Centralizer of the elementary subgroup of an isotropic reductive group

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1 Introduction

Let R be a commutative ring with 1, and let G be an isotropic reductive algebraic group over R. In [5] Victor Petrov and the second author introduced a notion of an elementary subgroup $E(R)$ of the group of points $G(R)$.

More precisely, assume that G is isotropic in the following strong sense: it possesses a parabolic subgroup that intersects properly any semisimple normal subgroup of G. Such a parabolic subgroup P is called *strictly proper*. Denote by $E_P(R)$ the subgroup of $G(R)$ generated by the R-points of the unipotent radicals of P and of an opposite parabolic subgroup P^-. The main theorem of [5] states that $E_P(R)$ does not depend on the choice of P, as soon as for any maximal ideal M of R all irreducible components of the relative root system of G_{R_M} (see [2, Exp. XXVI, §7] for the definition) are of rank ≥ 2. Under this assumption, we call $E_P(R)$ the *elementary subgroup* of $G(R)$ and denote it simply by $E(R)$. In particular, $E(R)$ is normal in $G(R)$. This definition of $E(R)$ generalizes the well-known definition of an elementary subgroup of a Chevalley group (or, more generally, of a split reductive group), as well as several other definitions of an elementary subgroup of isotropic classical groups and simple groups over fields. The group $E(R)$ is also perfect under natural assumptions on R [3]. Here we continue this theme by proving that the centralizer of $E(R)$ in $G(R)$ coincides with the group of R-points of the group scheme center $\mathrm{Cent}(G)$ (see [2, Exp. I 2.3] for the definition). Consequently, both these subgroups also coincide with the abstract group center of $G(R)$. Our result extends the respective theorem of E. Abe and J. Hurly for Chevalley groups [1]; see also [7, Lemma 2] for a slighly more general statement.

Theorem 1. *Let G be an isotropic reductive algebraic group over a commutative ring R having a strictly proper parabolic subgroup P. Assume that for any maximal ideal M of R all irreducible components of the relative root system of G_{R_M} are of rank ≥ 2. Then* $\mathrm{C}_{G(R)}(E(R)) = \mathrm{Cent}(G)(R) = \mathrm{C}(G(R))$.

Observe that the condition of the theorem ensures that the elementary subgroup $E(R)$ of $G(R)$ is correctly defined. We refer to [3] for its definition and basic properties, as well as for the preliminaries on relative root subschemes.

Remark. One may ask if the statement holds for $E_P(R)$ instead of $E(R)$, if we do not assume that the local relative rank is at least 2. This seems to hold always except for several natural exceptions, similar to the exception for PGL_2 described in [1]. We plan to address this case in the near future.

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2 Preliminary lemmas

We refer to [3] and [5] for the preliminaries and notation.

We include the following obvious lemma for the sake of completeness.

Lemma 1. *Let $X = \operatorname{Spec} A$ be an affine scheme over $Y = \operatorname{Spec} R$, and let Z be a closed subscheme of X. Take $g \in X(R)$. Then $g \in Z(R)$ if and only if $g \in Z(R_M)$ for any maximal ideal M of R.*

Proof. For any R-module V, the natural map $V \to \prod V \otimes R_M$, where the product runs over all maximal ideals M of R, is injective (e.g. [8, p. 104, Lemma]). Since $g \in Z(R)$ is equivalent to an inclusion between the respective ideals of A which are R-modules, the Lemma holds. □

Lemma 2. *Let R be any commutative ring, G an isotropic reductive group over R, P a strictly proper parabolic subgroup of G. Take any maximal ideal M of R and any strictly proper parabolic subgroup P' of G_{R_M} contained in P_{R_M}. Then for any $A \in \Phi_{P'}$ there is a system of generators e_{Ai}, $1 \le i \le n_A$, of the R_M-module V_A such that for all g in the image of $\operatorname{Cent}_{G(R)}(E_P(R))$ in $G(R_M)$, one has $[g, X_A(e_{Ai})] = 1$, $1 \le i \le n_A$.*

Proof. We assume from the very beginning that we have passed to a member of the disjoint union

$$\operatorname{Spec}(R) = \coprod_{i=1}^{m} \operatorname{Spec}(R_i),$$

so that the parabolic subgroup P is also provided with a relative root system Φ_P and corresponding relative root subschemes. Since for any $B \in \Phi_P$ elements of V_B generate $V_B \otimes_R R_M$ as an R_m-module, the claim of the lemma holds if $P' = P_{R_M}$.

