UNSTABLE CLASSIFICATION OF PROJECTIVE MODULES OVER AFFINE ALGEBRAS

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Abstract. We present a self-contained proof of a conjecture of Suslin about stably free modules.

Foreword

These notes contain the basic material I'm going to cover during the ICTP meeting. They are PRELIMINARY. Some proofs are missing and some important results around the subject are simply not stated. I intend to write down the missing proofs and extend the material covered in the future. In particular, one of my goals is to rewrite the classical results under a $\mathbb{A}^1$-homotopy category point of view. To my opinion, it is certainly the right framework to understand the isomorphism classes of projective modules over smooth algebras (over a field).

Conventions. All the rings considered in these lectures are supposed to be noetherian, and the projective modules of finite type. Similarly, all the schemes considered here are supposed to be of finite type and separated over some field k.

Basic definitions and results

Theorem 0.1 (Eisenbud-Evans-Plumstead)**.** *Let R be a ring and P be a projective R-module of rank r. Let $(\alpha, a) \in P^\vee \oplus R$. Then there exists $\beta \in P^\vee$ such that $\mathrm{ht}(I_a) \geq d$ where $I = (\alpha + a\beta)(P)$. In particular, if the ideal $(\alpha(P), a)$ has height $\geq d$ then $\mathrm{ht}(I) \geq d$. Further, if $(\alpha(P), a)$ is of height $\geq d$ and I is a proper ideal, then $ht(I) = d$.*

Theorem 0.2 (Swan-Bertini)**.** *Let R be a smooth affine algebra over an infinite field k, and let P be a projective R-module of rank r. Let $(\alpha, a) \in P^\vee \oplus R$ be a unimodular element. Then there exists $\beta \in P^\vee$ such that if $I = (\alpha + a\beta)(P)$ then*

1. *R/I is smooth of dimension $\dim(R) - r$ unless $I = R$.*
2. *R/I is integral if $\dim(R/I) \neq 0$.*

Proposition 0.3 (Suslin)**.** *For any $n \geq 3$ and any ring R, the subgroup $E_n(R)$ is normal in $GL_n(R)$.*

Lecture 0

0.1. **Stable versus unstable classification.** Let R be a ring. If P is a (finitely generated) projective R-module, we denote by $\{P\}$ its isomorphism class. The set of isomorphism classes $M(R)$ of projective R-modules is endowed the structure of an abelian monoid with operation defined by

$$\{P\} + \{Q\} = \{P \oplus Q\}$$

and neutral element the trivial module. The group $K_0(R)$ is the group completion of $M(R)$. More precisely, $K_0(R)$ is the free abelian group generated by the isomorphism classes $\{P\}$ quotiented by the subgroup generated by

$$\{P\}+\{Q\}-\{P\oplus Q\}$$

for any $\{P\}$ and $\{Q\}$. We denote by $[P]$ the class of $\{P\}$ in $K_0(R)$.

Proposition 0.4. *Let P and Q be projective R-modules be such that $[P]=[Q]$. Then there exists $n\in\mathbb{N}$ such that $P\oplus R^n\simeq Q\oplus R^m$.*

Suppose that R is such that $\mathrm{Spec}(R)$ is connected (otherwise we can decompose R as a product of such rings). Then we obtain a homomorphism

$$\rho: K_0(R)\to\mathbb{Z}$$

defined by $\rho([P])=\mathrm{rank}(P)$. We denote by $\widetilde{K}_0(R)$ the kernel of ρ.

For any $r\in\mathbb{N}$, let $\mathbf{P}_r(R)$ be the set of isomorphism classes of rank r projective R-modules, pointed by the class $\{R^r\}$. We define a map

$$s_r:\mathbf{P}_r\to\mathbf{P}_{r+1}$$

by $s_r(\{P\})=\{P\oplus R\}$ and we observe that s_r is a map of pointed sets. Let $\mathbf{P}(R):=\lim\mathbf{P}_r(R)$. For any $r\in\mathbb{R}$, we denote by

$$\pi_r:\mathbf{P}_r(R)\to\mathbf{P}(R)$$

the "limit" homomorphism. It follows that $\mathbf{P}(R)$ is pointed by the class of $\pi_0\{0\}$.

For any $r\in\mathbb{N}$ we define a map $f_r:\mathbf{P}_r(R)\to\widetilde{K}_0(R)$ by $f_r(\{P\})=[P]-[R^r]$. It is clear that the following diagram commutes for any $r\in\mathbb{N}$

$$\begin{tikzcd} \mathbf{P}(R) \arrow[rr, "s_r"] \arrow[dr, "f_r"'] & & \mathbf{P}_{r+1}(R) \arrow[dl, "f_{r+1}"] \\ & \widetilde{K}_0(R) & \end{tikzcd}$$

and we then obtain a map $f:\mathbf{P}(R)\to\widetilde{K}_0(R)$.

Proposition 0.5. *The map $f:\mathbf{P}(R)\to\widetilde{K}_0(R)$ is bijective.*

Proof. We first prove that f is surjective. Let $\alpha=[P]-[Q]\in\widetilde{K}_0(R)$. Then we have $\mathrm{rank}(P)=\mathrm{rank}(Q)$. Let Q' be such that $Q\oplus Q'=R^n$ for some $n\in\mathbb{N}$. We have

$$\alpha=[P]-[Q]=[P]+[Q']-[Q]-[Q']=[P\oplus Q']-[R^n].$$

Now $\mathrm{rank}(P\oplus Q')=n$ and it follows that $\alpha=f_n(\{P\oplus Q'\})$. Hence f is surjective.

Let β and γ in $\mathbf{P}(R)$ be such that $f(\beta)=f(\gamma)$. There exists therefore $r,s\in\mathbb{N}$ and $\{P\}\in\mathbf{P}_r(R)$, $\{Q\}\in\mathbf{P}_s(R)$ such that $f_r(\{P\})=f_s(\{Q\})$ with $\pi_r(\{P\})=\beta$ and $\pi_s(\{Q\})=\gamma$.

Since $f_r(\{P\})=f_s(\{Q\})$, we have $[P]-[R^r]=[Q]-[R^s]$ and it follows from Proposition 0.4 that $P\oplus R^{s+m}\simeq Q\oplus R^{r+m}$ for some $m\in\mathbb{N}$. This yields

$$\beta=\pi_r(\{P\}=\pi_{r+s+m}(\{P\oplus R^{s+m}\})=\pi_{r+s+m}(\{Q\oplus R^{r+m}\})=\pi_s(\{Q\})=\gamma.$$

□

Thus we see that K-theory studies the isomorphism classes of projective R-modules "at the limit". The goal of these lectures is to study the sets $\mathbf{P}_r(R)$ and the maps

$$s_r : \mathbf{P}_r(R) \to \mathbf{P}_{r+1}(R)$$

In lecture 1, we prove the well-known result that s_r is a bijection if $r \geq \dim(R)+1$. In lecture 2, we give a proof of Suslin's theorem saying that s_d is injective if R is an affine algebra of dimension d over an algebraically closed field. We also prove his subsequent result that s_d is a bijection in the same situation. In the following lectures, we introduce the necessary tools in order to prove that s_{d-1} has a trivial fiber (i.e. $s_{d-1}^{-1}(\{R^d\}) = \{R^{d-1}\}$) if R is a normal algebra of dimension d over an algebraically closed field k with $(d-1)! \in k^\times$.

1. Lecture 1

Definition 1.1. Let R be a ring, $X = \mathrm{Max}(R)$ and P be a projective R-module. We say that $p \in P$ is unimodular if one of the following equivalent conditions is satisfied

1. $P \simeq Rp \oplus P'$
2. For any $x \in X$, we have $p(x) \neq 0$ in P/xP.
3. There exists $\varphi : P \to R$ such that $\varphi(p) = 1$.

Theorem 1.2 (Serre)**.** *Let R be a commutative noetherian ring and $X = \mathrm{Max}(R)$. Suppose that X is connected of dimension d. Let P be a projective R-module of rank $r > d$. Then $P \simeq P' \oplus R$.*

Proof. For any $p_1, \ldots, p_n \in P$ and any $j \in \mathbb{N}$, let

$$F_j(p_1, \ldots, p_n) = \{x \in X \mid \dim(\langle p_1(x), \ldots, p_n(x)\rangle \subset P/xP) < j\}.$$

Obviously, we have $F_0(p_1, \ldots, p_n) = \emptyset$, $F_j(p_1, \ldots, p_n) \subset F_{j+1}(p_1, \ldots, p_n)$ for any $j \in \mathbb{N}$ and $F_{n+1}(p_1, \ldots, p_n) = X$. Also, it is clear that the subsets $F_j(p_1, \ldots, p_n)$ are closed in X (adding a complement to P, we may assume that P is free and the condition F_j expresses as the vanishing of some minors in a matrix).

Assertion 1. *For any integer $s \leq r$, there exists $p_1, \ldots, p_s \in P$ such that we have $\mathrm{codim}_X(F_j(p_1, \ldots, p_s)) \geq s + 1 - j$ for any $j = 1, \ldots, s$.*

Suppose first that $s = 1$. Let $X = X_1 \cup \ldots \cup X_m$ be the (non redundant) decomposition of X in irreducible components. Choose $x_i \in X_i \setminus (\cup_{s \neq i} X_s)$ for any $i = 1, \ldots, m$. Since $P(x_i) = P/x_iP$ is of dimension $r \geq 1$, there exists $v_i \in P(x_i)$ such that $v_i \neq 0$ for any $i = 1, \ldots, m$. By the Chinese remainder lemma, there exists $p_1 \in P$ such that $p_1 \equiv v_i \pmod{x_iP}$ for any i. Let $Y \subset F_1(p_1) \subset X$ be an irreducible closed subset. By construction, we have $Y \cap X_i \subset X_i$ is of codimension at least 1 (since $x_i \notin Y \cap X_i$). It follows that $F_1(p_1) \subset X$ is of codimension at least one and we are done in this case.

Suppose now that the result is proved for $s - 1 \leq r - 1$. There exists therefore $p_1, \ldots, p_{s-1} \in P$ such that $\mathrm{codim}_X(F_j(p_1, \ldots, p_{s-1})) \geq s - j$ for any $j = 1, \ldots, s-1$. We can decompose $F_j(p_1, \ldots, p_{s-1})$ as a union of closed subsets

$$F_j(p_1, \ldots, p_{s-1}) = Y_{j,1} \cup \ldots \cup Y_{j,u_j} \cup Y'_j$$

such that $Y_{j,i}$ is irreducible with $\mathrm{codim}_X(Y_{j,l}) = s - j$ for any $i = 1, \ldots, u_j$ and $\mathrm{codim}_X(Y'_j) > s - j$. Choose $y_{j,i} \in Y_{j,i} \setminus ((\cup_{l \neq i} Y_{j,l}) \cup Y'_j)$ for any $j = 1, \ldots, s-1$ and any $i = 1, \ldots, u_j$.

Since $y_{j,i} \in F_j(p_1,\ldots,p_{s-1})$, we have

$$\dim(\langle p_1(y_{j,i}),\ldots,p_{s-1}(y_{j,i})\rangle) < j \leq s-1 < r.$$

Yet $\dim P/y_{j,i}P = r \geq s$ and it follows that there exists $v_{j,i}$ linearly independent of $\{p_1(y_{j,i}),\ldots,p_{s-1}(y_{j,i})\}$. The Chinese remainder lemma shows that there exists p_s such that $p_s \equiv v_{j,i} \pmod{y_{j,i}P}$ for any j,i.

By definition, we have $F_j(p_1,\ldots,p_s) \subset F_j(p_1,\ldots,p_{s-1})$. Since $p_s(y_{j,i})$ is linearly independent of $\{p_1(y_{j,i}),\ldots,p_{s-1}(y_{j,i})\}$, it follows that $F_j(p_1,\ldots,p_s)$ doesn't contain any $Y_{j,i}$. Thus

$$\mathrm{codim}_X(F_j(p_1,\ldots,p_s)) \geq F_j(p_1,\ldots,p_{s-1}) + 1 \geq s+1-j.$$

Assertion 2. *Let $k \in \mathbb{N}$ be a fixed integer. For any $p_1,\ldots,p_s \in P$ such that $\mathrm{codim}_X(F_j(p_1,\ldots,p_s)) \geq k-j$ for any $j=1,\ldots,s$, there exists $a_1,\ldots,a_{s-1}$ such that*

$$\mathrm{codim}_X(F_j(p_1+a_1p_s,\ldots,p_{s-1}+a_{s-1}p_s)) \geq k-j$$

for any $j=1,\ldots,s-1$.

To prove the second assertion, write

$$F_{j+1}(p_1,\ldots,p_s) = Z_{j,1} \cup \ldots \cup Z_{j,u_j} \cup Z'_j$$

with $Z_{j,i}$ irreducible of codimension $k-j-1$ and Z'_j of codimension $\geq k-j$. Choose $z_{j,i} \in Z_{j,i} \setminus ((\cup_{l\neq i} Z_{j,l}) \cup Z'_j \cup F_j(p_1,\ldots,p_s))$ for any $j=1,\ldots,s-1$. Then $\dim(\langle p_1(z_{j,i}),\ldots,p_s(z_{j,i})\rangle) = j \leq s-1$. There exists therefore $a_1(z_{j,i}),\ldots,a_{s-1}(z_{j,i})$ in $R/z_{j,i}$ such that

$$\dim(\langle p_1(z_{j,i}) + a_1(z_{j,i})p_s(z_{j,i}),\ldots,p_{s-1}(z_{j,i}) + a_{s-1}(z_{j,i})p_s(z_{j,i})\rangle) = j.$$

The Chinese remainder lemma gives $a_1,\ldots,a_{s-1} \in R$ such that $a_l \equiv a_l(z_{j,i}) \pmod{z_{j,i}}$ for an j,i.

Since $F_{j+1}(p_1+a_1p_s,\ldots,p_{s-1}+a_{s-1}p_s) \subset F_{j+1}(p_1,\ldots,p_s)$, we get

$$\mathrm{codim}_X(F_{j+1}(p_1+a_1p_s,\ldots,p_{s-1}+a_{s-1}p_s)) \geq k-j-1.$$

If T is an irreducible component of $F_j(p_1+a_1p_s,\ldots,p_{s-1}+a_{s-1}p_s))$, we see that $z_{j,i} \notin T$. Thus

$$\mathrm{codim}_X(F_j(p_1+a_1p_s,\ldots,p_{s-1}+a_{s-1}p_s)) \geq k-j.$$

We can now finish the proof of Serre's theorem. Because P is of rank $r > d$, Assertion 1 shows that there exists $p_1,\ldots,p_{d+1}$ such that $F_j(p_1,\ldots,p_s)$ is of codimension $\geq d+2-j$ for any $j=1,\ldots,d+1$. We can apply Assertion 2 d times to get $p \in P$ such that $F_j(p)$ is of codimension $\geq d+2-j$ for $j=1$. Therefore $F_1(p) = \emptyset$ and $p(x) \neq 0$ for any $x \in X$. It follows that p is unimodular. □

Theorem 1.3 (Bass-Schanuel)**.** *Let R be a commutative noetherian ring and let $X = \mathrm{Max}(R)$. Suppose that X is connected of dimension d. Let P and P' be projective R-modules of rank $r > d$. Suppose that there exists a projective R-module Q such that $P \oplus Q \simeq P' \oplus Q$. Then $P \simeq P'$.*

Proof. Since Q is a projective R-module, there exists a projective module Q' such that $Q \oplus Q' = R^n$ for some $n \in \mathbb{N}$. Since $P \oplus Q \simeq P' \oplus Q$, it follows that $P \oplus R^n \simeq P' \oplus R^n$. By induction, we are reduced to prove the result for $n=1$, i.e. when $P \oplus R \simeq P' \oplus R$. Let $\phi : P' \oplus R \to P \oplus R$ be such an isomorphism. To prove the result, it suffices to show that there exists an automorphism τ of $P \oplus R$ such that $\tau\phi(0,1) = (0,1)$. Set $(p,a) = \phi(0,1)$.

