hopf map $\eta : S^3 \to S^2$. Since the topological Hopf map η becomes 2-torsion in $\pi_4(S^3)$, we expect that the factor of $\mathbf{I}^5$ appearing in $\boldsymbol{\pi}_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)$ will become trivial after a single simplicial suspension, i.e., in $\boldsymbol{\pi}_4^{\mathbb{A}^1}(\mathbb{P}^{1\wedge 3})$. In particular, it should follow that $\boldsymbol{\pi}_{4,5}^{\mathbb{A}^1}(\mathbb{P}^{1\wedge 3}) = \mathbb{Z}/24$.

The $\mathbb{A}^1$-homotopy type of Q_4

Let Q_4 be the quadric defined by $x_1y_1 + x_2y_2 = z(z+1)$. The $\mathbb{A}^1$-homotopy type of Q_4 was effectively described in [AD07]; we review the argument here. Consider the closed subscheme E_2 of Q_4 defined by $x_1 = 0, x_2 = 0, z = -1$. Observe that E_2 is isomorphic to $\mathbb{A}^2$. Let $Y_4 \subset Q_4$ be the (open) complement of E_2. In [AD07], it is observed that there is a $\mathbb{G}_a$-torsor $\mathbb{A}^5 \to Y_4$, and therefore Y_4 is $\mathbb{A}^1$-contractible (in fact, over $\operatorname{Spec} \mathbb{Z}$). There is a cofiber sequence of the form

$$Y_4 \longrightarrow Q_4 \longrightarrow Th(\nu_{E_2/Q_4}) \longrightarrow \Sigma^1_s Y_4 \longrightarrow \cdots .$$

The normal bundle to $E_2 \subset Q_4$ is trivial, and picking a trivialization of ν_{E_2/Q_4} yields an $\mathbb{A}^1$-weak equivalence $Th(\nu_{E_2/Q_4}) \cong \mathbb{P}^{1^{\wedge 2}} \wedge E_{2+}$. Since E_2 is itself $\mathbb{A}^1$-contractible we see that $E_{2+} \cong S^0_k$, and therefore $Th(\nu_{E_2/Q_4}) \cong \mathbb{P}^{1^{\wedge 2}}$. Thus, there is an induced map $Q_4 \to \mathbb{P}^{1^{\wedge 2}}$. Since Y_4 is $\mathbb{A}^1$-contractible, the next result follows from the fact that pushouts of $\mathbb{A}^1$-weak equivalences along cofibrations are again $\mathbb{A}^1$-weak equivalence, which is a consequence of the construction of the $\mathbb{A}^1$-homotopy category [MV99, §2 Theorem 3.2].

Proposition 4.6 (Asok, Doran)**.** *For any field k, the map $Q_4 \to \mathbb{P}^{1^{\wedge 2}}$ is an $\mathbb{A}^1$-weak equivalence.*

Remark 4.7. More generally, let Q_{2n} be the smooth affine quadric defined by the equation

$$\sum_i x_i x_{n+i} = x_{2n+1}(1 + x_{2n+1}).$$

It is straightforward to check that $Q_2 \cong SL_2/\mathbf{G}_m$ and is therefore $\mathbb{A}^1$-weakly equivalent to $\mathbb{P}^1$. Let $E_n \subset Q_{2n}$ be the closed subscheme defined by $x_1 = \cdots = x_n = 0, z = -1$, and let Y_{2n} be its open complement. The same argument as above gives a map $Q_{2n} \to \mathbb{P}^{1^{\wedge n}}$. If one knew that Y_{2n} was $\mathbb{A}^1$-contractible, then it would follow that $Q_{2n} \to \mathbb{P}^{1^{\wedge n}}$ is an $\mathbb{A}^1$-weak equivalence. It is known that Y_{2n} cannot be the base space of a unipotent group torsor, so the techniques of [AD07] cannot be applied.

A geometric Hopf map and fibration

Given a pair of 2×2-matrices A and B consider the equation $\det A - \det B = 1$; the result is a quadric $Q_7' \subset \mathbb{A}^8$. Let $\mu : M_2 \times M_2 \to M_2$ be multiplication of 2×2-matrices. Define a function

$$h_\mu(A, B) := (\mu(A, B), \det B).$$

Observe that if $\det A - \det B = 1$, then since $\det(AB) = \det(A)\det(B)$, it follows that $\det(AB) = \det(B)(1 + \det B)$, i.e., h_μ restricts to a morphism $Q_7' \longrightarrow Q_4$. If Q_7 is the standard quadric defined by the equation $\sum_{i=1}^4 x_i x_{4+i} = 1$, then there is an obvious isomorphism $Q_7 \cong Q_7'$ obtained by changing signs.

Definition 4.8. The Hopf map $\nu : Q_7 \to Q_4$ is the map h_μ precomposed with the isomorphism $Q_7 \cong Q_7'$ described above.

Define an action of SL_2 on pairs (A, B) by means of the formula

$$C \cdot (A, B) = (AC, C^{-1}B).$$

If $C \in SL_2$, then $\det(AC) - \det(C^{-1}B) = \det(A) - \det(B)$, so this action preserves Q_7'. Moreover, $\mu(AC, C^{-1}B) = \mu(A, B)$ and $\det(C^{-1}B) = \det B$. Therefore, $h_\mu(C \cdot (A, B)) = h_\mu(A, B)$. In fact, this action makes Q_7' into an SL_2-torsor over Q_4 (see the proof of [AD08, Corollary 3.1] for details). Because SL_2-torsors give rise to $\mathbb{A}^1$-fiber sequences, we deduce the following result.

Proposition 4.9. *There is an $\mathbb{A}^1$-fibration sequence of the form*

$$Q_3 \longrightarrow Q_7' \longrightarrow Q_4.$$

Remark 4.10. There is a homotopically simpler but less geometric description of the Hopf map. Indeed, the multiplication map $SL_2 \times SL_2 \to SL_2$ yields a morphism $\Sigma_s^1 SL_2 \wedge SL_2 \to \Sigma_s^1 SL_2$, which provides another candidate for ν (see [Mor12, p. 189] for more discussion of this map). The morphism we called ν above and this Hopf map should agree (perhaps up to a sign).