By [5, Lemma 12], for any two strictly proper parabolic subgroups $Q \le Q'$ of a reductive group scheme, one can find such $k > 0$ depending only on $\operatorname{rank} \Phi_Q$, that for any relative root $A \in \Phi_Q$ and any $v \in V_A$ there exist relative roots $B_i, C_{ij} \in \Phi_{Q'}$, elements $v_i \in V_{B_i}$, $u_{ij} \in V_{C_{ij}}$, and integers $k_i, n_i, l_{ij} > 0$ ($1 \le i \le m$, $1 \le j \le m_j$), which satisfy the equality

$$X_A(\xi\eta^k v) = \prod_{i=1}^{m} X_{B_i}(\xi^{k_i}\eta^{n_i}v_i)^{\prod\limits_{j=1}^{m_i} X_{C_{ij}}(\eta^{l_{ij}}u_{ij})},$$

where ξ, η are free variables. Taking $Q = P'$, $Q' = P_{R_M}$, $\xi = 1$, for any element v_i of a generating system of the R_m-module V_A we get a decomposition

$$X_A(\eta^k v) = \prod_{i=1}^{m} X_{B_i}(\eta^{n_i}v_i),$$

for some $B_i \in \Phi_P$ and $v_i \in V_{B_i} \otimes R_M$, $n_i > 0$. Clearly, for any v_i there is an element $s_i \in R \setminus M$ such that $s_i v_i$ belongs to V_{B_i} (strictly speaking, to the image of V_{B_i} in $V_{B_i} \otimes R_M$ under the localisation homomorphism; here and below we allow ourselves this freedom of speech). Set $\eta = s_1 \ldots s_m$. Then $X_A(\eta^k v) \in E_P(R)$, and hence $[g, X_A(\eta^k v)] = 1$ for any $g \in \operatorname{Cent}_{G(R)}(E_P(R))$. Thus, multiplying the elements of a generating system of V_A by certain invertible elements of R_M, we obtain a new generating system of V_A, which is centralized by $\operatorname{Cent}_{G(R)}(E_P(R))$. □

Lemma 3. *Let R be a local ring (in particular, R can be a field) with the maximal ideal M, and let G be a split reductive group over R. Let P be a parabolic subroup of G such that $\operatorname{rank} \Phi_P \ge 2$. Assume that $g \in G(R)$ is such that for any $A \in \Phi_P$ there is a system of generators e_{Ai}, $1 \le i \le n_A$, of V_A such that $[g, X_A(e_{Ai})] = 1$ for all i. Then $g \in U_P(M)L(R)U_{P^-}(M)$, where $U_{P^\pm}(M) = \langle X_A(MV_A),\ A \in \Phi^\pm \rangle$.*

Proof. First let R be a field. We need to show that $g \in L(R)$. We can assume that R is algebraically closed without loss of generality. Let $B^{\pm}$ be opposite Borel subgroups of G contained in $P^{\pm}$, $U^{\pm}$ be their unipotent radicals, and T their common maximal torus. Bruhat decomposition implies that $g = uhwv$, where $u \in U^+(R)$, $h \in T(R)$, w is a representative of the Weyl group, $v \in U^+_w(R) = \{x \in U^+(R) \mid w(x) \in U^-(R)\}$, and this decomposition is unique. We have $w \in L(R)$ if and only if w is a product of elementary reflections w_{α_i} for some simple roots α_i belonging to the root system of L.