Since P is of rank $r > d$, there exists $p_1 \in P$ such that $P = Rp_1 \oplus P_1$ by Serre's theorem. Since (p,a) is unimodular, there exists $p_2 \in P$ such that $p_3 := p + ap_2$ is unimodular in P by Bertini's theorem.

Define $g_1 : P \oplus R \to P \oplus R$ by $g_1(q,b) = (bp_2, 0)$. Since $g_1^2 = 0$, it follows that $\tau_1 := Id + g_1$ is an automorphism of $P \oplus R$ (with inverse $Id - g_1$) and we have

$$\tau_1(p,a) = (p,a) + (ap_2, 0) = (p_3, a).$$

Since p_3 is unimodular, there exists $\alpha : P \to R$ such that $\alpha(p_3) = 1$. Define $g_2 : P \oplus R \to P \oplus R$ by $g_2(q,b) = (0, (1-a)\varphi(q))$. Once again, we have $g_2^2 = 0$ and thus $\tau_2 := Id + g_2$ is an automorphism of $P \oplus R$. Moreover, we have

$$\tau_2\tau_1(p,a) = \tau_2(p_3,a) = (p_3,a) + (0, 1-a) = (p_3, 1).$$

Define finally $g_3 : P \oplus R \to P \oplus R$ by $g_3(q,b) = (-bq, 0)$. Since $g_3^2 = 0$, we see that $\tau_3 := Id + g_3$ is an automorphism of $P \oplus R$ and

$$\tau_3\tau_2\tau_1(p,a) = \tau_3(p_3,1) = (p_3,1) + (-p_3, 0) = (0,1).$$

Setting $\tau := \tau_3\tau_2\tau_1$, we see that $\tau(p,a) = (0,1)$ and the result is proved. $\square$

2. Lecture 2

Let R be a ring and let $a_1, \ldots, a_n, b_1, \ldots, b_n \in R$ with $n \in \mathbb{N}$. Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$. Following [14, §5], we define a matrix $\alpha_n(a,b) \in M_{2^{n-1}}(R)$ inductively starting with $\alpha_1(a_1,b_1) = a_1$ and

$$\alpha_n(a,b) = \begin{pmatrix} a_1 Id_{2^{n-2}} & \alpha_{n-1}(a',b') \\ -\alpha_{n-1}(b',a')^t & b_1 Id_{2^{n-2}} \end{pmatrix}$$

where $a' = (a_2, \ldots, a_n)$ and $b' = (b_2, \ldots, b_n)$.

Lemma 2.1. *We have:*

1. $\alpha_n(a,b) \cdot \alpha_n(b,a)^t = (\sum a_i b_i) \cdot Id_{2^{n-1}}$.
2. $\det \alpha_n(a,b) = (\sum a_i b_i)^{2^{n-2}}$ *for* $n \geq 2$.
3. *In particular,* $\alpha_n(a,b) \in SL_{2^{n-1}}(R)$ *if* $\sum a_i b_i = 1$.

Lemma 2.2. *Let R be a ring and let $(a_1, \ldots, a_n) \in Um_n(R)$. Let $m \in \mathbb{N}$. Then*

$$(a_1^m, a_2, \ldots, a_n) \sim_{E_n(R)} (a_1, a_2^m, a_3, \ldots, a_n).$$

Lemma 2.3. *Let R be a ring and $a_1, \ldots, a_n, b_1, \ldots, b_n \in R$ with $n \geq 2$ and $\sum a_i b_i = 1$. There exists an elementary matrix $E(a,b)$ such that*

$$\alpha_n(a,b)E(a,b) = \begin{pmatrix} \beta_n(a,b) & 0 \\ 0 & 1 \end{pmatrix}$$

where $\beta_n(a,b) \in SL_n(R)$ and $e_1\beta_n(a,b) = (a_1^{n-1}, a_2^{n-2}, \ldots, a_{n-2}^2, a_{n-1}, a_n)$.

Corollary 2.4. *Let R be a ring and let $(a_1, \ldots, a_n) \in Um_n(R)$. Then the unimodular row $(a_1^{(n-1)!}, a_2, \ldots, a_n)$ is completable in an invertible matrix.*

Proof. If $n \leq 2$, there is nothing to prove. We can thus suppose that $n \geq 3$. By the above lemma, we know that $((a_1^{n-1}, a_2^{n-2}, \ldots, a_{n-2}^2, a_{n-1}, a_n)$ is completable in an invertible matrix. The result follows then from Lemma 2.2. $\square$

We now have all the tools in hand to prove Suslin's theorem.

Theorem 2.5. *Let A be an affine algebra of dimension d over an algebraically closed field k. Then $Um_{d+1}(A) = SL_{d+1}(A)$. In particular any stably free module of rank d is free.*

Proof. Let P be a stably free module of rank d. There exists thus $n \in \mathbb{N}$ such that $P \oplus A^n \simeq A^{n+d}$. Bass' cancellation theorem 1.3, shows that the above isomorphism yields an isomorphism $P \oplus A \simeq A^{d+1}$. It follows that P is the projective module associated to a unimodular row $(a_1, \ldots, a_{d+1})$. Let $B = A_{red}$ be the reduced algebra associated to A. The equivalent conditions of Definition 1.1 shows that P is free if and only if $P \otimes_A A_{red}$ is free and therefore we can suppose that A is reduced. Let $J \subset A$ the singular locus of A. Because A is reduced, we see that $\mathrm{ht}(J) \geq 1$. There exists thus a non zerodivisor $s \in J$. Since A/sA is of dimension $\leq d-1$, we can use Theorem 0.1 to perform elementary operations on $(a_1, \ldots, a_{d+1})$ in order to find $a_1 \equiv 1 \pmod{sA}$ and $a_i \in sA$ for $i = 2, \ldots, d+1$. Using now Swan's Bertini theorem 0.2, we see that there exists $b_1, \ldots, b_d$ such that $A/(a_1 + b_1 a_{d+1}, \ldots, a_d + b_d a_{d+1})$ is smooth of dimension 0 outside J. Since $a_1 \equiv 1 \pmod{J}$ and $a_{d+1} \in J$, it follows that $A/(a_1 + b_1 a_{d+1}, \ldots, a_d + b_d a_{d+1})$ is actually smooth. Since k is algebraically closed, we get

$$A/(a_1 + b_1 a_{d+1}, \ldots, a_d + b_d a_{d+1}) = k \times \ldots \times k.$$

There exists thus $b \in A$ such that $b^{d!} \equiv a_{d+1} \pmod{(a_1 + b_1 a_{d+1}, \ldots, a_d + b_d a_{d+1})}$. Altogether, we proved that we can perform elementary operations on the row $(a_1, \ldots, a_{d+1})$ to obtain a row $(a'_1, \ldots, a'_d, b^{d!})$. It follows from Corollary 2.4 that this row is completable and therefore P is free. $\square$

Theorem 2.6. *Let A be an affine algebra of dimension d over an algebraically closed field k. Suppose that P, P' are projective A-module of rank d such that $P \oplus A \simeq P' \oplus A$. Then $P \simeq P'$.*

3. Lecture 3

Let R be a ring. We consider triples (P, f_1, f_2) where P is a finitely generated projective R-module and $f_1, f_2 : P \to P^\vee$ are skew-symmetric isomorphisms. Two triples (P, f_1, f_2) and (P', f'_1, f'_2) are isometric if there exists an isomorphism $\alpha : P \to P'$ such that $f_i = \alpha^\vee f'_i \alpha$ for $i = 1, 2$. We denote by $[P, f_1, f_2]$ the isometry class of a triple (P, f_1, f_2). We denote by $GW_1^3(R)$ the free abelian group on isometry classes of triples $[P, f_1, f_2]$ subject to the relations

1. $[P, f_1, f_2] + [P, f_2, f_3] - [P, f_1, f_3]$.
2. $[P, f_1, f_2] + [Q, g_1, g_2] - [P \oplus Q, f_1 \oplus g_1, f_2 \oplus g_2]$.

Recall from M. Schlichting notes that there is an exact sequence

$$(1) \qquad K_1 Sp(R) \xrightarrow{f'} K_1(R) \xrightarrow{h} GW_1^3(R) \xrightarrow{\eta} K_0 SP(R) \xrightarrow{f} K_0(R)$$

For any $n \in \mathbb{N}$, let $S'_{2n}(R)$ be the set of skew-symmetric invertible matrices of size $2n$. We define $\psi_{2n} \in S'_{2n}(R)$ inductively by $\psi_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and

$$\psi_{2n} := \psi_2 \perp \ldots \perp \psi_2.$$

We define a map $S'_{2n}(R) \to S'_{2n+2}(R)$ by $M \mapsto M \perp \psi_2$ for any $n \in \mathbb{N}$, and we set $S'(R) = \lim S'_{2n}(R)$. We say that $A \in S'_{2n}(R)$ and $B \in S'_{2m}(R)$ are equivalent and we write $A \sim B$ if there exists $t \in \mathbb{N}$ and a matrix $E \in E_{2n+2m+2t}(R)$ such that

$$E^t(A \perp \psi_{2m+2t})E = B \perp \psi_{2n+2t}.$$

Lemma 3.1. *The relation $\sim$ is an equivalence relation. Further, $S'(R)/\sim$ endowed with the operation $\perp$ is an abelian group.*

Proof. The first assertion is a straightforward exercise. It is clear that $\perp$ induces a well-defined operation on $S'(R)/\sim$, and the only non trivial thing to check is that every matrix $A \in S'_{2n}(R)$ has an inverse in $S'(R)/\sim$. We follow the proof of Suslin-Vaserstein in [15, §3].

Let $\sigma_{2n} \in GL_{2n}(R)$ be the matrix defined inductively by $\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$$\sigma_{2n} = \sigma_{2n-2} \perp \sigma_2.$$

Observe that $\sigma_{2n}^t = \sigma_{2n}$ and $\sigma_{2n}^2 = Id$. Let $G \in S'_{2n}(R)$ be a skew-symmetric invertible matrix. We can write $G = H - H^t$ for some matrix $H \in M_{2n}(R)$. We have

$$\begin{aligned}
\begin{pmatrix} G & 0 \\ 0 & \sigma_{2n}G^{-1}\sigma_{2n} \end{pmatrix} &\sim \begin{pmatrix} 1 & 0 \\ -\sigma_r G^{-1} & 1 \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & \sigma_{2n}G^{-1}\sigma_{2n} \end{pmatrix} \begin{pmatrix} 1 & G^{-1}\sigma_{2n} \\ 0 & 1 \end{pmatrix} \\
&\sim \begin{pmatrix} G & \sigma_{2n} \\ -\sigma_{2n} & 0 \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & H\sigma_{2n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} G & \sigma_{2n} \\ -\sigma_{2n} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sigma_{2n}H^t & 1 \end{pmatrix} \\
&\sim \begin{pmatrix} 0 & \sigma_{2n} \\ -\sigma_{2n} & 0 \end{pmatrix}.
\end{aligned}$$

Replacing now G by ψ_{2n} and observing that $\sigma_{2n}\psi_{2n}^{-1}\sigma_{2n} = \psi_{2n}$, we see that

$$\begin{pmatrix} G & 0 \\ 0 & \sigma_{2n}G^{-1}\sigma_{2n} \end{pmatrix} \sim \psi_{4m} \sim \psi_2.$$

It follows that $\sigma_{2n}G^{-1}\sigma_{2n}$ is an inverse of G. □

We denote by $W'_E(R)$ the group $S'(R)/\sim$. We now define an analogue of the long exact sequence (1) for $W'_E(R)$ before proving that this group coincides with $GW_1^3(R)$. We follow the steps of [7, §2].

We first define a homomorphism

$$\varphi : W'_E(R) \to K_0Sp(R)$$

as follows. Let $G \in S'_{2n}(R)$. Then G can be seen as a skew-symmetric form $G : R^{2n} \to (R^{2n})^\vee$ and we can consider its class $[R^{2n}, G]$ in $K_0Sp(R)$. Similarly, the matrix ψ_{2n} also defines a class $[R^{2n}, \psi_{2n}]$ and we set

$$\varphi(G) = [R^{2n}, G] - [R^{2n}, \psi_{2n}].$$

Suppose that $G \sim H$ for some $H \in S'_{2m}(R)$. There exists therefore $t \in \mathbb{N}$ and $E \in E_{2n+2m+2t}(R)$ such that $E^t(G \perp \psi_{2m+2t})E = H \perp \psi_{2n+2t}$. Since

$$[R^{2n+2m+2t}, G \perp \psi_{2m+2t}] = [R^{2n+2m+2t}, E^t(G \perp \psi_{2m+2t})E]$$

in $K_0Sp(R)$, we see that $\varphi(G) = \varphi(H)$ and therefore φ induces a well-defined map $W_E(R) \to K_0Sp(R)$. It is clear that φ is a homomorphism.

We now define a homomorphism

$$\theta : K_1(R) \to W'_E(R).$$

If $G \in GL_2n(R)$, then we define $\theta(G)$ to be the class of $G^t\psi_{2n}G$ in $W'_E(R)$. If now $G \in GL_{2n+1}(R)$, then we set $\theta(G) = (G \perp 1)^t\psi_{2n+2}(G \perp 1)$. We see that these maps induce a map $\theta : GL(R) \to W'_E(R)$.

Lemma 3.2. *The map $\theta : GL(R) \to W'_E(R)$ induces a homomorphism*

$$\theta : K_1(R) \to W'_E(R).$$

Proof. First observe that $\theta(E(R))$ is trivial by definition of $W'_E(R)$. If $G \in GL_n(R)$, then Whitehead lemma reads as

$$\begin{pmatrix} G^{-1} & 0 \\ 0 & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ G & 1 \end{pmatrix} \begin{pmatrix} 1 & 1-G^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1-G \\ 0 & 1 \end{pmatrix}$$

and shows that the matrix on the left is elementary. If G and H are invertible matrices, we can stabilize and thus we can suppose that $G, H \in GL_{2n}(R)$ for some $n \in \mathbb{N}$. Let $E = \begin{pmatrix} G^{-1} & 0 \\ 0 & G \end{pmatrix} \in E_{4n}(R)$. We find the following sequence of equalities in $W'_E(R)$

$$(HG)^t\psi_{2n}HG = (HG \perp 1)^t\psi_{4n}(HG \perp 1) = E^t(HG \perp 1)^t\psi_{4n}(HG \perp 1)E$$

Now $(HG \perp 1)E = H \perp G$ and we therefore see that $\theta(HG) = \theta(H) \perp \theta(G)$. This proves the lemma. □

Theorem 3.3. *The sequence*

$$K_1Sp(R) \xrightarrow{f'} K_1(R) \xrightarrow{\theta} W'_E(R) \xrightarrow{\varphi} K_0Sp(R) \xrightarrow{f} K_0(R)$$

is exact.