Splitting the geometric Hopf fibration

The fiber sequence $Q_3 \to Q_7' \to Q_4$ gives rise to a long exact sequence in homotopy sheaves. The (pointed) inclusion map $Q_3 \to Q_7'$ gives an element of $[Q_3, Q_7']_{\mathbb{A}^1} = \boldsymbol{\pi}_{1,2}^{\mathbb{A}^1}(\Sigma_s^3 \mathbf{G}_m^{\wedge 4})(k)$. However, since $\boldsymbol{\pi}_{1,2}^{\mathbb{A}^1}(\Sigma_s^3 \mathbf{G}_m^{\wedge 4}) = \boldsymbol{\pi}_1^{\mathbb{A}^1}(\Sigma_s^3 \mathbf{G}_m^{\wedge 4})_{-2} = 0$, it follows that the inclusion map $Q_3 \to Q_7'$ is null-homotopic. In particular, the induced maps $\boldsymbol{\pi}_{i,j}^{\mathbb{A}^1}(Q_3) \to \boldsymbol{\pi}_{i,j}^{\mathbb{A}^1}(Q_7')$ are zero for arbitrary i and j.

The suspension homomorphism $Q_3 \to \Omega_s^1 \Sigma_s^1 Q_3$ together with the $\mathbb{A}^1$-weak equivalence $\Sigma_s^1 Q_3 \xrightarrow{\sim} Q_4$ yields a homomorphism

$$\boldsymbol{\pi}_{i,j}^{\mathbb{A}^1}(Q_3) \to \boldsymbol{\pi}_{i+1,j}^{\mathbb{A}^1}(Q_4)$$

that provides a splitting of the connecting homomorphism $\boldsymbol{\pi}_{i+1,j}^{\mathbb{A}^1}(Q_4) \to \boldsymbol{\pi}_{i,j}^{\mathbb{A}^1}(Q_3)$ in the long exact sequence in homotopy sheaves associated with the Hopf fibration. Combining these two facts yields the following result.

Proposition 4.11. *For any integers i, j, the long exact sequence in homotopy sheaves associated with the $\mathbb{A}^1$-fibration $Q_3 \to Q_7' \to Q_4$ breaks into split short exact sequences of the form*

$$0 \longrightarrow \boldsymbol{\pi}_{i,j}^{\mathbb{A}^1}(Q_7) \longrightarrow \boldsymbol{\pi}_{i,j}^{\mathbb{A}^1}(Q_4) \longrightarrow \boldsymbol{\pi}_{i-1,j}^{\mathbb{A}^1}(Q_3) \longrightarrow 0.$$

Taking $i = 3$ in the above proposition, using the fact that $\boldsymbol{\pi}_3^{\mathbb{A}^1}(Q_7') = \mathbf{K}_4^{MW}$, contracting 4 times, and replacing SL_2 by BSL_2 via an index shift yields the following split short exact sequence:

$$0 \longrightarrow \mathbf{K}_0^{MW} \xrightarrow{\nu_*} \boldsymbol{\pi}_{3,4}^{\mathbb{A}^1}(\mathbb{P}^{1\wedge 2}) \longrightarrow \boldsymbol{\pi}_{3,4}^{\mathbb{A}^1}(BSL_2) \longrightarrow 0.$$

Evaluating on k, and precomposing with elements of $[Q_7, Q_7]_{\mathbb{A}^1}$ defines a $K_0^{MW}(k)$-module structure on each term of the exact sequence and since the morphisms are compatible with this $K_0^{MW}(k)$-module structure, the above sequence is a split short exact sequence of $K_0^{MW}(k)$-modules.

Remark 4.12. It is possible to identify more explicitly the $K_0^{MW}(k)$-module structure on $\boldsymbol{\pi}_{3,4}^{\mathbb{A}^1}(BSL_2)(k)$ using the computation of $\boldsymbol{\pi}_2^{\mathbb{A}^1}(SL_2)$ in [AF12b] and [AF12a]. Indeed, contracting four times, we saw that $\boldsymbol{\pi}_{2,4}^{\mathbb{A}^1}(SL_2) \cong \mathbb{Z}/12 \times_{\mathbb{Z}/2} \mathbf{W}$ in [AF12a]. Moreover, since $\boldsymbol{\pi}_{2,4}^{\mathbb{A}^1}(Sp_4) = (\mathbf{K}_3^{Sp})_{-4} = 0$, the connecting homomorphism $\xi : \Omega_s^1 Sp_4/SL_2 \to SL_2$ in the long exact sequence of the fiber sequence

$$SL_2 \to Sp_4 \to Sp_4/SL_2$$

yields a surjective homomorphism $\mathbf{K}_0^{MW} \to \boldsymbol{\pi}_{2,4}^{\mathbb{A}^1}(SL_2)$. This surjective homomorphism provides a $K_0^{MW}(k)$-module generator of $\boldsymbol{\pi}_{2,4}^{\mathbb{A}^1}(SL_2) = \boldsymbol{\pi}_{3,4}^{\mathbb{A}^1}(BSL_2)$ that we also refer to as ξ. From these facts, one can deduce that

$$\boldsymbol{\pi}_{3,4}^{\mathbb{A}^1}(\mathbb{P}^{1\wedge 2})(k) \cong K_0^{MW}(k)\nu \oplus (\mathbb{Z}/12 \times_{\mathbb{Z}/2} W(k))\xi.$$

The cone of ν

If $\eta : \mathbb{A}^2 \setminus 0 \to \mathbb{P}^1$ is the Hopf map given by the usual projection morphism, it is a classical fact that the cone of η, computed in $\mathscr{H}_\bullet(k)$ is isomorphic to $\mathbb{P}^2$. To see this, one takes $\mathbb{P}^2$ and considers the standard open cover by two open sets isomorphic to $\mathbb{A}^2$ and $\mathbb{P}^2 \setminus 0$. The inclusion of the intersection gives a map $\mathbb{A}^2 \setminus 0 \to \mathbb{P}^2 \setminus 0$ that under the $\mathbb{A}^1$-weak equivalence $\mathbb{P}^2 \setminus 0 \to \mathbb{P}^1$ coincides with the Hopf map. Since $\mathbb{A}^2$ is contractible, the Mayer-Vietoris square gives the required computation of the cone. The benefit of this computation is that the cohomology of $\mathbb{P}^2$ is well understood.

We now provide an analogous computation for ν. To this end, consider the spaces HP^i defined by Panin and Walter in [PW10b]. In terms of the notation of quaternionic Grassmannians introduced at the beginning of Section 2, we have

$$\mathrm{HP}^n = HGr(1, n+1).$$

The space HP^n is a smooth affine scheme of dimension $4n$ that behaves in a fashion very similar to the quaternionic projective spaces one considers in topology.