Assume first that $w \notin L$. Then there is a simple root α not belonging to the root system of L such that $w(\alpha) < 0$. Consider $A = \pi(\alpha)$. Let $e_A \in V_A$ be a vector from the generating set existing by the hypothesis of the Lemma such that $x_\alpha(\xi)$, $\xi \neq 0$, occurs in the canonic decomposition of $x = X_A(e_A)$ into a product of elementary root unipotents from U^+. Since $[g, x] = 1$, we have $x(uhwv) = (uhwv)x$. The rightmost factor in the Bruhat decomposition of $x(uhwv) = (xu)hwv$ equals v. However, since α is a positive root of minimal height, it is clear that the rightmost factor in the Bruhat decomposition of $(uhv)x$ contains $x_\alpha(\eta + \xi)$ in its canonic decomposition, if v contains $x_\alpha(\eta)$. Therefore, this rightmost factor is distinct from v, a contradiction.

Therefore, $w \in L(R)$. Then for any $x \in U_P(R)$ we have $wxw^{-1} \in U_P(R)$, hence by the definition of the Bruhat decomposition $v \in L(R) \cap U^+(R)$. This means that $g = uhwv \in U^+(R)L(R) = U_P(R)(U^+(R)\cap L(R))L(R) = U_P(R)L(R) = P(R)$. Since symmetric reasoning implies that $g \in P^-(R)$, we have $g \in P(R) \cap P^-(R) = L(R)$.

Now let R be any local ring. Recall that $\Omega_P = U_P L U_{P^-} \cong U_P \times L \times U_{P^-}$ is a principal open subscheme of G (e.g. [4, p. 9]). Therefore, if the image of $g \in G(R)$ under the natural homomorphism $G(R) \to G(R/M)$ is in $\Omega_P(R/M)$, then $g \in \Omega_P(R)$. Since by the above the image of g is in $L(R/M)$, and $\ker(U_{P^\pm}(R) \to U_{P^\pm}(R/M)) = U_{P^\pm}(M)$, we have $g \in U_P(M)L(R)U_{P^-}(M)$. □

Lemma 4. *Let G be an isotropic reductive group over a local ring R, M the maximal ideal of R, P a parabolic subgroup of G, P^- an opposite parabolic subgroup. For any $u \in U_{P^-}(M)$, $v \in U_P(R)$ there exist $u' \in U_{P^-}(M)$, $v' \in U_P(R)$, and $b \in L(R)$ such that $uv = v'bu'$.*

Proof. The image of $x = uv$ under $p : G(R) \to G(R/M)$ equals $p(v)$,and thus belongs to $\Omega_P(R/M)$, where $\Omega_P = U_P L U_{P^-}$. Since Ω_P is a principal open subscheme of G, this implies that $x \in \Omega_P(R)$, that is, $x = v'bu'$. Since $p(u') = 1$, we have $u' \in U_{P^-}(M)$. □

Lemma 5. *Let G be a reductive group over a commutative ring R, P a parabolic subgroup of G, $A, B \in \Phi_P$ two non-proportional relative roots such that $A + B \in \Phi_P$. Assume that $A - B \notin \Phi_P$, or A, B belong to the image of a simply laced irreducible component of the absolute root system of G. Take $0 \neq u \in V_B$. Any generating system $e_1, \ldots, e_n$ of the R-module V_A contains an element e_i such that $N_{AB11}(e_i, u) \neq 0$.*

Proof. Assume that $N_{AB11}(e_i, u) = 0$ for all $1 \leq i \leq n$. Consider an affine fpqc-covering $\coprod \operatorname{Spec} S_\tau \to \operatorname{Spec} R$ that splits G. There is a member $S_\tau = S$ of this covering such that the image of $X_B(u)$ under $G(R) \to G(S)$ is non-trivial. Write

$$X_B(u) = \prod_{\pi(\beta)=B} x_\beta(a_\beta) \cdot \prod_{i\geq 2} \prod_{\pi(\beta)=iB} x_\beta(c_\beta),$$