Proof. We first prove that the sequence is a complex. If $G \in Sp_{2n}(R)$, then $G^t\psi_{2n}G = \psi_{2n}$ by definition and it follows that $\theta f' = 0$. If $G \in GL_{2n}$, then $[R^{2n}, G^t\psi_{2n}G] = [R^{2n}, \psi_{2n}]$ in $K_0Sp(R)$. Therefore $\varphi\theta = 0$. Finally, the underlying modules of $[R^{2n}, G]$ and $[R^{2n}, \psi_{2n}]$ are the same for any $G \in S'_{2n}(R)$ and thus $f\varphi = 0$.

We now prove the exactness of the sequence. Suppose that $\theta(G) = 0$. We can suppose that $G \in GL_{2n}(R)$ for some $n \in \mathbb{N}$. Since $\theta(G) = 0$, there exists $t \in \mathbb{N}$ and $E \in E_{2n+2t}(R)$ such that

$$E^t(G \perp Id_{2t})^t\psi_{2n+2t}(G \perp Id_{2t})E = E^t(G^t\psi_{2n}G \perp \psi_{2t})E = \psi_{2n+2t}.$$

It follows that $H := (G \perp Id_{2t})E \in Sp_{2n+2t}(R)$. By definition, we have $G = f'(H) \in K_1(R)$ and the sequence is exact at $K_1(R)$.

Suppose next that $\varphi(G) = 0$ for some skew-symmetric $G \in S'_{2n}(R)$. Therefore we have $[R^{2n}, G] = [R^{2n}, \psi_{2n}]$ in $K_0Sp(R)$. Therefore, there exists $m \in \mathbb{N}$ and $H \in GL_{2n+2m}(R)$ such that

$$G \perp \psi_{2m} = M^t\psi_{2n+2m}M.$$

It follows that $G \sim M^t\psi_{2n+2m}M$ and then $G = \theta(M)$.

Let $\alpha \in K_0Sp(R)$. By definition of this group, we can write $\alpha = [P, f] - [Q, g]$ for some projective modules P, Q and skew-symmetric forms f, g. Let M be such that $Q \oplus M = R^{2n}$ for some $n \in \mathbb{N}$. Let $h_M : M \oplus M^\vee \to M^\vee \oplus M$ be given by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then $[Q, g] + [Q^\vee, ev_Q g^{-1}] + [M \oplus M^\vee, h_M] = [R^{4n}, \psi_{4n}]$, where $ev_Q : Q \to Q^{\vee\vee}$ is the canonical isomorphism (exercise). Therefore

$$\alpha = [P, f] + [Q^\vee, ev_Q g^{-1}] + [M \oplus M^\vee, h_M] - [R^{4n}, \psi_{4n}]$$

in $K_0Sp(R)$. Setting $P' = P \oplus Q^\vee \oplus M \oplus M^\vee$ and $f' = f \perp ev_Q g^{-1} \perp h_M$ we find $\alpha = [P', f'] - [R^{4n}, \psi_{4n}]$. Suppose that $f(\alpha) = 0$. It follows that P' is stably free. Adding if necessary $[R^{2m}, \psi_{2m}]$ for m big enough and using Bass' theorem 1.3 we can suppose that P' is free. It follows that f' is given by a skew-symmetric matrix G and therefore $\alpha = \varphi(G)$. □

Our next aim is to construct a homomorphism $\tau : W'_E(R) \to GW_1^3(R)$. Let $G \in S_{2n}(R)$. As seen before, G can be considered as a skew-symmetric isomorphism $G : R^{2n} \to (R^{2n})^\vee$, and so does ψ_{2n}. We set $\tau(G) = [R^{2n}, G, \psi_{2n}]$. Since $[R^{2m}, \psi_{2m}, \psi_{2m}] = 0$, it follows that τ induces a map $S'(R) \to GW_1^3(R)$.

Lemma 3.4. *The map τ induces a homomorphism $\tau : W'_E(R) \to GW_1^3(R)$.*

Proof. Let $G \in S'_{2n}(R)$ and $H \in S'_{2m}(R)$. By definition of $GW_1^3(R)$, we find that

$$[R^{2n+2m}, G \perp H, \psi_{2n+2m}] = [R^{2n}, G, \psi_{2n}] + [R^{2m}, H, \psi_{2m}].$$

Suppose that $G \sim H$. There exists therefore $t \in \mathbb{N}$ and $E \in E_{2n+2m+2t}(R)$ such that

$$E^t(G \perp \psi_{2m+2t})E = H \perp \psi_{2n+2t}$$

By the above remark, we have $\tau(H \perp \psi_{2n+2t}) = \tau(H)$ and $\tau(G \perp \psi_{2m+2t}) = \tau(G)$. We can therefore suppose that G and H are elements of $S_{2n}(R)$ and that there exists $E \in E_{2n}(R)$ such that $E^tGE = H$. Now we have

$$\tau(G) = [R^{2n}, G, \psi_{2n}] = [R^{2n}, E^tGE, E^t\psi_{2n}E] = [R^{2n}, H, \psi_{2n}] + [R^{2n}, \psi_{2n}, E^t\psi_{2n}E].$$

We are then reduced to show that $[R^{2n}, \psi_{2n}, E^t\psi_{2n}E] = 0$ for any elementary matrix $E \in E_{2n}(R)$. As

$$[R^{2n}, \psi_{2n}, E^t\psi_{2n}E] = [R^{2n}, (E^{-1})^t\psi_{2n}E^{-1}, \psi_{2n}]$$

we are going to prove instead that $[R^{2n}, E^t\psi_{2n}E, \psi_{2n}] = 0$ for any $E \in E_{2n}(R)$. In order to show this, recall that the homomorphism $h : K_1(R) \to GW_1^3(R)$ is defined by $h(G) = [R^{2n}, G^t\psi_{2n}G, \psi_{2n}]$ if $G \in GL_{2n}(R)$. It follows that $h(E) = 0$ and the result is proved. □

Theorem 3.5. *The homomorphism $\tau : W'_E(R) \to GW_1^3(R)$ is an isomorphism.*

Proof. It suffices to observe that the following diagram

$$\begin{tikzcd}
K_1Sp(R) \arrow[r] \arrow[d, equal] & K_1(R) \arrow[r, "\theta"] \arrow[d, equal] & W'_E(R) \arrow[r, "\varphi"] \arrow[d, "\tau"] & K_0Sp(R) \arrow[r] \arrow[d, equal] & K_0(R) \arrow[d, equal] \\
K_1Sp(R) \arrow[r] & K_1(R) \arrow[r, "h"'] & GW_1^3(R) \arrow[r, "\eta"'] & K_0Sp(R) \arrow[r] & K_0(R)
\end{tikzcd}$$

commutes by definition of the maps, and to use the five lemma. □

3.0.1. *Pfaffians.* We first recall the definition of the Pfaffian homomorphism. Let

$$B_{2n} = \mathbb{Z}[x_{ij} | 1 \leq i, j \leq 2n]/\langle x_{ij} + x_{ji}, x_{ii} | 1 \leq i < j \leq 2n \rangle$$

The determinant D of the matrix (x_{ij}) is a square in B_{2n}, i.e. there exists $Pf \in B_{2n}$ such that $(Pf)^2 = D$. The polynomial Pf is uniquely determined up to a factor ± 1. We can determine this sign by forcing $Pf(\psi_{2n}) = 1$. We call Pf the Pfaffian polynomial.

If R is a ring, then we see that $S'_{2n}(R) = \mathrm{Hom}_{\mathrm{rings}}(B_{2n}, R)$. If $M \in S'_{2n}(R)$ corresponds to a ring homomorphism $\varphi : B_{2n} \to R$, then we set $Pf(M) = \varphi(Pf)$. Since M is invertible, it follows that $Pf(M) \in R^\times$ and we obtain a homomorphism

$$Pf : S'_{2n}(R) \to R^\times.$$

Lemma 3.6. *Let R be a ring. Then*

1. *$Pf(H^t G H) = Pf(G) \cdot \det(H)$ for any $G \in S'_{2n}(R)$ and any $H \in GL_{2n}(R)$.*
2. *$Pf(G_1 \perp G_2) = Pf(G_1) \cdot Pf(G_2)$ for any $G_1, G_2 \in S'_{2n}(R)$.*

Proof. □

For any $n \in \mathbb{N}$, we define $S_{2n}(R)$ as the kernel of $Pf : S'_{2n}(R) \to R^\times$. Since $Pf(\psi_2) = 1$, it follows that we have a commutative diagram

$$\begin{tikzcd} S_{2n}(R) \arrow[r] \arrow[d, "\psi_2"'] & S'_{2n}(R) \arrow[d, "\psi_2"] \\ S_{2n}(R) \arrow[r] & S'_{2n}(R). \end{tikzcd}$$

Setting $S(R) = \cup S_{2n}(R)$, we thus get a map $S(R) \to S'(R)$. Using Lemma 3.6, we see that $\sim$ induces an equivalence relation on $S(R)$ and that $S(R)/\sim$ is an abelian group that we denote by $W_E(R)$. We obtain a short exact sequence

$$0 \longrightarrow W_E(R) \longrightarrow W'_E(R) \longrightarrow R^\times \longrightarrow 0$$

which is split by associating to any $a \in R^\times$ the skew-symmetric matrix $\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$ (exercise).

4. Lecture 4: The Vaserstein symbol

Let R be a ring and $(a_1, a_2, a_3) \in Um_3(R)$. Choose $(b_1, b_2, b_3) \in R^3$ such that $\sum a_i b_i = 1$. Following [15, §5], we define $V(a_1, a_2, a_3) \in W_E(R)$ to be the class of the matrix

$$\begin{pmatrix} 0 & -a_1 & -a_2 & -a_3 \\ a_1 & 0 & -b_3 & b_2 \\ a_2 & b_3 & 0 & -b_1 \\ a_3 & -b_2 & b_1 & 0 \end{pmatrix}.$$

It seems a priori that the definition of $V(a_1, a_2, a_3)$ depends on the choice of (b_1, b_2, b_3) such that $\sum a_i b_i = 1$. However, the next lemma proves that this is not the case.

Lemma 4.1. *Let R be a ring and $(a_1, a_2, a_3) \in Um_3(R)$. Let $(b_1, b_2, b_3) \in R^3$ and $(c_1, c_2, c_3) \in R^3$ be such that $\sum a_i b_i = \sum a_i c_i = 1$. Then the matrices*

$$\begin{pmatrix} 0 & -a_1 & -a_2 & -a_3 \\ a_1 & 0 & -b_3 & b_2 \\ a_2 & b_3 & 0 & -b_1 \\ a_3 & -b_2 & b_1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -a_1 & -a_2 & -a_3 \\ a_1 & 0 & -c_3 & c_2 \\ a_2 & c_3 & 0 & -c_1 \\ a_3 & -c_2 & c_1 & 0 \end{pmatrix}$$

are equivalent in $W_E(R)$.

Proof. We follow the proof of [15, Lemma 5.1]. Let $d_1 = c_3 b_2 - c_2 b_3, d_2 = c_1 b_3 - c_3 b_1$ and $d_3 = c_2 b_1 - c_1 b_2$. Let

$$\alpha = \begin{pmatrix} 1 & d_1 & d_2 & d_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then α is elementary and we have

$$\alpha^t \begin{pmatrix} 0 & -a_1 & -a_2 & -a_3 \\ a_1 & 0 & -b_3 & b_2 \\ a_2 & b_3 & 0 & -b_1 \\ a_3 & -b_2 & b_1 & 0 \end{pmatrix} \alpha = \begin{pmatrix} 0 & -a_1 & -a_2 & -a_3 \\ a_1 & 0 & -c_3 & c_2 \\ a_2 & c_3 & 0 & -c_1 \\ a_3 & -c_2 & c_1 & 0 \end{pmatrix}.$$

□

Lemma 4.2. *Let $a := (a_1, a_2, a_3) \in Um_3(R)$ and $G \in SGL_3(R)$. If $aG = (x_1, x_2, x_3)$ then*

$$\begin{pmatrix} 1 & 0 \\ 0 & G \end{pmatrix} V(a_1, a_2, a_3) \begin{pmatrix} 1 & 0 \\ 0 & G^t \end{pmatrix} = V(x_1, x_2, x_3)$$

in $W_E(R)$.

Proof. In view of the above lemma, it suffices to check that the first row and column of

$$\begin{pmatrix} 1 & 0 \\ 0 & G \end{pmatrix} V(a_1, a_2, a_3) \begin{pmatrix} 1 & 0 \\ 0 & G^t \end{pmatrix}$$

are respectively $(0, -x_1, -x_2, -x_3)$ and $(0, x_1, x_2, x_3)$. This is a direct computation. □

As a corollary of the two previous lemmas, we see that we obtain a well-defined map

$$V : Um_3(R)/E_3(R) \to W_E(R).$$

Theorem 4.3. *Suppose that R is noetherian of dimension 2. Then V is a bijection.*

We now pass to a special case of a result of R. Rao and W. van der Kallen [11].

Theorem 4.4. *Suppose that R is a smooth affine algebra of dimension 3 over an algebraically closed field. Then V is a bijection.*

In order to prove the theorem, we will need a few auxiliary results.