Remark 4.13. One can check that HP^1 coincides with the quadric Q_4 we considered. The varieties HP^n can all be constructed as quotients of the split smooth affine quadric Q_{4n+3} by a free action of SL_2, generalizing the construction of Q_4 as a quotient of Q_7 by a free action of SL_2. In fact, the varieties HP^n can all be seen to be smooth over $\operatorname{Spec}\mathbb{Z}$.

Roughly speaking, HP^n admits a "cell decomposition" with cells of dimension $4i$. More precisely, there exist smooth locally closed (in general, quasi-affine) subschemes Z_{2i} in HP^n of codimension $2i$, such i) $Z_{2n} = \mathbb{A}^{2n}$, ii) each Z_{2i} is an $\mathbb{A}^1$-contractible variety realized as the quotient $\mathbb{A}^{4n-2i+1}$ by a free action of $\mathbb{G}_a$, and iii) the closure $\overline{Z}_{2i}$ is a vector bundle of rank $2i$ over HP^{n-i} [PW10b, Theorem 1.1]. Given this notation, we can state the computation.

Proposition 4.14. *The cone of ν in $\mathscr{H}_\bullet(k)$ is* HP^2.

Proof. We know that HP^2 has a cell-decomposition with cells Z_0, Z_2 and Z_4, where Z_{2i} has codimension $2i$, and the closure of Z_2 is a rank 2 vector bundle over $Q_4 = \mathrm{HP}^1$ [PW10b, Theorem 3.2]. Since Z_0 is $\mathbb{A}^1$-contractible, with complement $\overline{Z_2}$, the Thom isomorphism theorem, combined with the cofiber sequence attached to the inclusion $Z_0 \hookrightarrow \mathrm{HP}^2$ yields an $\mathbb{A}^1$-weak equivalence

$$\mathrm{HP}^2 \cong Th(N_{\overline{Z_2}/\mathrm{HP}^2}).$$

Now, by definition the Thom space of $N_{\overline{Z_2}/\mathrm{HP}^2}$ is the quotient $N_{\overline{Z_2}/\mathrm{HP}^2}/N^{\circ}_{\overline{Z_2}/\mathrm{HP}^2}$, where $N^{\circ}_{\overline{Z_2}/\mathrm{HP}^2}$ denotes the complement of the zero section. We now describe these spaces more explicitly.

The total space $N_{\overline{Z_2}/\mathrm{HP}^2}$ is a rank 2 vector bundle over $\overline{Z_2}$, which is itself a rank 2 vector bundle over HP^1. Therefore, the composite map yields an $\mathbb{A}^1$-weak equivalence

$$N_{\overline{Z_2}/\mathrm{HP}^2} \longrightarrow \mathrm{HP}^1.$$

On the other hand, the space $N^{\circ}_{\overline{Z_2}/\mathrm{HP}^2}$ admits the following description. The space $\overline{Z_2}$ is affine and $\mathbb{A}^1$-weakly equivalent to Q_4. If ν is the Hopf map, then ν is an SL_2-torsor over Q_4, and we can form the associated $\mathbb{A}^2 \setminus 0$-bundle $Z := Q_7 \times^{SL_2} \mathbb{A}^2 \setminus 0 \longrightarrow Q_4$. The map

$$Q_7 \cong Q_7 \times^{SL_2} SL_2 \longrightarrow Q_7 \times^{SL_2} \mathbb{A}^2 \setminus 0$$

is Zariski locally trivial with fibers isomorphic to $\mathbb{A}^1$ and is therefore an $\mathbb{A}^1$-weak equivalence. Therefore, the induced map $Z \to Q_4$ coincides with ν up to $\mathbb{A}^1$-homotopy. One can check that $N^{\circ}_{\overline{Z_2}/\mathrm{HP}^2}$ is precisely the pullback of Z to $\overline{Z_2}$ along the vector bundle $\overline{Z_2} \to Q_4$.

Combining these two facts, we see that, up to $\mathbb{A}^1$-weak equivalence, the inclusion of the complement of the zero section of the normal bundle to $\overline{Z_2}$ into the total space is ν. $\square$

Stable non-triviality of ν

Since ν gives a morphism $Q_7 \to Q_4$, Tate suspension of ν yields, up to $\mathbb{A}^1$-homotopy, a map

$$\Sigma_{\mathbf{G}_m}\nu : \Sigma_s^3 \mathbf{G}_m^{\wedge 5} \longrightarrow \mathbb{A}^3 \setminus 0.$$

Therefore, $\Sigma_{\mathbf{G}_m}\nu$ is an element of $\boldsymbol{\pi}_{3,5}^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)$. By Proposition 4.1, it follows immediately that $\Sigma_{\mathbf{G}_m}\nu$ is a multiple of δ (for the $K_0^{MW}(k)$-module structure). To show that δ is stably non-trivial, we will show that ν is stably non-trivial.

If k is a field having characteristic zero, then stable non-triviality of δ follows from complex realization, but it is possible to give a purely algebraic argument for this fact. The purely algebraic argument is, unsurprisingly to anyone familiar with the classical topological story, related to the Hopf invariant. Recall that, in topology, given a map $g : S^{4n-1} \to S^{2n}$, one can form the CW complex $C(g) = D^{4n} \cup_f S^{2n}$, which has two cells of dimension $4n$ and $2n$. If g is homotopically trivial, this complex is simply $S^{4n} \vee S^{2n}$, and this completely determines (say) the cohomology of $C(g)$ (even, say, as modules over the Steenrod algebra). One way to detect that $C(g)$ is non-trivial is to study its cohomology ring or Steenrod operations.