where $\pi : \Phi \to \Phi_P$ is the canonical projection of the absolute root system of G onto the relative one, x_β are root subgroups of the split group G_S, and $a_\beta \in S$. Since $X_B(u) \neq 0$, the definition of X_B implies that there exists $a_\beta \neq 0$. Let $\beta_0 \in \pi^{-1}(B)$ be the root of minimal height with this property. By [5, Lemma 4] there exists a root $\alpha \in \pi^{-1}(A)$ such that $\alpha + \beta_0 \in \Phi$. Let $v \in V_A \otimes_R S$ be such that $X_A(v) = x_\alpha(1) \prod\limits_{i\geq 2} \prod\limits_{\pi(\gamma)=iA} x_\gamma(d_\gamma)$, for some $d_\gamma \in S$. Then the (usual) Chevalley commutator formula implies that $[X_A(v), X_B(u)]$

contains in its decomposition a factor $x_{\alpha+\beta}(\lambda a_{\beta_0})$, where $\lambda \in \{\pm 1, \pm 2, \pm 3\}$. However, since either α, β belong to a simply laced irreducible component of Φ, or $A - B \notin \Phi_P$, we have $\lambda = \pm 1$. Then $N_{AB11}(v, u) \neq 0$, a contradiction. $\square$

Recall [5] that any relative root $A \in \Phi_{J,\Gamma}$ can be represented as a (unique) linear combination of simple relative roots. The *level* $\operatorname{lev}(A)$ of a relative root A is the sum of coefficients in this decomposition.

Lemma 6. *Let R be a local ring with the maximal ideal M, and let G be a reductive group over R. Let P be a parabolic subgroup of G such that* $\operatorname{rank}\Phi_P \geq 2$*, and the type of P occurs as the type of a minimal parabolic subgroup of some reductive group over a local ring (not necessarily over R). Assume that $g \in G(R)$ is such that for any $A \in \Phi_P$ there is a system of generators e_{Ai}, $1 \leq i \leq n_A$, of V_A such that $[g, X_A(e_{Ai})] = 1$ for all i. If $g \in U_P(M)L(R)U_{P^-}(M)$, then $g \in L(R)$.*

Proof. Write $g = xhy$, where $x \in U_P(M)$, $h \in L(R)$, $y \in U_{P^-}(M)$. We have $\prod_{A\in\Phi_P^+} X_A(u_A)$, $y = \prod_{A\in\Phi_P^-} X_A(u_A)$, where the product is taken in any fixed order.

Let $A \in \Phi_P$ be such that $u_A \neq 0$, and $|\operatorname{lev}(A)|$ is minimal among the levels of relative roots with this property. We are going to deduce a contradiction, thus showing that A cannot occur in the decomposition of g.

Assume that $A \in \Phi_P^+$; the other case is treated symmetrically. Since the type of P coincides with the type of a minimal parabolic subgroup, Φ_P is isomorphic to a root system as a set with two partially defined operations—addition and multiplication by integers. Then the standard properties of a root system imply that one can find a simple root or a minus simple root $B \in \Phi_P$, non-proportional to A, such that $A + B \in \Phi_P$. Moreover, if the irreducible component of Φ_P containing A is not of type G_2, we can, and we will, choose B so that $A - B \notin \Phi_P$. If it is of type G_2, this may be impossible; then we stipulate that we take B positive. The classification of Tits indices over local rings [6] also implies that in this case the respective irreducible component of the absolute root system of G is either simply laced or itself of type G_2. Assume for now that the latter does not take place; we will treat this exceptional case in the very end of this proof. Then by Lemma 5 one can find an element e of a generating system of V_B centralized by g such that $N_{AB11}(u_A, e) \neq 0$.