Lemma 4.5. *Let R be a smooth affine algebra of dimension d over an algebraically closed field, and let $s \in R$. Suppose that $v \in Um_{d+1}(R)$ is congruent to $e_1 := (1, 0 \ldots, 0)$ modulo sR. Then v can be completed in a matrix $G \in SL_{d+1}(R)$ such that $G \equiv Id_{d+1} \pmod{sR}$.*

Proof. If $\mathrm{Spec}(R)$ is not connected, we write $R = R_1 \times \ldots \times R_n$ and we work component by component. We can thus suppose that R is integral. Let $A = R[X]/\langle X^2 - sX\rangle$. Since $p = X^2 - sX$ is not a zerodivisor in $R[X]$, and is not invertible, it follows that A is affine of dimension d. As v is congruent to e_1 modulo sR, we can write $v = e_1 + sw$ for some $w \in R^d$. We claim that $u(X) := e_1 + Tw$ is unimodular. Indeed, let $v' \in R^n$ be such that $v(v')^t = 1$. Since $v = e_1 + sw$, we can write $v' = (1 + sa_1, a_2, \ldots, a_{d+1})$. The equality $v(v')^t = 1$ yields

$$s(a_1 + w_1) + s^2 a_1 w_1 + s a_2 w_2 + \ldots + s a_{d+1} w_{d+1} = 0.$$

Since R is integral and $s \neq 0$ (otherwise there is nothing to do), it follows that

$$(a_1 + w_1) + s a_1 w_1 + a_2 w_2 + \ldots + a_{d+1} w_{d+1} = 0.$$

Consider $u'(X) = (1 + Xa_1, a_2, \ldots, a_{d+1})$. Then

$$u(X)(u'(X))^t = 1 + X(a_1 + w_1) + X^2 a_1 w_1 + X a_2 w_2 + \ldots + X a_{d+1} w_{d+1}.$$

But $X^2 = sX$ in A and thus $X(a_1 + w_1) + X^2 a_1 w_1 + X a_2 w_2 + \ldots + X a_{d+1} w_{d+1} = 0$. Therefore $u(X)$ is unimodular. By Suslin's theorem 2.5, there exists $H(X)$ in $SL_{d+1}(A)$ such that $e_1 H(X) = u(X)$. In particular, we have $e_1 H(0) = u(0) = e_1$ and $e_1 H(s) = u(s) = v$. The matrix $G = H(0)^{-1} H(s)$ is such that $e_1 G = v$ and $G \equiv Id_{d+1} \pmod{sR}$. $\square$

Proposition 4.6. *Let R be a smooth affine algebra over an algebraically closed field k. Let $G \in SL_{d+1}(R) \cap E(R)$. Then there exists $H \in SL_{d+1}(R[X])$ such that $H(0) = Id_{d+1}$ and $H(1) = G$.*

Proof. By Vaserstein stability theorem, we see that $(1 \perp G) \in E_{d+2}(R)$. There exists thus $M \in E_{d+2}(R[X])$ such that $M(0) = Id_{d+2}$ and $M(1) = 1 \perp G$. Set $R' = R[X]$ and $s = X^2 - X$. Then R' is smooth of dimension $d+1$ and we can apply the above lemma with $v = e_1 M \in Um_{d+2}(R')$. There exists thus $N \in SL_{d+2}(R')$ such that $e_1 N = v = e_1 M$ and $N \equiv Id_{d+2} \pmod{sR'}$. The matrix MN^{-1} satisfies $e_1 MN^{-1} = e_1$ and therefore there exists $H \in SL_{d+1}(R')$ and $b \in (R')^{d+1}$ such that

$$MN^{-1} = \begin{pmatrix} 1 & 0 \\ b^t & H \end{pmatrix}.$$

Since $N(0) = N(1) = M(0) = Id_{d+2}$ and $M(1) = 1 \perp G$, we see that $H(0) = Id_{d+1}$ and $H(1) = G$. $\square$

Corollary 4.7. *Let R be a smooth affine algebra over an algebraically closed field k. Then $SL_{d+1}(R) \cap E(R) = E_{d+1}(R)$.*

Proof. In view of the above proposition, we know that for any $G \in SL_{d+1}(R) \cap E(R)$ there exists $H \in SL_{d+1}(R[X])$ such that $H(0) = Id$ and $H(1) = G$. By Vorst's result [17], we get that $H \in E_{d+1}(R[X])$. It follows that $G \in E_{d+1}(R)$ and the result is proved. $\square$

proof of Theorem 4.4. We first show that V is surjective. Let $G \in S_{2n}(R)$ with $n \geq 3$. Since G is skew-symmetric, its first row is of the form $(0, w)$ for some $w \in Um_{2n-1}(R)$. Since $2n - 1 > 3 + 1$, it follows from Swan's Bertini theorem that there exists an elementary matrix $E \in E_{2n-1}(R)$ such that $vE = (1, 0, \ldots, 0) := e_1$. Set $F = \begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix} \in E_{2n}(R)$. Then

$$E^t G E = \begin{pmatrix} 0 & e_1 \\ -e_1^t & M \end{pmatrix}$$

for some skew-symmetric matrix M (not necessarily invertible). There exists then an elementary matrix $E_2 \in E_{2n}(R)$ such that

$$E_2^t E^t G E E_2 = \psi_2 \perp H$$

for some $H \in S_{2n-2}(R)$. It follows that any $G \in S_{2n}(R)$ for $n \geq 3$ has a representative in $S_4(R)$.

If $G \in S_4(R)$, then we can write

$$G = \begin{pmatrix} 0 & -a_1 & -a_2 & -a_3 \\ a_1 & 0 & -b_3 & b_2 \\ a_2 & b_3 & 0 & -b_1 \\ a_3 & -b_2 & b_1 & 0 \end{pmatrix}$$

for some $a_1, a_2, a_3, b_1, b_2, b_3 \in R$. Since G is of Pfaffian 1, it follows that $(a_1, a_2, a_3) \in Um_3(R)$ and that $\sum a_i b_i = 1$. Therefore V is surjective.

Following the arguments of [15, Theorem 5.2(c)], we now show that it is enough to prove that $SL_4(R) \cap E(R) = E_4(R)$ in order to prove that V is injective. In view of Corollary 4.7, this would conclude the proof. $\square$

Corollary 4.8. *Let R be a smooth affine threefold over an algebraically closed field. Then $Um_3(R)/E_3(R)$ is endowed with the structure of an abelian group.*

We now prove some results in order to understand the group law in $Um_3(R)/E_3(R)$ a bit better. We begin with Vaserstein rule.

Lemma 4.9 (Vaserstein rule)**.** *Let R be a ring and let (a, b, c) and (a, d, e) be unimodular rows. Let $d', e' \in R$ be such that $dd' + ee' \equiv 1 \pmod{aR}$. Then*

$$V(a, b, c) \perp V(a, d, e) = V(a, \begin{pmatrix} b & c \end{pmatrix} \cdot \begin{pmatrix} d & e \\ -e' & d' \end{pmatrix})$$

in $W_E(R)$.

Lemma 4.10 (Rao's antipodal Lemma)**.** *Let R be a ring and $(a, b, c) \in Um_3(R)$. Suppose that (a, b, c) and $(-a, b, c)$ are in the same elementary orbit. Then for any $n \in \mathbb{N}$, we have*

$$nV(a, b, c) = V(a^n, b, c)$$

in $W_E(R)$.

Corollary 4.11. *Let R be a ring such that there exists $u \in R$ with $u^2 = 1$. Then for any $n \in \mathbb{N}$, we have*

$$nV(a, b, c) = V(a^n, b, c)$$

in $W_E(R)$.

Proof. $\square$

5. Lecture 5: Around the cohomology of some sheaves

5.1. **Milnor K-theory.** Let F be a field, and $F^\times = F \setminus \{0\}$ the (abelian) group of invertible elements in F. Recall that the Milnor K-theory groups $K_n^M(F)$ of F are defined as follows (see [9]).

Consider the tensor algebra

$$T(F^\times) = \mathbb{Z} \oplus F^\times \oplus (F^\times \otimes F^\times) \oplus \ldots$$

with its obvious graduation and the ideal $I = \langle a \otimes (1-a) \mid a \neq 0, 1\rangle$. It is clear that I is generated by homogeneous elements, and therefore the quotient algebra $K_*^M(F) := T(F^\times)/I$ has a natural gradation. We set $K_n^M(F)$ to be the n-th graded piece of this algebra.

Alternatively, we can define $K_*^M(F)$ to be the free associative graded $\mathbb{Z}$-algebra generated by the elements $\{a\}$ with $a \in F^\times$ in degree one subject to the relation

$$\{a\} \cdot \{1-a\} = 0$$

for any $a \neq 0, 1$. Given $a_1, \ldots, a_n \in F^\times$, we usually write $\{a_1, \ldots, a_n\}$ instead of $\{a_1\} \cdot \ldots \cdot \{a_n\}$ in $K_n^M(F)$.

Lemma 5.1. *We have*

1. $\{a, a\} = \{a, -1\}$ *for any* $a \in F^\times$.
2. $\{a_1, \ldots, a_n\} = 0$ *if* $\sum a_i = 0$ *or* $\sum a_i = 1$.
3. $\{a, b\} = -\{b, a\}$ *for any* $a, b \in F^\times$.

Proof. □

If $F \subset L$ is a field extension, there is an obvious homomorphism $K_*^M(F) \to K_*^M(L)$ induced by $\{a\} \mapsto \{a\}$. If $i : F \to L$ is the inclusion, we often denote by i^* the induced map on Milnor K-theory.

5.1.1. *Residue homomorphisms.* Suppose that F is endowed with a discrete valuation $v : F \to \mathbb{Z} \cup \infty$ with valuation ring $\mathcal{O}_v$, maximal ideal $\mathfrak{m}_v$ and quotient field $k(v) = \mathcal{O}_v/\mathfrak{m}_v$. Choose a uniformizing parameter $\pi_v \in \mathfrak{m}_v$ and set $U = \mathcal{O}_v \setminus \mathfrak{m}_v$. If $u \in U$, we denote by $\overline{u}$ its class in $k(v)^\times$.

Lemma 5.2. *There exists a unique homomorphism* $\partial : K_n^M(F) \to K_{n-1}^M(k(v))$ *such that*

1. $\partial(\{\pi_v, u_2, \ldots, u_n\}) = \{\overline{u}_2, \ldots, \overline{u}_n\}$ *for any* $u_2, \ldots, u_n \in U$.
2. $\partial(\{u_1, u_2, \ldots, u_n\}) = 0$ *for any* $u_1, \ldots, u_n \in U$.

Proof. We follow Serre's idea. Consider the free associative graded $\mathbb{Z}$-algebra generated by the symbols $\{\overline{a}\}$ with $\overline{a} \in k(v)^\times$ and ξ in degree one subject to the relations

1. $\{\overline{a}, \overline{1-a}\} = 0$ for any $\overline{a} \neq 0, 1$.
2. $\xi \cdot \{\overline{a}\} = -\{\overline{a}\} \cdot \xi$ for any $\overline{a} \in k(v)^\times$.
3. $\xi^2 = \xi \cdot \{-\overline{1}\}$.

It is clear that this algebra is of the form $K_*^M(k(v))[\xi]$ and that $1, \xi$ form a basis of this algebra as a $K_*^M(k(v))$-module.

If $a \in F^\times$, we can write uniquely $a = \pi_v^i u$ with $i \in \mathbb{Z}$ and $u \in U$. Let A be the free graded associative algebra A generated by the elements $\{a\}$ with $a \in F^\times$. We can then define a map

$$\theta : A \to K_*^M(k(v))[\xi]$$

by $\theta(\{a\}) = i\xi + \{\overline{u}\}$. Let $a \in F^\times$ be such that $a \neq 1$. We now check that $\theta(\{a, 1-a\}) = 0$. Suppose that $a = \pi_v^i u$ for $u \in U$. We distinguish three cases. Suppose first that $i > 0$. In that case, $1 - a = 1 - \pi_v^i u$ is invertible and $\overline{1-a} = \overline{1}$. It follows immediately that $\theta(\{1-a\}) = 0$ and thus $\theta(\{a, 1-a\}) = 0$. If $i = 0$, then $1-a = \pi_v^j w$ for some $w \in U$. If $j > 0$, then we are reduced to the previous case and we can suppose that $j = 0$ as well. Since $\{\overline{a}, \overline{1-a}\} = 0$, we get $\theta(\{a, 1-a\}) = 0$. Suppose finally that $a = u/\pi_v^i$ for $i > 0$. In this case, $1 - a = (\pi_v^i - u)/\pi_v^i$. We have

$$\theta(\{a, 1-a\}) = \theta(\{a\})\theta(\{1-a\}) = (-i\xi + \{\overline{u}\})(-i\xi + \{-\overline{u}\}).$$

Using relations 2. and 3. above, we get

$$(-i\xi + \{\overline{u}\})(-i\xi + \{-\overline{u}\}) = i^2\{-\overline{1}\}\xi - i\xi\{-\overline{u}\} + i\xi\{\overline{u}\} + \{\overline{u}\}\{-\overline{u}\}.$$

By Lemma 5.1, we have $\{\overline{u}\}\{-\overline{u}\} = 0$. Moreover, since $\{-\overline{u}\} = \{-\overline{1}\} + \{\overline{u}\}$, the right-hand term becomes $i^2\{-\overline{1}\}\xi + i\{-\overline{1}\}\xi$. Now $\{-\overline{1}\}$ is 2-torsion, the assertion follows from the fact that $i^2 + i \equiv 0 \pmod 2$ for any $i \in \mathbb{Z}$.

Therefore, the homomorphism θ induces a homomorphism of graded algebras

$$\theta : K_*^M(F) \to K_*^M(k(v))[\xi].$$

It follows that $\theta(\alpha) = \psi(\alpha) + \xi\partial(\alpha)$ for any $\alpha \in K_*^M(F)$. The homomorphism ∂ does the job.

It remains to prove that ∂ is unique. This is clear, since Lemma 5.1 shows that any $\alpha \in K_n^M(F)$ can be written as a sum of symbols of the form $\{\pi_v, u_2, \ldots, u_n\}$ and $\{u_1, u_2, \ldots, u_n\}$. □

Remark 5.3. Observe that ∂ doesn't depend on the choice of the uniformizing parameter π_v. This follows from the simple fact that if π_v' is another uniformizing parameter, then $\pi_v' = \pi_v u$ for some unit u. It follows that $\{\pi_v'\} = \{\pi_v\} + \{u\}$ and the homomorphism obtain through π_v' and π_v are the same. On the other hand, the homomorphism ψ obtained using Serre's trick is dependent on the choice of π_v.

Let F be a field, and let $F(X)$ the field in one indeterminate over F. Let $\mathcal{V}$ be the set of irreducible monic polynomials in $F[X]$. Any $P \in \mathcal{V}$ induces a valuation that we call P-adic valuation with residue field $F(P)$ and associated residue homomorphism $\partial_P : K_*^M(F(X)) \to K_{*-1}^M(F(P))$.

There is yet another interesting valuation on $F(X)$. Associating to a polynomial $Q \in F[X]$ minus its degree yields a valuation $v_\infty : F(X) \to \mathbb{Z} \cup \infty$ that we call valuation at infinity. We denote by $\partial_\infty : K_*^M(F(X)) \to K_{*-1}^M(F)$ the associated residue map. It is clear that any $\alpha \in K_*^M(F(X))$ vanishes under the associated residue maps for all valuations $P \in \mathcal{V}$ except a finite number of them.

Theorem 5.4. *Let $i : F \to F(X)$ be the inclusion. Then the following sequence is split exact for any $n \in \mathbb{N}$*

$$0 \longrightarrow K_n^M(F) \xrightarrow{i^*} K_n^M(F(X)) \xrightarrow{\sum_{P\in\mathcal{V}} \partial_v} \bigoplus_{P\in\mathcal{V}} K_{n-1}(F(P)) \longrightarrow 0.$$

Proof. We first show that i^* is injective by exhibiting a retraction r. Consider the homomorphism

$$\{X\} : K_n^M(F(X)) \to K_{n+1}^M(F(X))$$

defined by $\alpha \mapsto \{X\} \cdot \alpha$. We define r as the following composite

$$K_n^M(F(X)) \xrightarrow{\{X\}} K_{n+1}^M(F(X)) \xrightarrow{\partial_X} K_n^M(F) \ .$$

Since X is a uniformizing parameter of the X-adic valuation, it follows essentially from Lemma 5.2 that $ri^* = Id$.