In algebraic geometry, one may replace $C(g)$ by the cone of the map g and perform all the same arguments. Suppose given an element of $f \in [\mathbb{A}^{2n} \setminus 0, \mathbb{P}^{1^{\wedge n}}]_{\mathbb{A}^1}$. We can form the cone $C(f)$ in $\mathcal{H}_\bullet(k)$. If f is $\mathbb{A}^1$-homotopically constant, then $C(f) \cong \mathbb{P}^{1^{\wedge n}} \vee \Sigma_s^1 \mathbb{A}^{2n} \setminus 0$. In particular, we have

$$\tilde{H}^{*,*}(\mathbb{P}^{1^{\wedge n}} \vee \Sigma_s^1 \mathbb{A}^{2n} \setminus 0, \mathbb{Z}/2) \cong \tilde{H}^{*,*}(\mathbb{P}^{1^{\wedge n}}, \mathbb{Z}/2) \oplus \tilde{H}^{*,*}(\Sigma_s^1 \mathbb{A}^{2n} \setminus 0, \mathbb{Z}/2).$$

where we write $\tilde{H}^{*,*}(-, \mathbb{Z}/2)$ for reduced motivic cohomology with $\mathbb{Z}/2$-coefficients. If $A^{*,*}$ is the (mod 2) motivic Steenrod algebra studied in [Voe03, §11], then $\tilde{H}^{*,*}(-, \mathbb{Z}/2)$ is a module over $A^{*,*}$, and the above direct sum decomposition is a decomposition as modules over $A^{*,*}$.

Note that $H^{*,*}(\Sigma_s^1 \mathbb{A}^{2n} \setminus 0, \mathbb{Z}/2) \cong H^{*,*}(\operatorname{Spec} k)[\xi]/\xi^2$, where ξ is a class of bidegree $(4n, 2n)$, and that $H^{*,*}(\mathbb{P}^{1\wedge n}, \mathbb{Z}/2) \cong H^{*,*}(\operatorname{Spec} k)[\tau]/\tau^2$, where τ is a class of bidegree $(2n, n)$. For this reason, we will limit our attention to the subring $H^{2*,*}(-, \mathbb{Z}/2)$, which we view as a $\mathbb{Z}/2$-vector space. Now, the algebra $A^{*,*}$ does not preserve $\tilde{H}^{2*,*}(-, \mathbb{Z}/2)$. However, if we write $A^{*,*}/\beta$ for the quotient of $A^{*,*}$ by the 2-sided ideal generated by the Bockstein, then $A^{*,*}/\beta$ actually does preserve $\tilde{H}^{2*,*}$ (see [Bro03, §11] for a discussion of this fact, in our context it follows from [Voe03, Theorem 10.2] upon observing that $Sq^{2i+1} = \beta Sq^{2i}$). The action of $A^{*,*}/\beta$ on $\tilde{H}^{2*,*}(-, \mathbb{Z}/2)$ is $\mathbb{Z}/2$-linear by construction.

If $f \in [\mathbb{A}^{2n} \setminus 0, \mathbb{P}^{1\wedge n}]_{\mathbb{A}^1}$ is $\mathbb{A}^1$-homotopically constant, then the $A^{*,*}/\beta$-module structure on $\tilde{H}^{2*,*}(\Sigma_s^1 \mathbb{A}^{2n} \setminus 0, \mathbb{Z}/2)$ is trivial, since every Steenrod operation acts trivially on ξ. Similarly, the $A^{*,*}/\beta$-module structure on $\tilde{H}^{*,*}(\mathbb{P}^{1\wedge n}, \mathbb{Z}/2)$ is trivial. Thus, to prove non-triviality of f, it suffices to prove that the action of $A^{*,*}/\beta$ on $C(f)$ is non-trivial. Since the operations we consider are all stable with respect to both simplicial and $\mathbf{G}_m$-suspension, it follows that if the $A^{*,*}/\beta$-module structure on $C(f)$ is non-trivial, then f remains non-trivial after both simplicial and $\mathbf{G}_m$-suspension, so is non-trivial in the stable $\mathbb{A}^1$-homotopy category of $\mathbb{P}^1$-spectra (see [Mor04a] for details regarding the latter category).

Remark 4.15. One would like to just describe the $A^{*,*}$-module structure on the motivic cohomology of $H^{*,*}(\operatorname{Spec} k)[\xi]/\xi^2$ directly, but there are some technical difficulties preventing an easy statement. The main problem is that if X is a scheme, the action of $A^{*,*}$ on $\tilde{H}^{*,*}(X, \mathbb{Z}/2)$ is not $H^{*,*}(\operatorname{Spec} k, \mathbb{Z}/2)$-linear (see [Voe03, p. 41]). This is the reason we consider $A^{*,*}/\beta$.

Remark 4.16. If $f \in \boldsymbol{\pi}_{2n-1,2n}(\mathbb{P}^{1\wedge n})$ is as above, and we look at $H^{2*,*}(C(f), \mathbb{Z})$ instead, then we see that

$$H^{2*,*}(C(f), \mathbb{Z}) \cong \mathbb{Z}[\xi, \tau]/\langle \xi^2, \xi\tau, \tau^2 - h_f \xi \rangle$$

with $h_f \in \mathbb{Z}$ for dimensional reasons. One can check that the function

$$H : \boldsymbol{\pi}^{\mathbb{A}^1}_{2n-1,2n}(\mathbb{P}^{1\wedge n}) \longrightarrow \mathbb{Z}$$

given by the assignment $f \mapsto h_f$ is actually a group homomorphism, just as in topology and defines a motivic analog of the classical Hopf invariant [Whi50]. Since this invariant depends only the ring structure of the motivic cohomology of $C(f)$, it is an unstable invariant.

Loosely following the notation of Morel [Mor04a, §5], we write

$$\boldsymbol{\pi}^{s\mathbb{A}^1}_{i,j}(\mathbb{S}^0_k) := \operatorname{colim}_n \boldsymbol{\pi}^{\mathbb{A}^1}_{i+n,j+n}(\mathbb{P}^{1\wedge n});$$

in words, this sheaf is the bidegree (i, j)-stable $\mathbb{A}^1$-homotopy sheaf of the motivic sphere spectrum. Iterated $\mathbb{P}^1$-suspension of ν gives rise to an element of $\boldsymbol{\pi}^{s\mathbb{A}^1}_{1,2}(\mathbb{S}^0_k)(k)$.