We have $1 = [X_B(e), g] = [X_B(e), x](x[X_B(e), hy]x^{-1})$. This is equivalent to

$$1 = (x^{-1}[X_B(e), x]x)[X_B(e), hy] = [x^{-1}, X_B(e)][X_B(e), hy]. \tag{1}$$

By [5, Th. 2] we can write

$$x^{-1} = X_A(-u_A) \prod_{\substack{C\in\Phi_P^+,\ C\neq A,\\ \operatorname{lev}(C)\geq\operatorname{lev}(A)}} X_C(v_C) = X_A(-u_A)\cdot x_1,$$

and thus

$$\begin{aligned}[x^{-1}, X_B(e)] &= [X_A(-u_A)x_1, X_B(e)]\\ &= [X_A(-u_A), [x_1, X_B(e)]]\cdot[x_1, X_B(e)]\cdot[X_A(-u_A), X_B(e)].\end{aligned} \tag{2}$$

Case 1: B is positive, that is, B is a simple root. We study the factor $[X_B(e), hy]$ of (1). Write $[X_B(e), hy] = X_B(e)h(yX_B(e)^{-1}y^{-1})h^{-1}$, and

$$y = \prod_{C\in\Phi_P^-,\ C\nparallel B} X_C(v_c)\cdot\prod_{i>0} X_{-iB}(v_{-iB}) = y_1y_2.$$

Using Lemma 4 we obtain $yX_B(e)^{-1} = y_1(y_2\cdot X_B(e)^{-1}) = y_1\cdot\prod_{i>0} X_{iB}(w_{iB})\cdot b\cdot\prod_{i>0} X_{-iB}(w_{iB})$, where $b \in L(R)$. Since relative roots proportional to B does not occur in the decomposition of y_1, and B is a simple root, the generalized Chevalley commutator formula implies that $y_1 \cdot \prod\limits_{i>0} X_{iB}(w_{iB}) = \Big(\prod\limits_{i>0} X_{iB}(w_{iB})\Big) y_3$, where $y_3 \in U_{P^-}(R)$. Hence $yX_B(e)^{-1} \in \Big(\prod\limits_{i>0} X_{iB}(w_{iB})\Big) P^-(R)$, and also

$$[X_B(e), hy] \in X_B(e)h\Big(\prod_{i>0} X_{iB}(w_{iB})\Big)h^{-1}P^-(R) = \Big(\prod_{i>0} X_{iB}(z_{iB})\Big)P^-(R).$$

Now we consider the first factor $[x^{-1}, X_B(e)]$ of the right side of (1). The generalized Chevalley commutator formula, applied to (2), says that

$$[x^{-1}, X_B(e)] = \prod_{D\in\Phi_P^+} X_D(w_D).$$

Moreover, $D = A+B$ is a root of minimal height in the decomposition (2) satisfying $w_D \neq 0$; in fact, $w_{A+B} = N_{AB11}(-u_A, e)$. Hence, the whole product

$$[x^{-1}, X_B(e)] \cdot [X_B(e), hy] \in X_{A+B}(N_{AB11}(-u_A, e)) \cdot \Big(\prod_{i>0} X_{iB}(z_{iB})\Big) \cdot \prod_{\substack{C\in\Phi_P^+,\\ \mathrm{lev}(C)>\mathrm{lev}(A+B)}} X_C(t_C) \cdot P^-(R)$$

does not equal 1, a contradiction.

Case 2: B is negative, that is $B' = -B$ is a simple root. In this case the generalized Chevalley commutator formula immediately implies $[X_B(e), hy] \in P^-(R)$. We study (2). Note that the decomposition of x_1 does not contain $X_{B'}(v_{B'})$, and, if $2B' \in \Phi_P$, also does not contain $X_{2B'}(v_{2B'})$. Indeed, in the first case we would have $\mathrm{lev}(A) = 1$, hence A is a simple relative root, hence $A + B = A - B'$ is not a relative root. In the second case we would have $\mathrm{lev}(A) = 2$, and, since $A + B \in \Phi_P$, $A = A' + B'$ for a simple relative root A'. Since in this case we are in the irreducible component of Φ_P of type BC_n, and B' is an extra-short simple root, we also have $A' + 2B' = A - B \in \Phi_P$. But then by our algorithm we would have taken $(-A')$ instead of B, since $A - (-A') = 2A' + B' \notin \Phi_P$.