We now follow Milnor to prove at the same time that the sequence is exact in the middle and on the right. Let $L_d \subset K_n^M(F(X))$ be the subgroup generated by elements of the form $\{f_1, \ldots, f_n\}$ with f_i of degree $\leq d$ for any $i = 1, \ldots, n$. Observe that L_0 is precisely $i^*(K_n^M(F))$ (and is therefore a direct factor) and that we have a filtration

$$L_0 \subset L_1 \subset L_2 \subset \ldots$$

with $\cup L_i = K_n^M(F(X))$.

Let P be an irreducible monic polynomial of degree d. Consider the map

$$h_P : F(P)^\times \times \ldots \times F(P)^\times \to L_d/L_{d-1}$$

defined by $h_P(\overline{g}_1, \ldots, \overline{g}_{n-1}) = \{P, g_1, \ldots, g_{n-1}\}$ where g_i is the unique polynomial of degree $< d$ representing $\overline{g}_i$ for any $i = 1, \ldots, n-1$. We first prove that h_P induces a homomorphism

$$h_P : F(P)^\times \otimes \ldots \otimes F(P)^\times \to L_d/L_{d-1}$$

Suppose that $\overline{g}_1 = \overline{g}_1' \overline{g}_1''$. Let g_1' and g_1'' be the unique polynomials of degree $< d$ representing these polynomials. Then $g_1 = g_1' g_1'' + fP$ for some polynomial f, which is easily seen to be of degree $< d$. If $f = 0$ there is nothing to do, and we may suppose that $f \neq 0$. It follows that $1 = g_1' g_1''/g_1 + fP/g_1$ and thus Lemma 5.1 gives

$$(\{f\} + \{P\} - \{g_1\})(\{g_1'\} + \{g_1''\} - \{g_1\}) = 0$$

in $K_2^M(F(X))$. Multiplying on the right by $\{g_2, \ldots, g_{n-1}\}$ and reducing modulo L_{d-1}, we obtain

$$\{P, g_1', g_2, \ldots, g_{n-1}\} + \{P, g_1'', g_2, \ldots, g_{n-1}\} - \{P, g_1, g_2, \ldots, g_{n-1}\} = 0.$$

Therefore we get the desired homomorphism

$$h_P : F(P)^\times \otimes \ldots \otimes F(P)^\times \to L_d/L_{d-1}.$$

Suppose that $\overline{g}_j + \overline{g}_{j+1} = 1$ for some $j = 1, \ldots, n-2$. It follows that $g_j + g_{j+1} = 1$ and we therefore see that h_P induces a homomorphism

$$h_P : K_{n-1}^M(F(P)) \to L_d/L_{d-1}.$$

We thus obtain a homomorphism

$$h_d : \bigoplus_{P \in \mathcal{V} \text{ of degree } d} K_{n-1}^M(F(P)) \to L_d/L_{d-1}.$$

On the other hand, it is clear that $\sum_{P \in \mathcal{V}} \partial_P : K_n^M(F(X)) \to \bigoplus_{P \in \mathcal{V}} K_{n-1}^M(F(P))$ induces a homomorphism

$$\partial_d : L_d/L_{d-1} \to \bigoplus_{P \in \mathcal{V} \text{ of degree } d} K_{n-1}^M(F(P))$$

and it is easy to check that $\partial_d h_d = Id$. Therefore h_d is an isomorphism if and only if it is surjective.

Any generator of L_d can be expressed as $\{f_1, \ldots, f_s, g_{s+1}, \ldots, g_n\}$ with $\deg(f_i) = d$ and $\deg(g_j) < d$. We can further suppose that f_i is monic for any $i = 1, \ldots, s$. Suppose that $s \geq 2$. In that case, we can write $f_2 = f_1 + g$ for some g of degree $< d$. If $g = 0$, we can use Lemma 5.1 to get rid of f_2. We suppose therefore that $g \neq 0$. Since $f_1/f_2 + g/f_2 = 1$, we can use Lemma 5.1 to get

$$0 = (\{f_1\} - \{f_2\})(\{g\} - \{f_2\}) = \{f_1, g\} - \{f_1, f_2\} - \{f_2, g\} + \{f_2, -1\}.$$

It follows that L_d is generated by elements of the form $\{f, g_2, \ldots, g_n\}$ with $\deg(f) = d$ and $\deg(g_i) < d$. If f is reducible, then $\{f, g_2, \ldots, g_n\}$ splits as a sum of elements of L_{d-1} and then vanishes in L_d/L_{d-1}. It follows that L_d/L_{d-1} is generated by elements of the form $\{f, g_2, \ldots, g_n\}$ with f irreducible and monic. Therefore h_d is an isomorphism.

We conclude by induction that $\sum_{P \in \mathcal{V}} \partial_v$ induces an isomorphism

$$K_n^M(F(X))/K_n^M(F) \to \bigoplus_{P \in \mathcal{V}} K_{n-1}(F(P)).$$

The theorem follows. $\square$

5.1.2. *The transfer maps.* Let P be an irreducible monic polynomial in $F[X]$. Let $i : F \to F(P)$ be the inclusion. For any $n \in \mathbb{N}$, we define a homomorphism

$$i_* : K_n^M(F(P)) \to K_n^M(F)$$

as the composite

$$K_n^M(F(P)) \longrightarrow \bigoplus_{P \in \mathcal{V}} K_{n-1}(F(P)) \xrightarrow{s} K_{n+1}^M(F(X)) \xrightarrow{-\partial_\infty} K_n^M(F)$$

where s is any section of $\sum_{P \in \mathcal{V}} \partial_v$. Since $\partial_\infty i^* = 0$, it follows from Theorem 5.4 that i_* doesn't depend on the choice of s.

Suppose now that $F \subset K$ is a finitely generated algebraic field extension. Denote by $i : F \to K$ the inclusion. We have a filtration

$$F = K_0 \subset K_1 \subset \ldots \subset K_d = K$$

where $K_j = K_{j-1}[X]/P_j$ for some irreducible monic polynomial P_j for any $j = 1, \ldots, d$. Composing the successive residue maps, we obtain a homomorphism

$$i_* : K_n^M(K) \to K_n^M(F).$$

Theorem 5.5. *Let $i : F \to K$ be a field homomorphism such that K is finitely generated and algebraic over F. Then $i_* : K_n^M(K) \to K_n^M(F)$ doesn't depend on the choice of a filtration*

$$F = K_0 \subset K_1 \subset \ldots \subset K_d = K$$

such that $K_j = K_{j-1}(\theta_j)$ for any $j = 1, \ldots, d$.

5.1.3. *The Gersten complex.* Let X be a smooth scheme over a field k. For any $i \in \mathbb{N}$, we denote by $X^{(i)}$ the set of points $x \in X$ of codimension i (i.e. $\dim(\mathcal{O}_{X,x}) = i$). Let $x \in X^{(i)}$ for some $i \in \mathbb{N}$ and let $y \in X^{(i+1)}$ be in the closure of x. Then $\mathcal{O}_{X,y}/x\mathcal{O}_{X,y}$ is a one dimensional local k-algebra with quotient field $k(x)$. Let $B(y)$ be the integral closure of $\mathcal{O}_{X,y}/x\mathcal{O}_{X,y}$ in $k(x)$. Then $B(y)$ is a finite $\mathcal{O}_{X,y}/x\mathcal{O}_{X,y}$-module and the finitely many closed points $z_1, \ldots, z_n$ of $\mathrm{Spec}(B(y))$ yields finitely generated field extensions $k(y) \subset k(z_j)$ for any $j = 1, \ldots, n$.

Since $B(y)$ is of dimension 1 and normal, it follows that it is regular. Therefore, we see that $B(y)_{z_j}$ is a valuation ring for any z_j with residue field $k(z_j)$. We get a residue map

$$\partial : K_n^M(k(x)) \to \bigoplus_{z_j} K_{n-1}^M(k(z_j))$$

and a transfer map

$$\bigoplus_{z_j} K_{n-1}^M(k(z_j)) \to K_{n-1}^M(k(y)).$$

We define the residue homomorphism

$$\partial_x^y : K_n^M(k(x)) \to K_{n-1}^M(k(y))$$

as the composite of the two previous maps. If y is not in the closure of x, we then set $\partial_x^y = 0$.

It is clear that given an element $\alpha \in K_n^M(k(x))$, there are only a finite number of $y \in X^{(i+1)}$ such that $\partial_x^y(\alpha) \neq 0$ and we see that we finally get a homomorphism

$$\partial^i : \bigoplus_{x\in X^{(i)}} K_n^M(k(x)) \to \bigoplus_{x\in X^{(i+1)}} K_{n-1}^M(k(x)).$$

For any smooth scheme X of dimension d and any $n \in \mathbb{N}$, we denote by $C(X, K_n^M)$ the sequence of abelian groups

$$\bigoplus_{x\in X^{(0)}} K_n^M(k(x)) \xrightarrow{\partial^0} \bigoplus_{x\in X^{(1)}} K_{n-1}^M(k(x)) \xrightarrow{\partial^1} \dots \xrightarrow{\partial^{d-1}} \bigoplus_{x\in X^{(d)}} K_{n-d}^M(k(x))$$

We see thus sequence as a cochain complex with the gradation given by the codimension.

Theorem 5.6. *For any smooth scheme X and any $n \in \mathbb{N}$, the sequence $C(X, K_n^M)$ is a complex.*

In view of the theorem, we can consider the cohomology groups $H^i(X, K_n^M)$ of the complex $C(X, K_n^M)$.

5.2. **Galois cohomology.**

5.2.1. *The Galois symbol.* Let p be a prime number, F be a field of characteristic different from p and let F_{sep} be a separable closure of F. We denote by G_F the Galois group of F_{sep} over F. Suppose that $m \in \mathbb{N}$ is prime to the characteristic of F. The Kummer exact sequence of (continuous) G_F-modules

$$0 \longrightarrow \mu_n \longrightarrow F_{sep}^\times \xrightarrow{n} F_{sep}^\times \longrightarrow 0$$

yields an exact sequence

$$F^\times \xrightarrow{n} F^\times \xrightarrow{\delta} H^1(G_F, \mu_n) \longrightarrow H^1(G_F, F^\times).$$

By Hilbert Theorem 90, we have $H^1(G_F, F^\times) = 0$ and therefore δ induces an isomorphism $F^\times/(F^\times)^n \to H^1(G_F, \mu_2)$. If $a \in F^\times$, we denote by $(a) \in H^1(G_F, \mu_n)$ its image under δ. We obtain an isomorphism

$$\chi(n)_1 : K_1^M(F)/n \to H^1(G_F, \mu_n)$$

given by $\{a\} \mapsto (a)$.

Lemma 5.7. *The map $\chi(n)_1$ extends to a homomorphism of graded rings*

$$\chi(n)_* : K_*^M(F)/n \to H^*(G_F, \mu_n^{\otimes *}).$$

Proof. It suffices to prove that $(a) \cup (1-a) = 0$ in $H^2(G_F, \mu_n^{\otimes 2})$. Let

$$X^n - a = \prod_{i=1}^k p_i(X)^{n_i}$$

by the decomposition of $X^n - a$ in irreducible (monic) polynomials over $F[X]$. Let $\alpha_i \in F_{sep}$ be such that $F(\alpha_i) = 0$ and let $F_i = F(\alpha_i) = F[X]/p_i(X)$. Then $N_{F_i/F}(1-\alpha_i) = p_i(1)$ (exercise) and we find

$$1 - a = \prod_{i=1}^k p_i(1)^{n_i} = \prod_{i=1}^k N_{F_i/F}(1-\alpha_i)^{n_i}.$$

It follows that $(1-a) = \sum_{i=1}^k n_i(N_{F_i/F}(1-\alpha_i)) = \sum_{i=1}^k n_i N_{F_i/F}(1-\alpha_i)$ where

$$N_{F_i/F} : H^1(G_{F_i}, \mu_n) \to H^1(G_F, \mu_n)$$

is the corestriction map. It follows from the projection formula that

$$(a) \cup (1-a) = (a) \cup \sum_{i=1}^k n_i N_{F_i/F}(1-\alpha_i) = \sum_{i=1}^k n_i N_{F_i/F}((a) \cup (1-\alpha_i)).$$

Now $a = \alpha_i^n$ in F_i and thus $(a) = (\alpha_i)^n = 0$ in $K_1^M(F_i)/n$. The result is therefore proved. □

It follows directly from the definition that $\chi(n)$ induces an isomorphism in degree ≤ 1. The question to know if $\chi(n)_i$ is also an isomorphism for $i \geq 2$ is known as the Block-Kato conjecture. The case $i = 2$ was proved by Merkurjev and Suslin, and recently the general case was proved by Voevodsky-Suslin, together with Weibel's patch. We will only need the result for $i = 2$ and we state it for further reference.

Theorem 5.8 (Merkurjev-Suslin)**.** *Let F be a field and $n \in \mathbb{N}$ be an integer prime to* $\mathrm{char}(k)$*. Then the Galois symbol*

$$\chi(n)_2 : K_2^M(F)/n \to H^2(G_F, \mu_n^{\otimes 2})$$

is an isomorphism.

5.2.2. *Cohomological dimension.* Let F be a field and let G_F be its absolute Galois group and let p be a prime number. We say that F is of p-cohomological dimension $\leq n$ and we write $cd_p(F) \leq n$ if $H^q(G_F, A) = 0$ for any $q > n$ and any discrete G_F-module A whose torsion is p-primary. We say that F is of p-cohomological dimension n and we write $cd_p(F) = n$ if $cd_p(F) \leq n$ and $cd_p(F) \not\leq n-1$. We write $cd(F) = \sup(cd_p(F))$.

We recall the following results due to Serre:

Proposition 5.9. *Suppose that F is of characteristic $p > 0$. Then $cd_p(F) \leq 1$.*

Proposition 5.10. *Let $F \subset F'$ be a field extension of transcendence degree n, and let p be a prime number. Then*

$$cd_p(F') \leq cd_p(F) + n.$$

The inequality is an equality if F' is finitely generated over F, $cd_p(F) < \infty$ and $p \neq$ $\mathrm{char}(k)$*.*

Corollary 5.11. *Let F be an algebraically closed field and let $F \subset F'$ be a finitely generated field extension. Then $cd(F') = trdeg(F'/F)$.*

Proof. Since F is algebraically closed, if follows that $cd(F) = 0$. If F' is algebraic over F, then there is nothing to prove since $F = F'$. Suppose then that $F \subset F'$ is of transcendence degree $n \geq 1$. If $p \neq \mathrm{char}(F)$, then Proposition 5.10 shows that $cd_p(F') = n$. If $p = \mathrm{char}(F)$, then $cd_p(F') = 1$ by Proposition 5.9 and it follows that $cd(F') = n = trdeg(F'/F)$. □

5.2.3. *Etale cohomology.* Let X be a smooth scheme over a field k, and let Y be a scheme. Recall that a morphism $f : Y \to X$ is said to be étale if the following conditions are satisfied:

1. f is flat.
2. f is of constant relative dimension 0.
3. $\Omega_{Y/X} = 0$.