Theorem 4.17. *The element $\nu \in \boldsymbol{\pi}^{s\mathbb{A}^1}_{1,2}(\mathbb{S}^0_k)(k)$ is non-trivial.*

Proof. Since ν is defined over $\operatorname{Spec} \mathbb{Z}$, it suffices by a base-change argument to show that it is non-trivial over the prime field. Since the prime field is perfect, we can use motivic cohomology to detect non-triviality. We saw that $C(\nu) = \mathrm{HP}^2$ in Proposition 4.14. By [PW10b, Theorem 8.1], we know that $H^{*,*}(C(\nu), \mathbb{Z}/2) \cong \mathbb{Z}/2[\zeta]/\zeta^3$, where ζ, a class of bidegree $(4, 2)$, is the first Pontryagin class of a canonical symplectic line bundle over HP^2. In particular, $Sq^4(\zeta) = \zeta^2$ by [Voe03, Lemma 9.8], so $\tilde{H}^{*,*}(\mathrm{HP}^2, \mathbb{Z}/2)$ has a non-trivial $A^{*,*}/\beta$-module structure, and the required stable non-triviality of ν follows. $\square$

Corollary 4.18. *The element δ is $\mathbb{P}^1$-stably non-trivial.*

Proof. Since the element $\Sigma_{\mathbf{G}_m}\nu$ is non-trivial by the above argument, it follows that δ is a non-trivial multiple of $\Sigma_{\mathbf{G}_m}\nu$. The result follows immediately from Theorem 4.17. $\square$

Finally, note that δ can actually be defined over $\operatorname{Spec}\mathbb{Z}$. Every indication suggests that the following conjecture is true.

Conjecture 4.19. *The elements δ and $\Sigma_{\mathbf{G}_m}\nu$ are $\mathbb{A}^1$-weakly equivalent (over $\operatorname{Spec}\mathbb{Z}$).*

Remark 4.20. If one knew that the (pointed) endomorphisms of $\mathbb{A}^n \setminus 0$ ($n \geq 2$) in the $\mathbb{A}^1$-homotopy category over $\operatorname{Spec}\mathbb{Z}$ were given by $GW(\mathbb{Z})$, one could use the two realization computations above to establish this conjecture for fields having characteristic 0. One can construct a split injective map from $GW(\mathbb{Z})$ to the above homotopy endomorphisms, but we do not know how to prove this map is surjective. Alternatively, as mentioned in Remark 4.5, it seems reasonable to expect that $\boldsymbol{\pi}^{\mathbb{A}^1}_{4,5}(\mathbb{P}^{1\wedge 3}) \cong \mathbb{Z}/24$. From this, one could easily deduce that $\Sigma_s\delta = \Sigma_{\mathbb{P}^1}\nu$ in $\boldsymbol{\pi}^{\mathbb{A}^1}_{4,5}(\mathbb{P}^{1\wedge 3})$ for any field having characteristic 0 by appealing to complex realization.

5 Obstruction theory and the splitting problem

In this section, we explain in detail the obstruction theoretic computations required to reduce the splitting problem to the computation of $\mathbb{A}^1$-homotopy sheaves. We then explain how the computation of $\boldsymbol{\pi}^{\mathbb{A}^1}_3(\mathbb{A}^3 \setminus 0)$ yields the statement of the introduction. By Morel's results, we know that if X is a smooth affine k-scheme, then $[X, BGL_n]_{\mathbb{A}^1}$ is canonically in bijection with the set of isomorphism classes of rank n vector bundles on X. Consider the morphism $GL_n \to GL_{n+1}$ that sends an invertible matrix M to the block diagonal matrix with diagonal blocks M and 1. By functoriality, there is an induced morphism $BGL_n \to BGL_{n+1}$.

The image of a vector bundle $\mathcal{E}$ on X under the induced morphism

$$[X, BGL_n]_{\mathbb{A}^1} \longrightarrow [X, BGL_{n+1}]_{\mathbb{A}^1}$$

is a vector bundle of the form $\mathcal{E} \oplus \mathcal{O}_X$. To understand whether a given vector bundle $\mathcal{E}'$ on X of rank $n+1$, classified by an element $\xi \in [X, BGL_{n+1}]_{\mathbb{A}^1}$ splits off a trivial rank 1 summand, it therefore suffices to determine whether ξ lies in the image of $[X, BGL_n]_{\mathbb{A}^1}$, and this question can be studied by means of the Moore-Postnikov tower of the morphism $BGL_n \to BGL_{n+1}$.

Remark 5.1. Strictly speaking, if we are to work with the Moore-Postnikov factorization, then we must work in the category of pointed maps. This presents no real difficulty since we can replace X by X_+ and use the fact that the space of pointed maps between X_+ and BGL_n is canonically identified with the space of unpointed maps between X and BGL_n. Throughout this section, we will implicitly make this choice and avoid further discussion of base-points.

The $\mathbb{A}^1$-homotopy fiber of $BGL_n \to BGL_{n+1}$ is precisely $\mathbb{A}^n \setminus 0$, so the obstructions to lifting are controlled by homotopy sheaves of this space. However, for any integer n, we know that $\boldsymbol{\pi}^{\mathbb{A}^1}_1(BGL_n) = \mathbf{G}_m$ (induced by the determinant homomorphism $GL_n \to \mathbf{G}_m$), and the sheaf $\boldsymbol{\pi}^{\mathbb{A}^1}_1(BGL_n)$ acts non-trivially on $\boldsymbol{\pi}^{\mathbb{A}^1}_i(BGL_n)$ in general, so the obstruction theory is slightly more complicated.

We refer the reader to [AF12b, §6] for a general discussion of $\mathbb{A}^1$-Postnikov towers; the Moore-Postnikov factorization of a morphism $f : \mathcal{X} \to \mathcal{Y}$ of spaces comes from applying the $\mathbb{A}^1$-Postnikov tower discussed there to the $\mathbb{A}^1$-homotopy fiber of f. The identification $BGL_n^{(1)} = B\mathbf{G}_m$ yields a map $[X, BGL_n]_{\mathbb{A}^1} \to [X, B\mathbf{G}_m]_{\mathbb{A}^1}$ that sends a vector bundle to its determinant line bundle. Given an element $\xi \in [X, BGL_{n+1}]_{\mathbb{A}^1}$, since the action of $\boldsymbol{\pi}_1^{\mathbb{A}^1}(BGL_n) = \boldsymbol{\pi}_1^{\mathbb{A}^1}(BGL_{n+1})$ on the total space and fiber are compatible, there is an induced action of $\boldsymbol{\pi}_1^{\mathbb{A}^1}(BGL_{n+1})$ on the $\mathbb{A}^1$-homotopy sheaves $\boldsymbol{\pi}_i^{\mathbb{A}^1}(\mathbb{A}^{n+1} \setminus 0)$. Therefore, there are inductively defined obstructions

$$o_{i,n+1}(\xi) \in H^{i+1}(X, \boldsymbol{\pi}_i^{\mathbb{A}^1}(\mathbb{A}^{n+1} \setminus 0)(\det \xi)).$$