The above, together with the fact that $B' = -B$ is a simple root, and the generalized Chevalley commutator formula, implies that $[x_1, X_B(e)] = \prod\limits_{D\in\Phi_P^+} X_D(w_D)$. Moreover, if $w_D \neq 0$, then $D \neq A+B$, since $A-B$ is not a relative root by our assumptions, and obviously D is not proportional to B. Further, we see that for any relative root D, occuring in the decomposition of $[X_A(-u_A), [x_1, X_B(e)]]$ or $[X_A(-u_A), X_B(e)]$, the coefficient near any simple root $A_0 \neq B'$ in the decomposition of D is greater or equal to that in the decomposition of A. Summing up, the only factor of the form $X_{A-B}(u)$ in the decompositions of the expressions $[X_A(-u_A), [x_1, X_B(e)]]$, $[x_1, X_B(e)]$, $[X_A(-u_A), X_B(e)]$ is the factor $X_{A-B}(N_{AB11}(-u_A, e))$ in the third one, and no commutator of the factors can give a new factor of the form $X_{A-B}(u)$ with $u \neq 0$. Hence, $[x^{-1}, X_B(e)]$ contains $X_{A-B}(N_{AB11}(-u_A, e)) \neq 1$ in its decomposition, and

$$[x^{-1}, X_B(e)][X_B(e), hy] \in X_{A-B}(N_{AB11}(-u_A, e)) \cdot \prod_{\substack{F\in\Phi_P^+,\\ F\neq A-B}} X_F(t_F) \cdot P^-(R)$$

cannot equal 1, a contradiction.

Case G_2. We are left with the case when Φ_P is of type G_2, and moreover the relevant component of the absolute root system of G is also of type G_2. Then we can assume without loss of generality that all components of the absolute root system are of type G_2, and consequently G is quasi-split. There exists a canonical étale extension R' of R such that G is a Weil restriction of a split group G' of type G_2 over R', see [2, Exp. XXIV Prop. 5.9]. Then $G_{R'}$ is a direct product of k split groups G_i of type G_2. To show that $g \in L(R)$, it is enough to show that the image g' of g in $G(R')$ is in $L(R')$. We know that $P_{R'}$ is a Borel subgroup of $G_{R'}$, and, since Φ_P has no multiple roots, for any $A \in \Phi_P$ we can identify the root subscheme $X_A(V_A \otimes R')$ with the direct product of k elementary root subgroups $x_\alpha(R')$ of the groups G_i. Considering the relevant projections of g and the generating systems of V_A, we are reduced to proving the following: if a point $h \in H(S)$ of a split reductive group H of type G_2 centralizes $x_\alpha(u_\alpha)$ for some $u_\alpha \in S^\times$, for any root $\alpha \in \Psi$, where Ψ is the root system of H, then h belongs to the corresponding split maximal torus. By Lemmas 1 and 3 we can also assume that the ring S is local with the maximal ideal N, and $h = \prod_{\alpha\in\Psi^+} x_\alpha(a_\alpha) \cdot h \cdot \prod_{\alpha\in\Psi^-} x_\alpha(a_\alpha)$, where all $a_\alpha \in N$. Then the proof goes exactly as in [1, Prop. 3], substituting the elements $x_\beta(1)$ and $w_\beta(1)$ by $x_\beta(u_\beta)$ and $w_\beta(u_\beta) = x_\beta(u_\beta)x_{-\beta}(-u_\beta^{-1})x_\beta(u_\beta)$. □

Lemma 7. *Let G be an isotropic reductive algebraic group over a commutative ring R, P a parabolic subgroup of G, L a Levi subgroup of P. Assume that $g \in G(R)$ is such that for any $A \in \Phi_P$ there is a system of generators e_{Ai}, $1 \le i \le n_A$, of V_A such that $[g, X_A(e_{Ai})] = 1$ for all i. If $g \in L(R)$, then $[g, E_P(R)] = 1$.*