In particular, Y is smooth if f is étale. We can form the category $Et(X)$ whose objects are the schemes Y over X such that the structural morphism $p : Y \to X$ is étale and whose morphisms are morphisms of X-schemes. It follows from ?? that morphisms in $Et(X)$ are indeed étale morphisms. If $Y \in Et(X)$ is a scheme, then a covering of Y is a family of étale morphisms $(g_i : U_i \to Y)_{i \in I}$ such that $\cup g_i(U_i) = Y$. This defines a topology on $Et(X)$, the *étale topology.*

An étale presheaf of abelian groups on $Et(X)$ is a contravariant functor

$$F : Et(X) \to \mathcal{A}b$$

such that $F(\emptyset) = 0$. As usual, if $f : U \to V$ is a morphism in $Et(X)$ we denote by $s_{|U}$ the element $F(f)(s)$ for any $s \in V$.

A morphism of presheaves is a natural transformation $F \to G$. We can define in an obvious way the notions of kernel and cokernel and we obtain an abelian category $Psh(X)$ of presheaves on $Et(X)$. A sheaf on $Et(X)$ is a presheaf F such that the following sequence of abelian groups

$$F(U) \longrightarrow \prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j \in I} F(U_i \times U_j)$$

is exact (i.e. $F(U)$ is the equalizer of the diagram on the right) for any covering $(g_i : U_i \to U)_{i \in I}$ of U. We denote by $Sh(X)$ the category of étale sheaves on $Et(X)$.

Remark 5.12. Let $X = \mathrm{Spec}(k)$, where k is a field. Then $Y \in Et(X)$ if and only if $Y = \coprod_{i=1}^{n} L_i$ for separable algebraic extensions L_i/k. Let $\overline{k}$ be the separable closure of k, and $G_k = \mathrm{Gal}(\overline{k}/k)$ be its Galois group.

Suppose that F is a sheaf on $Et(X)$ and that L/k is a a finite separable field extension. Then G acts on L (say on the left) and therefore also on $F(L)$. If L'/L is also a finite separable extension, then the map

$$F(L) \to F(L')$$

is a homomorphism of G-modules. We can thus define a G-module M_F by setting

$$M_F := \lim F(L)$$

where L runs through the finite separable extensions of k in $\overline{k}$. Let $x \in M_F$. There exists then L/k finite and separable such that x is in the image of the canonical morphism $F(L) \to M_F$. It follows that $H := \mathrm{Gal}(\overline{k}/L)$ acts trivially on x and by consequence the stabilizer of x is open in G since it contains H. Therefore the action of G on M_F is continuous.

Conversely, let M be a discrete abelian group endowed with a continuous action of G. We can define a presheaf F_M as follows. If L/k is a finite separable extension and $H := \mathrm{Gal}(\overline{k}/L)$ then we set $F(L) = M^H$. We extend this definition to schemes $Y = \coprod_{i=1}^n L_i$ with L_i finite and separable over k by putting $M(Y) = \prod_{i=1}^n F(L_i)$. It is easy to check that F is indeed a presheaf. It is even a sheaf by [8, Chapter II, Lemma 1.8].

We can easily check that the correspondences $F \to M_F$ and $M \to F_M$ are inverse to each other.

Theorem 5.13. *Let F be a presheaf on $Et(X)$. Then there exists a sheaf aF and a morphism of presheaves $\phi : F \to aF$ such that for any morphism of presheaves $F \to G$ with G a sheaf there exists a unique morphism of sheaves $\psi : aF \to G$ such that the diagram*

$$\begin{tikzcd} F \arrow[r, "\phi"] \arrow[dr] & aF \arrow[d, "\psi"] \\ & G \end{tikzcd}$$

commutes.

Recall that a sheaf I on $Et(X)$ is injective if the functor $F \to \hom_{Sh(X)}(F, I)$ is exact.

Proposition 5.14. *The category $Sh(X)$ has enough injectives.*

Proof. For any $x \in X$, choose a separable closure $\overline{k(x)}$ of $k(x)$. Let $\overline{x} := \mathrm{Spec}(\overline{k(x)})$ and $u_x : \overline{x} \to X$ be the associated morphism.

Let F be a sheaf on $Et(X)$. The stalk u_x^*F of F at $\overline{x}$ is the limit $\lim F(U)$ on étale morphisms $f : U \to X$ such that the map $u_x : \overline{x} \to X$ factors through U (or more precisely the sheaf associated to this limit). Since the category $Et(\overline{x})$ is equivalent to the category of abelian groups, it follows that there exists an injective abelian group I_x and a monomorphism $j_x : u_x^*F \to I_x$.

For any sheaf $G \in Et(\overline{x})$, denote by $(u_x)_*G$ the sheaf whose sections on U in $Et(X)$ is the abelian group $G(U \times_X \overline{x})$. It is a straightforward exercise to check that $(u_x)_*$ is left exact, and it follows that $j_x : u_x^*F \to I_x$ induces a monomorphism $i_x : (u_x)_*u_x^*F \to (u_x)_*I_x$. The functors u_x^* and $(u_x)_*$ are adjoint to each other, and the unit of the adjunction reads as a morphism $\eta_x : F \to (u_x)_*u_x^*F$. We obtain a sequence of morphisms

$$F \xrightarrow{\prod \eta_x} \prod_{x\in X}(u_x)_*u_x^*F \xrightarrow{\prod i_x} \prod_{x\in X}(u_x)_*I_x.$$

We claim that the composite is a monomorphism and that $\prod_{x\in X}(u_x)_*I_x$ is injective. The first assertion follows from the fact that a morphism of sheaves $f : F \to G$ is a monomorphism if and only if $u_x^*f : u_x^*F \to u_x^*G$ is a monomorphism for any geometric point $\overline{x} \to X$. Indeed, if $y \in X$ is a point and $\overline{y}$ is the geometric point associated to y, then $u_y^* \prod_{x\in X}(u_x)_*u_x^*F = u_y^*(u_y)_*u_y^*F$ and $u_y^*\eta_y : u_y^*F \to u_y^*(u_y)_*u_y^*F$ is split injective. The second assertion follows from the fact that $(u_x)_*I_x$ is injective for any $x \in X$ by adjunction and that a product of injectives is still injective. $\square$

We can this define the cohomology groups of a sheaf $F \in Et(X)$ using the usual procedure. Namely, choose an injective resolution

$$0 \longrightarrow F \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \ldots$$

of F and define the cohomology groups $H^i_{\mathrm{et}}(X,F)$ as the cohomology groups of the complex

$$0 \longrightarrow I_0(X) \longrightarrow I_1(X) \longrightarrow \ldots$$

If $X = \mathrm{Spec}(k)$ for some field k with Galois group G_k, then the equivalence of categories between sheaves on $Et(X)$ and discrete Galois modules yields a canonical isomorphism $H^i_{\mathrm{et}}(X,F) = H^i(G_k, M_F)$ for any $i \in \mathbb{N}$.

Let $Z \subset X$ be a closed subset, and $U = X \setminus Z$. We define the functor

$$\Gamma_Z(X, \_) : Sh(X) \to \mathcal{A}b$$

by $\Gamma_Z(X,F) := \ker(F(X) \to F(U))$. A simple diagram chase shows that $\Gamma_Z(X, \_)$ is left exact and we define $H^i_{\mathrm{et},Z}(X,F)$ as the derived functors of $\Gamma_Z(X, \_)$.

Proposition 5.15. *For any $F \in Sh(X)$, we have a long exact sequence*

$$0 \to H^0_{\mathrm{et},Z}(X,F) \longrightarrow F(X) \longrightarrow F(U) \to H^1_{\mathrm{et},Z}(X,F) \to \ldots \to H^i_{\mathrm{et}}(X,F) \longrightarrow$$

$$\longrightarrow H^i_{\mathrm{et}}(U,F) \longrightarrow H^{i+1}_{\mathrm{et},Z}(X,F) \longrightarrow \ldots$$

Proof. □

Corollary 5.16 (Excision)**.** *Let $Z \subset X$ be a closed subset and X' be a scheme and $f : X' \to X$ be an étale morphism. Let $Z' = f^{-1}(Z)$ and suppose that f induces an isomorphism $f : Z' \to Z$. Then f induces isomorphisms*

$$f^* : H^i_{\mathrm{et},Z}(X,F) \to H^i_{\mathrm{et},Z'}(X', f^*F)$$

for any $i \in \mathbb{N}$ and any sheaf F on $Et(X)$.

Proof. For any sheaf $F \in Sh(X)$, the morphism of sheaves $F \to f_*f^*F$ yields a commutative diagram

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(U) \arrow[d] \\ f^*F(X') \arrow[r] & f^*F(U'). \end{tikzcd} \tag{2}$$

where $U' = X' \times_X U$, and therefore a homomorphism $\phi : \Gamma_Z(X,F) \to \Gamma_{Z'}(X', f^*F)$. Using the adjunction between f_* and f^*, it is easy to see that f^* preserves injective sheaves and it suffices therefore to prove that ϕ is an isomorphism to conclude. In order to do this, it suffices to prove that diagram (2) is Cartesian.

Since f is étale, we have $f^*F(X') = f(X')$ and $f^*F(U') = f(U')$. Thus diagram (2) becomes

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(U) \arrow[d] \\ F(X') \arrow[r] & F(U') \end{tikzcd}$$

which is clearly Cartesian since (X', U) is a covering of X. □

We now deal with dévissage questions. Let $n \geq 2$ be an integer prime to the characteristic of the base field. We consider the sheaves $F(i)$ for $i \in \mathbb{Z}$ on $Et(X)$ defined by

$$F(i) = \begin{cases} \mu_n^{\otimes i} & \text{if } i \geq 1. \\ \mathbb{Z}/n & \text{if } i = 0. \\ \mathrm{Hom}_{Sh(X)}(\mu_n^{\otimes -i}, \mathbb{Z}/n) & \text{if } i \leq 1. \end{cases}$$

$F = \mu_n^{\otimes i}$ where n is prime to the characteristic of the base field and $i \geq 1$ is an integer. If k contains a primitive n-th root of unity ξ, then $\mu_n \simeq \mathbb{Z}/n$ and thus $F(i) \simeq \mathbb{Z}/n$ for any $i \in \mathbb{Z}$.

It is straightforward to check that we have isomorphisms $F(i) \otimes F(j) \simeq F(i+j)$ for any $i, j \in \mathbb{Z}$.

Theorem 5.17 (Dévissage)**.** *Suppose that $Z \subset X$ is a closed subset of pure codimension c. Suppose moreover that Z is smooth. Then we have isomorphisms*

$$H^{i-2c}_{\mathrm{et}}(Z, F(j-c)) \to H^i_{\mathrm{et},Z}(X, F(j)).$$

for any $i \in \mathbb{N}$ and any $j \in \mathbb{Z}$.

5.2.4. *The weak Lefschetz theorem.* Let X be a scheme and p be a prime number. A sheaf $F \in Sh(X)$ is said to be p-torsion if $F(U)$ is a p-primary torsion abelian group for any quasi-compact $U \in Et(X)$. The scheme X is of p-cohomological dimension $n \in \mathbb{N}$ if $H^i_{\mathrm{et}}(X, F) = 0$ for any $i > n$ and any p-torsion sheaf F. We write $cd_p(X) = n$ in that case, and $cd_p(X) = \infty$ if such an integer n doesn't exist. We define the cohomological dimension of X as $cd(X) := \sup_p cd_p(X)$. The purpose of this section is to prove the following theorem:

Theorem 5.18. *Let X be a smooth affine scheme of dimension d over a separably closed field k. Then $cd(X) = \dim(X) = d$.*

5.2.5. *The Bloch-Ogus spectral sequence.* In this section, we recall the construction of the Bloch-Ogus spectral sequence following [5, §1]. To avoid overloading the notations, we simply write $H^i(X, F)$ and $H^i_Z(X, F)$ instead of $H^i_{\mathrm{et}}(X, F)$ and $H^i_{\mathrm{et},Z}(X, F)$. We also assume that X is equidimensional.

Let X be a scheme and let

$$\vec{Z} : \emptyset \subset Z_d \subset Z_{d-1} \subset \ldots \subset Z_1 \subset Z_0 = X$$

be a filtration of X by closed subsets $Z_i \subset X$. For convenience, we set $Z_i = \emptyset$ if $i > d$ and $Z_i = X$ if $i < 0$. We also assume that $\mathrm{codim}_X(Z_p) \geq p$ for any $p \in \mathbb{Z}$.

For any $p \in \mathbb{Z}$, any $F \in Sh(X)$ and any pair (Z_{p+1}, Z_p), Proposition 5.15 yields a long exact sequence of localization

$$\ldots \longrightarrow H^{p+q}_{Z_{p+1}}(X, F) \xrightarrow{i^{p+1,q-1}} H^{p+q}_{Z_p}(X, F) \xrightarrow{j^{p,q}}$$

$$\xrightarrow[j^{p,q}]{} H^{p+q}_{Z_p \setminus Z_{p+1}}(X \setminus Z_{p+1}, F) \xrightarrow[k^{p,q}]{} H^{p+q+1}_{Z_{p+1}}(X, F) \xrightarrow[i^{p+1,q}]{} \ldots$$

Setting $D^{p,q} := H^{p+q}_{Z_p}(X,F)$ and $E^{p,q} := H^{p+q}_{Z_p\setminus Z_{p+1}}(X\setminus Z_{p+1},F)$, we obtain an exact couple in the sense of [18, §5.9]

$$\begin{tikzcd} D^{p+1,q-1} \arrow[rr, "i^{p+1,q-1}"] & & D^{p,q} \arrow[dl, "j^{p,q}"] \\ & E^{p,q} \arrow[ul, "k^{p,q}"] & \end{tikzcd}$$

with $k^{p,q}$ of degree $(0,1)$. We thus get a spectral sequence

$$E^{p,q} \Longrightarrow H^{p+q}(X,F).$$

By definition $E_1^{p,q} = H^{p+q}_{Z_p\setminus Z_{p+1}}(X\setminus Z_{p+1},F)$ and the differential $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ is the composite

$$H^{p+q}_{Z_p\setminus Z_{p+1}}(X\setminus Z_{p+1},F) \xrightarrow{k^{p,q}} H^{p+q+1}_{Z_{p+1}}(X,F) \xrightarrow{i^{p+1,q}} H^{p+q+1}_{Z_{p+1}\setminus Z_{p+2}}(X\setminus Z_{p+2},F)$$

If

$$\vec{Z}' : \emptyset \subset Z'_d \subset Z'_{d-1} \subset \ldots \subset Z'_0 = X$$

is another filtration of X, then we say that $\vec{Z} \le \vec{Z}'$ if $Z_p \subset Z'_p$ for any $p\in\mathbb{Z}$. It is clear that the exact couple above is functorial in $\vec{Z}$ with respect to this ordering. We can pass to the limit and obtain a new exact couple with $D^{p,q} := \lim_{\vec{Z}} H^{p+q}_{Z_p}(X,F)$ and $E^{p,q} := \lim_{\vec{Z}} H^{p+q}_{Z_p\setminus Z_{p+1}}(X\setminus Z_{p+1},F)$.