Since $\mathbb{A}^{n+1} \setminus 0$ is $(n-1)$-$\mathbb{A}^1$-connected, the first potentially non-trivial obstruction is $o_{n,n+1}(\xi)$, which is an element of $H^{n+1}(X, \mathbf{K}^{MW}_{n+1}(\det \xi))$. If X has dimension $n+2$, then for reasons of cohomological dimensions, if $o_{n,n+1}(\xi)$ vanishes, the only further possible non-trivial obstruction is $o_{n+1,n+1} \in H^{n+2}(X, \boldsymbol{\pi}_{n+1}^{\mathbb{A}^1}(\mathbb{A}^{n+1} \setminus 0)(\det \xi))$. Therefore, if we understand the sheaf $\boldsymbol{\pi}_{n+1}^{\mathbb{A}^1}(\mathbb{A}^{n+1} \setminus 0)$ (together with the induced $\mathbf{G}_m$-action), we completely understand the splitting problem for rank $n+1$-vector bundles on a smooth affine $(n+2)$-fold.

The primary obstruction and Murthy's splitting conjecture

Proposition 5.2. *If k is an algebraically closed field having characteristic unequal to 2, $o_{n,n+1}(\xi)$ vanishes if and only if $c_{n+1}(\xi) = 0$.*

Proof. In [AF12b, Corollary 5.9], we showed that, under the stated hypotheses, the canonical morphism

$$H^{n+1}(X, \mathbf{K}^{MW}_{n+1}(\xi)) \longrightarrow H^{n+1}(X, \mathbf{K}^{M}_{n+1}) = CH^{n+1}(X)$$

is an isomorphism. In particular, the element $o_{n,n+1}$ is canonically determined by an element of $CH^{n+1}(X)$ (always under the stated hypotheses).

The obstruction class on X is pulled-back from a universal class on BGL_{n+1}, induced by the identity map on BGL_{n+1}. If ν is the universal bundle on BGL_{n+1} with determinant $\det \nu$, then there is a commutative diagram of the form

$$\begin{tikzcd} H^{n+1}(BGL_{n+1}, \mathbf{K}^{MW}_{n+1}(\det \nu)) \arrow[r] \arrow[d] & H^{n+1}(X, \mathbf{K}^{MW}_{n+1}(\det \xi)) \arrow[d] \\ H^{n+1}(BGL_{n+1}, \mathbf{K}^{M}_{n+1}) \arrow[r] & H^{n+1}(X, \mathbf{K}^{M}_{n+1}). \end{tikzcd}$$

In particular, since $o_{n,n+1}(\xi)$ is uniquely determined by its image in $H^{n+1}(X, \mathbf{K}^M_{n+1})$, it suffices to understand the image $o_{n,n+1}$ of $o_{n,n+1}(\nu)$ in $H^{n+1}(BGL_{n+1}, \mathbf{K}^M_{n+1})$. Now, if we identify BGL_{n+1} with the infinite Grassmannian Gr_{n+1}, it follows that the image of $o_{n,n+1}(\nu)$ in $H^{n+1}(BGL_{n+1}, \mathbf{K}^M_{n+1})$ is given by an element of $H^{n+1}(Gr_{n+1}, \mathbf{K}^M_{n+1}) = CH^{n+1}(Gr_{n+1}) = \mathbb{Z}$. It follows that $o_{n,n+1}$ is a multiple of $c_{n+1}(\nu)$, which is a generator of $CH^{n+1}(Gr_{n+1})$.

In fact, we will see that $o_{n,n+1} = c_{n+1}(\nu)$ and it follows by functoriality of the obstruction class and the Chern class that the same result holds for an arbitrary smooth scheme. Topologically, the fact that the top Chern class is an Euler class is the definition taken in Milnor-Stasheff [MS74, §14], and the fact that the Euler class is the obstruction class in question is [MS74, Theorem 12.5].

First, we use an alternate identification of c_{n+1}. Up to $\mathbb{A}^1$-weak equivalence, we can identify BGL_n with the complement of the zero section in the universal bundle over BGL_{n+1}. If $\mathbf{V}(\nu)$ is the associated geometric vector bundle (we can think of an inductive limit of vector bundles over finite dimensional Grassmannians), and we write $\mathbf{V}(\nu) \setminus 0$ for the complement of the zero section, then there is a cofiber sequence of the form

$$\mathbf{V}(\nu) \setminus 0 \longrightarrow \mathbf{V}(\nu) \longrightarrow Th(\nu).$$

Now, the group $H^{n+1}(Th(\nu), \mathbf{K}^M_{n+1})$ is precisely isomorphic to $H^0(BGL_{n+1}, \mathbb{Z}) \cong \mathbb{Z}$ by means of the Thom isomorphism, and c_{n+1} is precisely the image of $1 \in \mathbb{Z}$ under this morphism.

On the other hand, the obstruction class is an element of $H^{n+1}(BGL_{n+1}, \mathbf{K}^{MW}_{n+1}(\det \nu))$. The cofiber sequence above gives rise to a long exact sequence, a portion of which takes the form:

$$H^{n+1}(Th(\nu), \mathbf{K}^{MW}_{n+1}(\det \nu)) \longrightarrow H^{n+1}(\mathbf{V}(\nu), \mathbf{K}^{MW}_{n+1}(\det \nu)) \longrightarrow H^{n+1}(\mathbf{V}(\nu)\setminus 0, \mathbf{K}^{MW}_{n+1}(\det \nu)).$$

By the identifications discussed in the previous paragraph, up to $\mathbb{A}^1$-weak equivalence, the second arrow (from the left) in the above sequence coincides with the morphism

$$H^{n+1}(BGL_{n+1}, \mathbf{K}^{MW}_{n+1}(\det \nu)) \longrightarrow H^{n+1}(BGL_n, \mathbf{K}^{MW}_{n+1}(\det \nu))$$

induced by the inclusion $BGL_n \to BGL_{n+1}$. Since the pullback of ν to BGL_n splits off a free rank 1 summand by construction, it follows by functoriality of the obstruction class that the image of $o_{n,n+1}$ in $H^{n+1}(BGL_n, \mathbf{K}^{MW}_{n+1}(\det \nu))$ is 0. Therefore, $o_{n,n+1}$ comes from an element of $H^{n+1}(Th(\nu), \mathbf{K}^{MW}_{n+1})$.