Proof. We show that $[g, X_A(V_A)] = 0$ for any $A \in \Phi_P^+$ by descending induction on the hight of A; the case $A \in \Phi_P^-$ is symmetric. By [5, Th. 2] for any $g \in L(S)$ and any $A \in \Phi_P$ there exists a set of homogeneous polynomial maps $\varphi^i_{g,A} : V_A \to V_{iA}$, $i \ge 1$, such that for any $v \in V_A$ one has

$$gX_A(v)g^{-1} = \prod_{i\ge1} X_{iA}(\varphi^i_{g,A}(v)).$$

Since $\varphi^i_{g,A}$ are homogeneous, $[g, X_A(v)] = 1$ for $v \in V_A$ implies $[g, X_A(\lambda v)] = 1$ for any $\lambda \in R$. Also by [5, Th. 2], there exist a set of homogeneous polynomial maps $q^i_A : V_A \times V_A \to V_{iA}$, $i > 1$, such that

$$X_A(v)X_A(w) = X_A(v + w) \prod_{i>1} X_{iA}(q^i_A(v, w))$$

for all $v, w \in V_A$. Assume that $[g, X_A(v)] = [g, X_A(w)] = 1$. Then

$$gX_A(v + w)g^{-1} = gX_A(v)X_A(w)g^{-1} \cdot g\Big(\prod_{i>1} X_{iA}(q^i_A(v, w))\Big)^{-1} g^{-1} = 1,$$

since by inductive hypothesis g centralizes $X_{iA}(V_{iA})$ for all $i > 0$. □

3 The proof

Proof of Theorem 1. Let $g \in G(R)$ centralize $E(R) = E_Q(R)$, where Q a strictly proper parabolic subgroup of G. We are going to show that $g \in \mathrm{Cent}(G)(R)$. By Lemma 1 it is enough to show that $g \in \mathrm{Cent}(G)(R_M)$ for any maximal ideal M of R. Fix an ideal M, and set $R' = R_M$. Let P be a minimal parabolic subgroup of $G_{R'}$. By Lemma 2 for any $A \in \Phi_P$ there is a system of generators e_{Ai}, $1 \le i \le n_A$, of the R'-module V_A such that one has $[g, X_A(e_{Ai})] = 1$, $1 \le i \le n_A$. Note that Φ_P is a root system by [2, Exp. XXVI, §7], and by the assumption of the theorem all irreducible components of Φ_P are of rank ≥ 2.

Let $\coprod \mathrm{Spec}\, S_\tau \to \mathrm{Spec}\, R'$ be an fpqc-covering such that G splits over each $\mathrm{Spec}\, S_\tau$. It is enough to check that $g \in \mathrm{Cent}(G)(S_\tau)$ for every τ (here we identify g with its image under $G(R') \to G(S_\tau)$). Fix one τ, and set $S = S_\tau$ for short. Again by Lemma 1 it is enough to show that $g \in \mathrm{Cent}(G)(S_N)$ for any maximal ideal N of S.

Since a system of generators e_{Ai}, $1 \le i \le n_A$, of the R'-module V_A, also generates $(V_A \otimes_{R'} S) \otimes_S S_N$ as an S_N-module, g satisfies the conditions of Lemmas 3 and 6 (for the base ring S_N); hence $g \in L(S_N)$, where L is a Levi subgroup of P. By Lemma 7 this implies that g centralizes $E(S_N)$. Since G_{S_N} is split, it has a Borel subgroup B, and $E(S_N) = E_B(S_N)$. Applying Lemmas 3 and 6 to B instead of P, we get that $g \in T(S_N)$ for a split maximal subtorus T of G_{S_N}. Hence $g \in \mathrm{Hom}\,(\Lambda/\Lambda_r, S_N) \subseteq \mathrm{Hom}\,(\Lambda, S_N) = T(S_N)$, where Λ is the weight lattice of G, and Λ_r is the root sublattice. Therefore, $g \in \mathrm{Cent}(G)(S_N)$. $\square$

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