If $x\in X^{(p)}$, we set $H^{p+q}_x(X,F) := \lim_{x\in U} H^{p+q}_{\overline{\{x\}}\cap U}(U,F)$ where $\overline{\{x\}}$ is the closure of x.

Lemma 5.19. *We have*

$$\lim_{\vec{Z}} H^{p+q}_{Z_p\setminus Z_{p+1}}(X\setminus Z_{p+1},F) = \bigoplus_{x\in X^{(p)}} H^{p+q}_x(X,F).$$

Proof. Let $Y_1,\ldots,Y_n$ be the irreducible components of codimension p in Z_p. Then $Y_i\cap Y_j$ is of codimension $\ge p+1$ for any i,j and we can refine Z_{p+1} by adding these intersections and the higher codimensional components of Z_p. We can thus suppose that $Y_i\cap Y_j\subset Z_{p+1}$ for any i,j. We then have $Z_p\setminus Z_{p+1} = \coprod Y_i\setminus Z_{p+1}$. The lemma follows then from the following assertion.

Assertion 3. *Let $T_1,T_2\subset X$ be closed subsets such that $T_1\cap T_2=\emptyset$. Then $H^i_{T_1\cup T_2}(X,F) = H^i_{T_1}(X,F)\oplus H^i_{T_2}(X,F)$.*

Indeed, the localization sequences for $T_1\subset T_1\cup T_2$ and $T_2\subset T_1\cup T_2$ yield a diagram

$$\begin{tikzcd} & H^i_{T_1}(X,F) \arrow[d] \arrow[dr] & \\ H^i_{T_2}(X,F) \arrow[r] \arrow[dr] & H^i_{T_1\cup T_2}(X,F) \arrow[r] \arrow[d] & H^i_{T_1}(X\setminus T_2,F) \\ & H^i_{T_2}(X\setminus T_1,F) & \end{tikzcd}$$

The diagonal arrows are isomorphism by excision. $\square$

It follows from the lemma that the spectral sequence above has a first page whose line q is equal to

$$\bigoplus_{x\in X^{(0)}} H^q_x(X,F) \longrightarrow \bigoplus_{x\in X^{(1)}} H^{q+1}_x(X,F) \longrightarrow \ldots \longrightarrow \bigoplus_{x\in X^{(d)}} H^{q+d}_x(X,F).$$

We now assume that the base field k is perfect, X is smooth and that $F=\mu_n^{\otimes m}$ for some integer n prime to $\mathrm{char}(k)$ and some $m\in\mathbb{N}$. If $x\in X^{(p)}$, then there exists $U\subset X$ such that $\overline{\{x\}}\cap U$ is smooth. In view of Theorem 5.17, we have an isomorphism

$$H^{p+q}_{\overline{\{x\}}\cap U}(U,\mu_n^{\otimes m})\simeq H^{p-q}(k(x),\mu_n^{\otimes(m-p)})$$

and it follows that the q-th line at the first page of the spectral sequence looks like

$$\bigoplus_{x\in X^{(0)}} H^q(k(x),\mu_n^{\otimes m}) \longrightarrow \ldots \longrightarrow \bigoplus_{x\in X^{(d)}} H^{q-d}(k(x),\mu_n^{\otimes(m-d)}). \tag{3}$$

If moreover k contains a primitive n-th root of unity ξ, then we can replace $\mu_n^{\otimes(m-i)}$ by $\mathbb{Z}/n$ everywhere in the sequence.

Theorem 5.20. *Let n be an integer prime to* $\mathrm{char}(k)$*. The Galois symbols*

$$\chi(n)_{q-p}: K^M_{q-p}(k(x))/n\to H^{q-p}(k(x),\mu_n\otimes^{(q-p)})$$

induce a morphism of complex

$$\begin{tikzcd}
\bigoplus_{x\in X^{(0)}} K^M_q(k(x))/n \arrow[r] \arrow[d, "\chi(n)_q"] & \ldots \arrow[r] & \bigoplus_{x\in X^{(d)}} K^M_{q-d}(k(x))/n \arrow[d, "\chi(n)_{q-d}"] \\
\bigoplus_{x\in X^{(0)}} H^q(k(x),\mu_n^{\otimes q}) \arrow[r] & \ldots \arrow[r] & \bigoplus_{x\in X^{(d)}} H^{q-d}(k(x),\mu_n^{\otimes(q-d)})
\end{tikzcd}$$

Proof. □

Suppose that $V\in Et(X)$. Then we can consider the group $H^q(V,\mu_n^{\otimes m})$. If $f:V'\to V$ is a morphism in $Et(X)$ then f is in particular étale and induces a homomorphism

$$f^*:H^q(V,\mu_m^{\otimes n})\to H^q(V',f^*\mu_m^{\otimes n})=H^q(V',\mu_m^{\otimes n}).$$

We therefore obtain a presheaf

$$\mathcal{H}^q(n,m):Et(X)\to\mathcal{A}b$$

defined by $V\mapsto H^q(V,\mu_n^{\otimes m})$. We also denote by $\mathcal{H}^q(n,m)$ the *Zariski* sheaf associated to the presheaf $\mathcal{H}^q(n,m)$.

Theorem 5.21 (Gersten conjecture)**.** *For any $q\geq 0$ the complex (3) is a flabby resolution of the sheaf $\mathcal{H}^q(n,m)$. In particular, its cohomology groups compute the cohomology of the sheaf $\mathcal{H}^q(n,m)$ and $H^i(\mathcal{H}^q(n,m))=0$ if $i\geq q+1$.*

Proof. □

Corollary 5.22. *Let k be a field and $n \in \mathbb{N}$ be prime to* $\mathrm{char}(k)$. *Let X be a smooth scheme over k. Then the morphism of complexes χ of Theorem 5.20 induces isomorphisms*
$$H^i(X, K_j/n) \simeq H^i(X, \mathcal{H}^j(n,j))$$
for any $i, j \in \mathbb{N}$ such that $i \geq j-1$.

Proof. This is a straightforward consequence of Theorems 5.20 and 5.8. □

Remark 5.23. Of course, the positive answer to the Bloch-Kato conjecture implies that χ induces isomorphisms for any $i, j \in \mathbb{N}$.

Corollary 5.24. *Let X be a smooth affine scheme of dimension $d \geq 2$ over an algebraically closed field k. Let $n \in \mathbb{N}$ be such that $(n, \mathrm{char}(k)) = 1$. Then*
$$H^{d-1}(X, K_d^M/n) = H^d(X, K_d^M/n) = 0.$$

Proof. Since k is algebraically closed, we have $\mu_n = \mathbb{Z}/n$ and therefore we have $H^i(k(x), \mu_n^{\otimes m}) = H^i(k(x), \mathbb{Z}/n)$ for any $m \in \mathbb{N}$. In view of this the sheaves $\mathcal{H}^i(n,m)$ are all isomorphic to $\mathcal{H}^i(n,0)$, that we write $\mathcal{H}^i(n)$ to lighten the notations.

By Corollary 5.11, we know that $cd(k(x)) = d-p$ for any $x \in X^{(p)}$. It follows that $H^i(k(x), \mathbb{Z}/n) = 0$ for any $i > d-p$. We can therefore write the non trivial group appearing at page 2 of the Bloch-Ogus spectral sequence as

$$\begin{array}{cccc}
H^0(X,\mathcal{H}^d(n)) & H^1(X,\mathcal{H}^d(n)) & \ldots & H^{d-1}(X,\mathcal{H}^d(n)) \quad H^d(X,\mathcal{H}^d(n)) \\
 & & & \\
H^0(X,\mathcal{H}^{d-1}(n)) & H^1(X,\mathcal{H}^{d-1}(n)) & \ldots & H^{d-1}(X,\mathcal{H}^{d-1}(n)) \\
 & & & \\
\vdots & \ddots & \ddots & \\
 & & & \\
H^0(X,\mathcal{H}^0(n)) & & &
\end{array}$$

Therefore $H^{d-1}(X, \mathcal{H}^d(n))$ and $H^{d-1}(X, \mathcal{H}^d(n))$ cannot be neither the target nor the source of any non trivial differential. Thus $E_\infty^{d-1,d} = H^{d-1}(X, \mathcal{H}^d(n))$ and $E_\infty^{d,d} = H^d(X, \mathcal{H}^d(n))$. Now the Bloch-Ogus spectral sequence converges to the groups $H_{\mathrm{et}}^{p+q}(X, \mathbb{Z}/n)$. Looking at the diagonal $p+q = d-1$ and $p+q = d$, we thus see that $H_{\mathrm{et}}^{2d-1}(X, \mathbb{Z}/n) = H^{d-1}(X, \mathcal{H}^d(n))$ and $H_{\mathrm{et}}^{2d}(X, \mathbb{Z}/n) = H^d(X, \mathcal{H}^d(n))$. Since $d \geq 2$, we have $2d-1 > d$ and we can apply Theorem 5.18 to conclude. □

We can now state the following result, which is one of the main ingredients of Theorem 7.1.

Theorem 5.25. *Let X be a smooth affine scheme of dimension d over an algebraically closed field k. Then $H^{d-1}(X, K_d^M)$ is divisible prime to the characteristic of k.*

Proof. In order to prove the theorem, it suffices to show that if $n \in \mathbb{N}$ is prime to $\mathrm{char}(k)$, the multiplication by n
$$H^{d-1}(X, K_d^M) \xrightarrow{n} H^{d-1}(X, K_d^M)$$

is surjective. Consider the multiplication by n morphism

$$C(X, K_d^M) \xrightarrow{n} C(X, K_d^M).$$

The cokernel of this multiplication is the complex $C(X, K_d^M/n)$ and we denote by $B(n)$ the kernel. We have therefore a commutative diagram

$$\begin{tikzcd}
B(n)_{d-2} \arrow[r] \arrow[d] & B(n)_{d-1} \arrow[r] \arrow[d] & B(n)_d \arrow[d] \\
\bigoplus_{x\in X^{(d-2)}} K_2^M(k(x)) \arrow[r] \arrow[d, "n"] & \bigoplus_{x\in X^{(d-1)}} K_1^M(k(x)) \arrow[r] \arrow[d, "n"] & \bigoplus_{x\in X^{(d)}} K_0^M(k(x)) \arrow[d, "n"] \\
\bigoplus_{x\in X^{(d-2)}} K_2^M(k(x)) \arrow[r] \arrow[d] & \bigoplus_{x\in X^{(d-1)}} K_1^M(k(x)) \arrow[r] \arrow[d] & \bigoplus_{x\in X^{(d)}} K_0^M(k(x)) \arrow[d] \\
\bigoplus_{x\in X^{(d-2)}} K_2^M(k(x))/n \arrow[r] & \bigoplus_{x\in X^{(d-1)}} K_1^M(k(x))/n \arrow[r] & \bigoplus_{x\in X^{(d)}} K_0^M(k(x))/n
\end{tikzcd}$$

with exact columns. Since $K_0(F) = \mathbb{Z}$ for any field F, we see that $B(n)_d = 0$. By Corollary 5.24, we know that the bottom sequence is exact in the middle. A simple diagram chase shows that the multiplication by n

$$n : \bigoplus_{x\in X^{(d-1)}} K_1^M(k(x)) \to \bigoplus_{x\in X^{(d-1)}} K_1^M(k(x))$$

induces the required surjection after taking cohomology. □

6. Lecture 6: The Gersten-Grothendieck-Witt spectral sequence

We refer here to M. Schlichting lectures for the general framework of higher Grothendieck-Witt groups of schemes.

Let k be a field of characteristic different from 2 and let X be a quasi-projective smooth scheme over k. Let $\mathrm{Vect}(X)$ be the category of coherent locally free $\mathcal{O}_X$-modules. We denote by $Ch^b(X)$ the category of bounded complexes of objects in $\mathrm{Vect}(X)$. This category is endowed with a tensor product $\_ \otimes \_$ and an internal hom object $[\_, \_]$ defined as follows. Let E and F be objects of $Ch^b(X)$. Then $E \otimes F$ is the complex such that

$$(E \otimes F)_i = \bigoplus_{m+n=i} (E_m \otimes F_n)$$

and differential $d(x \otimes y) = dx \otimes y + (-1)^m x \otimes y$ if $x \otimes y \in E_m \otimes F_n$, and $[E, F]$ is the complex

$$[E, F]_i = \mathrm{Hom}(E[i], F)$$

where $E[i]$ is the complex such that $E[i]_j := E_{j-i}$ and the differential is given by $d_{E[i]} = (-1)^i d_E$. The differential

$$d : [E, F]_i \to [E, F]_{i-1}$$

is defined by $d(\varphi) = d_F \circ \varphi + (-1)^i \varphi \circ d_{E[i]} = d_F \circ \varphi + \varphi \circ d_E$.

We say that a morphism of complexes $f : E \to F$ is a quasi-isomorphism if it induces an isomorphism in homology. We denote by quis the class of quasi-isomorphisms in $Ch^b(X)$.

Suppose that L is a line bundle over X. We consider L as a complex concentrated in degree 0. To avoid complicated notation, we denote by $(\_)^{\sharp_L^n}$ the contravariant functor

$$[\_, L[n]] : Ch^b(X) \to Ch^b(X),$$

which defines a duality on $Ch^b(X)$. A direct computation shows that this duality preserves the quasi-isomorphisms. We have a natural isomorphism

$$\varpi_L : 1 \to (\_)^{\sharp_L^n \sharp_L^n}$$

given by the evaluation isomorphisms $ev : E_i \to \mathrm{Hom}(\mathrm{Hom}(E_i, L), L)$. We set $\varpi_L^n = (-1)^{\frac{n(n-1)}{2}} \varpi_L$

Altogether, $(Ch^b(X), \mathrm{quis}, \sharp_L^n, \varpi_L^n)$ is a dg-category with weak-equivalences and duality in the sense of [12]. We can thus consider its n-th shifted Grothendieck-Witt spectrum with coefficients in L denoted by $GW^{[n]}(X, L)$.

We denote by $GW_i^n(X, L)$ the homotopy groups $\pi_i(GW^{[n]}(X, L))$ (beware that i might be negative). When $L = \mathcal{O}_X$, we omit it from the notation.

We now collect some results in order to define the spectral sequence we will need. For simplicity, we suppose that X is integral.