The twisted Thom isomorphism gives an identification

$$H^{n+1}(Th(\nu), \mathbf{K}^{MW}_{n+1}(\det \nu)) \xrightarrow{\sim} H^0(BGL_{n+1}, \mathbf{K}^{MW}_0) \cong GW(k)$$

[AH11, Theorem 4.2.7]. We claim that $o_{n,n+1}$ is the image of $\langle 1 \rangle \in GW(k)$ under this map. Indeed, by the self-intersection formula, the morphism $GW(k) \to H^{n+1}(BGL_n, \mathbf{K}^{MW}_{n+1}(\det \nu))$ sends 1 to the (twisted) Euler class of ν, which coincides with the obstruction class by definition.

The canonical homomorphism $\mathbf{K}^{MW}_{n+1}(\det \nu) \to \mathbf{K}^M_{n+1}$, when combined with the Thom isomorphisms, yields a homomorphism

$$H^0(BGL_{n+1}, \mathbf{K}^{MW}_0) \to H^0(BGL_{n+1}, \mathbb{Z})$$

that corresponds to the degree homomorphism $GW(k) \to \mathbb{Z}$, which is split surjective sending $\langle 1 \rangle$ to 1. □

As a consequence of this identification of the primary obstruction class, we see that Murthy's splitting conjecture is equivalent to a cohomological vanishing statement.

Corollary 5.3. *If X is a smooth affine scheme of dimension $d+1$ over an algebraically closed field having characteristic unequal to 2, then Murthy's splitting conjecture holds if and only if for any rank d vector bundle ξ on X, such that $c_d(\xi) \in CH^d(X) = 0$,*

$$o_{d,d}(\xi) \in H^{d+1}_{\mathrm{Nis}}(X, \boldsymbol{\pi}^{\mathbb{A}^1}_d(\mathbb{A}^d \setminus 0)(\det \xi)) = 0.$$

Murthy's splitting conjecture for smooth affine 4-folds

In this section, we will establish Theorem 2 from the introduction. In light of Corollary 5.3, we would like to study the group $H^{d+1}_{\mathrm{Nis}}(X, \boldsymbol{\pi}^{\mathbb{A}^1}_d(\mathbb{A}^d \setminus 0)(\det \xi))$. To this end, we need to understand the Gersten resolution for the sheaf $\boldsymbol{\pi}^{\mathbb{A}^1}_d(\mathbb{A}^d \setminus 0)(\det \xi)$, which is complicated by the non-triviality of the twist by $\det \xi$. In the case $d = 3$, the sheaf $\boldsymbol{\pi}^{\mathbb{A}^1}_3(\mathbf{A}^3 \setminus 0)$ is an extension of $\mathbf{GW}^3_4$ by the sheaf $\mathbf{F}_5$. We will understand the $\mathbf{G}_m$-action on each of these components separately.

Lemma 5.4. *If X is a smooth affine 4-fold over an algebraically closed field k having characteristic unequal to 2, and ξ is a rank 3-vector bundle on X, then the group $H^4(X, \mathbf{F}_5(\det \xi)) = 0$.*

Proof. The sheaf $\mathbf{F}_5$ is defined as the cokernel of $\chi_5 : \mathbf{GW}^3_5 \longrightarrow \mathbf{K}^{MW}_5$. Given a line bundle $\mathcal{L}$, we can define the sheaf $\mathbf{K}^{MW}_n(\mathcal{L})$ on the small Nisnevich site of X. The Gersten resolution of such a sheaf is described in [Mor12, Remark 5.13], and unwinding the definitions, one sees that $\mathbf{F}_5(\det \xi)$ is a quotient of $\mathbf{K}^{MW}_5(\det \xi)$.

There is a short exact sequence of the form

$$0 \longrightarrow \mathbf{I}^6(\det \xi) \longrightarrow \mathbf{K}^{MW}_5(\det \xi) \longrightarrow \mathbf{K}^M_5 \longrightarrow 0,$$

i.e., the induced action of $\det \xi$ on the Milnor K-theory quotient of $\mathbf{K}^{MW}_5$ is trivial. Now, the description of $\mathbf{F}_5$ we gave in Theorem 3.7 shows that it admits an epimorphism from $\mathbf{T}_5$, which is itself a fiber product of $\mathbf{I}^5$ and a quotient of $\mathbf{K}^M_5/24$.

Since k is algebraically closed, it follows from [AF12b, Proposition 5.8] that $H^4(X, \mathbf{I}^5(\det \xi)) = 0$. Likewise, it follows from [AF12b, Proposition 5.10] that $H^4(X, \mathbf{K}^M_5/24) = 0$. Therefore, $H^4(X, \mathbf{T}_5(\det \xi)) = 0$. For reasons of cohomological dimension, there is a surjective homomorphism $H^4(X, \mathbf{T}_5(\det \xi)) \longrightarrow H^4(X, \mathbf{F}_5(\det \xi))$, and so we conclude that the latter vanishes as well. $\square$

Our next aim is to compute the group $H^d(X, \mathbf{GW}^{d-1}_d(\det \xi))$. We start with two lemmas.

Lemma 5.5. *Let F be a field of characteristic different from 2 and let L be a F-vector space of rank one. Then the hyperbolic map $\mathbb{Z} = K_0(F) \to GW^3(F, L)$ induces an isomorphism*

$$\mathbb{Z}/2 \to GW^3(F, L).$$

Proof. Choosing a generator of L, we get a commutative diagram

$$\begin{tikzcd} GW^0(F) \arrow[r, "f"] \arrow[d] & K_0(F) \arrow[r, "H"] \arrow[d, equal] & GW^3(F) \arrow[d] \\ GW^0(F, L) \arrow[r, "f"] & K_0(F) \arrow[r, "H"] & GW^3(F, L) \end{tikzcd}$$

where the vertical maps are isomorphisms. The result follows then from [FS08, Lemma 4.1]. $\square$

Lemma 5.6. *Let F be a field of characteristic different from 2 and let L be a F-vector space of rank one. Then Karoubi periodicity yields a split exact sequence*

$$0 \longrightarrow K_1(F)/2 \xrightarrow{H} GW^0_1(F, L) \xrightarrow{\eta} GW^3_0(F, L) \longrightarrow 0$$

Proof. As above, choosing a generator of L yields an isomorphism $F \to L$ and a commutative diagram

$$\begin{tikzcd} GW^0_1(F) \arrow[r, "\eta"] \arrow[d] & GW^3_0(F) \arrow[d] \\ GW^0_1(F,L) \arrow[r, "\eta"] & GW^3_0(F,L) \end{tikzcd}$$

where the vertical maps are isomorphisms. The result follows therefore from [AF12b, Lemma 4.9]. □

Theorem 5.7. *Let X be a smooth d-fold over a field k of characteristic different form 2 and $\mathcal{L}$ be a line bundle over X. Then there is an exact sequence*

$$Ch^{d-1}(X) \xrightarrow{Sq^2_{\mathcal{L}}} Ch^d(X) \longrightarrow H^d(X, \mathbf{GW}^{d-1}_d(\mathcal{L})) \longrightarrow 0,$$

where $Sq^2_{\mathcal{L}}$ is the Steenrod square operation twisted by $\mathcal{L}$, i.e. $Sq^2_{\mathcal{L}}(\alpha) = Sq^2(\alpha) + \alpha \cdot l$ with l the class of $\mathcal{L}$ in $Ch^1(X)$.

Proof. The proof follows the lines of [AF12b, Theorem 4.11]. There, we proved in particular that if Y is a smooth (connected) curve over k and $\mathcal{N}$ is a line bundle over Y, then $H^1(Y, \mathbf{GW}^0_1(\mathcal{N}))$ is precisely the cokernel of the map

$$Sq^2_{\mathcal{N}} : Ch^0(Y) \to Ch^1(Y).$$

Consider now the Gersten-Grothendieck-Witt spectral sequence $E(d-1)^{p,q}_{\mathcal{L}}$ twisted by $\mathcal{L}$ ([FS09, §3]). Its groups at page 1 are of the form

$$E(d-1)^{p,q}_{\mathcal{L}} = \bigoplus_{x_p \in X^{(p)}} GW^{d-1-p}_{d-1-p-q}(k(x_p), \omega^{\mathcal{L}}_{x_p})$$

and it abuts to $GW^{d-1}_{d-1-p-q}(X, \mathcal{L})$. The Gersten conjecture being true for Grothendieck-Witt groups, its line $q = -1$ is a flasque resolution of the sheaf $\mathbf{GW}^{d-1}_d(\mathcal{L})$. In particular, we have an exact sequence

$$\bigoplus_{x_{d-1} \in X^{(d-1)}} GW^0_1(k(x_{d-1}), \omega^{\mathcal{L}}_{x_{d-1}}) \xrightarrow{d^{\mathcal{L}}} \bigoplus_{x_d \in X^{(d)}} GW^3_0(k(x_d), \omega^{\mathcal{L}}_{x_d}) \to H^d(X, \mathbf{GW}^{d-1}_d(\mathcal{L})) \to 0.$$

Using Lemmas 5.5 and 5.6, we can argue as in [AF12b, Theorem 4.11] to get an exact sequence

$$\bigoplus_{x_{d-1} \in X^{(d-1)}} \mathbb{Z}/2 \xrightarrow{\chi_{\mathcal{L}}} Ch^d(X) \longrightarrow H^d(X, \mathbf{GW}^{d-1}_d(\mathcal{L})) \longrightarrow 0$$

with a map $\chi_{\mathcal{L}}$ that we have to identify.

Let $x_{d-1} \in X^{(d-1)}$ be any point, and let Y be the normalization of the closure Z of x_{d-1} in X. Then the morphism $F : Y \to Z \subset X$ is finite, and we can compute the differential $d_{\mathcal{L}}$ (restricted to $GW^0_1(k(x_{d-1}, \omega^{\mathcal{L}}_{x_{d-1}})))$ using the transfer map. If $\mathcal{N} := \omega_{Y/k} \otimes f^*\omega^{\vee}_{X/k}$, we get a commutative diagram

$$\begin{tikzcd} GW^0_1(k(x_{d-1}), \omega^{\mathcal{L}}_{x_{d-1}}) \arrow[r, "d^{\mathcal{L}}_X"] \arrow[d, equal] & \bigoplus_{x_d \in X^{(d)} \cap Z} GW^3_0(k(x_d), \omega^{\mathcal{L}}_{x_d}) \\ GW^0_1(k(y_0), \omega^{(f^*\mathcal{L})\otimes\mathcal{N}}_{y_0}) \arrow[r, "d^{(f^*\mathcal{L})\otimes N}_Y"'] & \bigoplus_{y_1 \in Y^{(1)}} GW^3_0(k(y_1), \omega^{(f^*\mathcal{L})\otimes\mathcal{N}}_{y_1}) \arrow[u, "f_*"'] \end{tikzcd}$$

and thus a commutative diagram

$$\begin{tikzcd} \mathbb{Z}/2 \arrow[r, "\chi_{\mathcal{L}}"] \arrow[d, equal] & Ch^d(X) \\ \mathbb{Z}/2 \arrow[r, "\chi_{f^*\mathcal{L}\otimes\mathcal{N}}"'] & Ch^1(Y). \arrow[u, "f_*"'] \end{tikzcd}$$

We find therefore

$$\chi_L(\overline{1}) = f_*(\chi_{f^*\mathcal{L}\otimes\mathcal{N}}(\overline{1})) = f_*(c_1((f^*\mathcal{L})\otimes\mathcal{N})) = f_*(f^*l + n) = l \cdot [Y] + f_*Sq^2(\overline{1}) = Sq^2_{\mathcal{L}}([Y]).$$

$\square$

Combining the above results, we can deduce Theorem 2 from the introduction.

Proof of Theorem 2. By Corollary 5.3 to prove Murthy's splitting conjecture in dimension 4, it suffices to prove that $H^4(X, \boldsymbol{\pi}^{\mathbb{A}^1}_3(\mathbb{A}^3 \setminus 0)(\lambda))$ vanishes for an arbitrary line bundle λ on X. By Lemma 5.4 and the long exact sequence in cohomology associated with the extension describing $\boldsymbol{\pi}^{\mathbb{A}^1}_3(\mathbb{A}^3 \setminus 0)(\lambda)$, we are reduced to proving vanishing of $H^4(X, \mathbf{GW}^3_4(\mathcal{L}))$. However, if X is a smooth affine 4-fold, we know that $Ch^4(X)$ is trivial as a consequence of Roitman's theorem on unique divisibility of the Chow group of zero cycles [Sri89]. Combining this observation with Theorem 5.7, we conclude that if X is a smooth affine 4-fold over an algebraically closed field, then $H^4(X, \mathbf{GW}^3_4(\mathcal{L}))$ vanishes. $\square$

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