For any $i \in \mathbb{N}$, let $Ch^b(X)^i$ be the full subcategory of $Ch^b(X)$ of objects E whose homology is supported in degree $\geq i$. It is clear that we have a filtration

$$0 \subset Ch^b(X)^d \subset Ch^b(X)^{d-1} \subset \ldots \subset Ch^b(X)^1 \subset Ch^b(X)$$

where $d = \dim(X)$. Let $i \in \mathbb{N}$ and let $j \in \mathbb{N}$ be such that $j \geq i$. We denote by quis^j the class of morphisms in $Ch^b(X)^i$ whose cone lies in $Ch^b(X)^j$. Observe that a quasi-isomorphism in $Ch^b(X)$ belongs to quis^j for any $j \in \mathbb{N}$ and we obtain a filtration

$$\mathrm{quis} = \mathrm{quis}^{d+1} \subset \mathrm{quis}^d \subset \ldots \subset \mathrm{quis}^{i+1} \subset \mathrm{quis}^i$$

for any $i \in \mathbb{N}$.

Lemma 6.1. *Let L be a line bundle over X and $n \in \mathbb{N}$. For any $i, j \in \mathbb{N}$ such that $j \geq i$, the duality $\sharp(n)_L$ preserves the category $Ch^b(X)^i$ and the class quis^j. In particular, the quadruple $(Ch^b(X)^i, \mathrm{quis}^j, \sharp_L^n, \varpi_L^n)$ is a dg-category with weak-equivalences and duality.*

Proof. Let E be an object of $Ch^b(X)^i$. Let $\mathrm{Supp}(E) = \{x \in X | H_*(E_x) \neq 0\}$. We first prove that $\mathrm{Supp}(E^{\sharp_L^n}) \subset \mathrm{Supp}(E)$. Indeed, suppose that $x \in X$ is such that E_x is exact. Then $(E^{\sharp_L^n})_x = ([E, L[n]]_x \simeq [E_x, L_x[n]]$ is also exact. Dualizing once again, we see that $\mathrm{Supp}(E^{\sharp_L^n \sharp_L^n}) \subset \mathrm{Supp}(E^{\sharp_L^n})$. Since $E^{\sharp_L^n \sharp_L^n} \simeq E$ it follows that $\mathrm{Supp}(E^{\sharp_L^n}) = \mathrm{Supp}(E)$.

Let now $f : E \to F$ in quis^j. Let $C(f)$ be the cone of f. By definition, there is an exact sequence of chain complexes

$$0 \longrightarrow F \longrightarrow C(f) \longrightarrow E[1] \longrightarrow 0.$$

Accordingly, the cone of $f^{\sharp_L^n}$ fits into the exact sequence

$$0 \longrightarrow E^{\sharp_L^n} \longrightarrow C(f^{\sharp_L^n}) \longrightarrow F^{\sharp_L^n}[1] \longrightarrow 0$$

The functor $\sharp_L^n$ is exact on $Ch^b(X)^i$ and we can dualize the first sequence to obtain a sequence

$$0 \longrightarrow E^{\sharp_L^n}[-1] \longrightarrow C(f)^{\sharp_L^n} \longrightarrow F^{\sharp_L^n} \longrightarrow 0$$

Comparing the two last exact sequences, we see that $C(f)^{\sharp_L^n}[1] \simeq C(f^{\sharp_L^n})$. Thus

$$\operatorname{Supp}(C(f^{\sharp_L^n})) = \operatorname{Supp}(C(f)^{\sharp_L^n}[1]) = \operatorname{Supp}(C(f)^{\sharp_L^n}) = \operatorname{Supp}(C(f)).$$

Therefore $\sharp_L^n$ preserves quis^j. □

For any $j \geq i$, we therefore get a sequence of dg categories with weak-equivalences and duality

$$(Ch^b(X)^j, \mathrm{quis}, \sharp_L^n, \varpi_L^n) \to (Ch^b(X)^i, \mathrm{quis}, \sharp_L^n, \varpi_L^n) \to (Ch^b(X)^i, \mathrm{quis}^j, \sharp_L^n, \varpi_L^n)$$

For any $i, j \in \mathbb{N}$ such that $j \geq i$, let $D^b(X)^i$ be the triangulated category obtained by formally inverting the weak-equivalences, and $D^b(X)^{i/j}$ be the triangulated category obtained from $Ch^b(X)^i$ by inverting the class quis^j.

Lemma 6.2. *The sequence*

$$D^b(X)^j \to D^b(X)^i \to D^b(X)^{i/j}$$

is an exact sequence of triangulated categories.

Proof. We observe first that $D^b(X)^j \subset D^b(X)^i$ is a full thick subcategory by very definition. The quotient $D^b(X)^i/D^b(X)^j$ is the triangulated category obtained from $D^b(X)^i$ by inverting the class of morphisms whose cone is in $D^b(X)^j$. The functor $D^b(X)^i \to D^b(X)^{i/j}$ therefore induces a functor $D^b(X)^i/D^b(X)^j \to D^b(X)^{i/j}$. On the other hand, the functor $Ch^i(X)^i \to D^b(X)^i/D^b(X)^j$ induces a functor $D^b(X)^{i/j} \to D^b(X)^i/D^b(X)^j$ and we check that the two functors are mutually inverse to each other. □

It follows from [12, Theorem 6.6] that we obtain a homotopy fibration of Grothendieck-Witt spaces

$$\begin{array}{ccc} GW^{[n]}(Ch^b(X)^j, \mathrm{quis}, \sharp_L, \varpi_L) & \longrightarrow & GW^{[n]}(Ch^b(X)^i, \mathrm{quis}, \sharp_L, \varpi_L) \\ & & \downarrow \\ & & GW^{[n]}(Ch^b(X)^i, \mathrm{quis}^j, \sharp_L, \varpi_L) \end{array}$$

We state this result in a slightly different form in the next result. We denote by $GW^n_m(Ch^b(X)^i, L)$ the homotopy groups $\pi_m(GW^{[n]}(Ch^b(X)^i, \mathrm{quis}, \sharp_L, \varpi_L))$ and by $GW^n_m(Ch^b(X)^{i/j}, L)$ the homotopy groups $\pi_m(GW^{[n]}(Ch^b(X)^i, \mathrm{quis}^j, \sharp_L, \varpi_L))$.

Proposition 6.3. *Let L be a line bundle and let $n \in \mathbb{N}$. Let $i, j \in \mathbb{N}$ be such that $j \geq i$. Then we have a long exact sequence*

$$\ldots \longrightarrow GW^n_m(Ch^b(X)^j, L) \longrightarrow GW^n_m(Ch^b(X)^i, L) \longrightarrow GW^n_m(Ch^b(X)^{i/j}, L) \longrightarrow$$

$$\longrightarrow GW^n_{m-1}(Ch^b(X)^j, L) \to GW^n_{m-1}(Ch^b(X)^i, L) \longrightarrow \ldots$$

We now have everything in hand to construct the spectral sequence we need.

Theorem 6.4. *Let X be a smooth scheme of dimension d over a field k such that* $\mathrm{char}(k) \neq 2$ *. For any $n \in \mathbb{Z}$ and any line bundle L over X, there is a spectral sequence of Grothendieck-Witt groups converging to $E(n)^m = GW^n_{n-m}(X, L)$ with terms on the first page*

$$E(n)^{pq}_1 = \begin{cases} GW^n_{n-p-q}(Ch^b(X)^{p/p+1}, L) & \text{if } 0 \leq p \leq d. \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Consider the filtration of the category $Ch^b(X)$:

$$0 = Ch^b(X)^{d+1} \subset Ch^b(X)^d \subset \ldots \subset Ch^b(X)^1 \subset Ch^b(X)^0 = Ch^b(X).$$

We know that for any $i, j \in \mathbb{N}$ such that $j \geq i$ we have a sequence of dg-categories with weak-equivalences and duality

$$(Ch^b(X)^j, \mathrm{qis}, \sharp^n_L, \varpi^n_L) \to (Ch^b(X)^i, \mathrm{qis}, \sharp^n_L, \varpi^n_L) \to (Ch^b(X)^i, \mathrm{qis}^j, \sharp^n_L, \varpi^n_L).$$

By Proposition 6.3, we have a long exact sequence of Grothendieck-Witt groups associated to this sequence.

If $A(n)^{p,q} := GW^n_{n-p-q}(Ch^b(X)^p, L)$ and $E(n)^{p,q} := GW^n_{n-p-q}(Ch^b(X)^{p/p+1}, L)$, this long exact sequence reads as

$$\ldots \to A(n)^{p+1,q-1} \to A(n)^{p,q} \to E(n)^{p,q} \to A(n)^{p+1,q} \to \ldots$$

Let $A(n) := \bigoplus_{p,q} A(n)^{p,q}$ and $E(n) := \bigoplus_{p,q} E(n)^{p,q}$. We obtain an exact couple ([18, §5.9])

$$\begin{tikzcd} A(n) \arrow[rr] & & A(n) \arrow[dl] \\ & E(n) \arrow[ul] & \end{tikzcd}$$

which gives a spectral sequence starting with $E(n)^{p,q}_1 = GW^n_{n-p-q}(D^b(X)^{p/p+1}, L)$. This exact couple is bounded below because $A(n)^{p,q} = 0$ if $p < 0$ or $p \geq d$. Therefore the spectral sequence converges to $E(n)^m := \lim A(n)^{-p,m+p} = GW^n_{n-m}(X, L)$ by [18, Theorem 5.9.7]. $\square$

Under this form, the spectral sequence is quite abstract, since it is hard to grasp the groups $GW^n_{n-p-q}(Ch^b(X)^{p/p+1}, L)$ involved. Our next aim is to provide a computation of these groups.

If $x \in X^{(p)}$, we denote by $\omega^{\sharp_L}_x$ the $k(x)$-vector space $\mathrm{Ext}^p_{\mathcal{O}_{X,x}}(k(x), L \otimes \mathcal{O}_{X,x})$ and we can consider the Grothendieck-Witt groups $GW^n_m(k(x), \omega^{\sharp_L}_x)$ for any $m, n \in \mathbb{Z}$.

Proposition 6.5 (Dévissage)**.** *We have isomorphisms*

$$GW^n_m(Ch^b(X)^{p/p+1}, L) \simeq \bigoplus_{x \in X^{(p)}} GW^{n-p}_m(k(x), \omega^{\sharp_L}_x)$$

for any $p \in \mathbb{N}$ and any $m, n \in \mathbb{Z}$.

Proof. Let $x \in X^{(p)}$ be a point of codimension p. Denote by $Ch^b_{fl}(\mathcal{O}_{X,x})$ the category of bounded complexes of free (coherent) $\mathcal{O}_{X,x}$-modules whose homology is of finite length. The duality $\mathrm{Hom}_{\mathcal{O}_{X,x_p}}(\_, L \otimes \mathcal{O}_{X,x})$ on free $\mathcal{O}_{X,x}$-modules induces a duality on $Ch^b_{fl}(\mathcal{O}_{X,x})$ that we still denote by $\sharp_L$. It is straightforward to check that $(Ch^b_{fl}(\mathcal{O}_{X,x}), \mathrm{quis}, \sharp^n_L, \varpi^n_L)$ is a dg-category with weak-equivalences and duality. Moreover, the localization functor (at x)

$$(Ch^b(X)^p, \mathrm{quis}^{p+1}, \sharp^n_L, \varpi^n_L) \to (Ch^b_{fl}(\mathcal{O}_{X,x}), \mathrm{quis}, \sharp^n_L, \varpi^n_L)$$

is a functor of dg-categories with weak-equivalences and duality. Summing these functors for any $x \in X^{(p)}$, we get a functor

$$(Ch^b(X)^p, \mathrm{quis}^{p+1}, \sharp^n_L, \varpi^n_L) \to \coprod_{x\in X^{(p)}} (Ch^b_{fl}(\mathcal{O}_{X,x}), \mathrm{quis}, \sharp^n_L, \varpi^n_L)$$

which induces an equivalence of triangulated categories with duality ([2, Proposition 7.1])

$$D^b(X)^{p/p+1} \simeq \coprod_{x\in X^{(p)}} D^b_{fl}(\mathrm{Spec}(\mathcal{O}_{X,x})).$$

Such an equivalence induces an isomorphism

$$(4)\quad GW^n_m(Ch^b(X)^{p/p+1}, L) \simeq \bigoplus_{x\in X^{(p)}} GW^n_m(Ch^b_{fl}(\mathrm{Spec}(\mathcal{O}_{X,x})), L\otimes \mathcal{O}_{X,x})$$

by [12, Theorem 6.5].

Consider next the exact category $\mathcal{O}_{X,x} - fl$ of finite length $\mathcal{O}_{X,x}$-modules and the exact category $Ch^b(\mathcal{O}_{X,x} - fl)$ of bounded complexes of finite length modules. The duality $\flat(\_) := \mathrm{Ext}^p_{\mathcal{O}_{X,x}}(\_, L\otimes\mathcal{O}_{X,x})$ and the natural isomorphism (see [2, §6 (18),§6 (19)])

$$\varpi_{ext} : 1 \to \mathrm{Ext}^p_{\mathcal{O}_{X,x}}(\mathrm{Ext}^p_{\mathcal{O}_{X,x}}(\_, L\otimes\mathcal{O}_{X,x}), L\otimes\mathcal{O}_{X,x})$$

induces a duality on $Ch^b(\mathcal{O}_{X,x} - fl)$ and a natural isomorphism. It turns out that $(Ch^b(\mathcal{O}_{X,x} - fl), \mathrm{quis}, \flat, \varpi_{ext})$ is a dg-category with weak-equivalences and duality.

Let $\mathbf{C}$ be the dg-category whose objects $P_{\bullet,\bullet} \to M_\bullet$ are bounded bicomplexes of the form

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & M_{i+1} & \xrightarrow{d} & M_i & \xrightarrow{d} & M_{i-1} & \longrightarrow & \dots \\
 & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
\dots & \longrightarrow & P_{i+1,0} & \xrightarrow{\partial} & P_{i,0} & \xrightarrow{\partial} & P_{i-1,0} & \longrightarrow & \dots \\
 & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
\dots & \longrightarrow & P_{i+1,1} & \xrightarrow{\partial} & P_{i,1} & \xrightarrow{\partial} & P_{i-1,1} & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & \dots
\end{array}$$

where the M_i are finite length $\mathcal{O}_{X,x_p}$-modules, the P_{ij} are free $\mathcal{O}_{X,x_p}$-modules such that each column is a (bounded) free resolution of M_i and the mapping complex $\mathcal{C}(P_{\bullet,\bullet} \to M_\bullet, Q_{\bullet,\bullet} \to N_\bullet)$ is given in degree $n \in \mathbb{N}$ by

$$\mathcal{C}(P_{\bullet,\bullet} \to M_\bullet, Q_{\bullet,\bullet} \to N_\bullet)_n := \bigoplus_{i,j} \mathrm{Hom}(P_{i-n,j}, Q_{i,j}) \oplus \bigoplus_i \mathrm{Hom}(M_{i-n}, N_i)$$

with obvious differentials. The projection of such a bicomplex to the complex $M_\bullet$ yields a dg-functor $p : \mathbf{C} \to Ch^b(\mathcal{O}_{X,x_p} - fl)$. We say that a morphism f in $\mathbf{C}$ is a weak-equivalence if $p(f)$ is a weak equivalence in $Ch^b(\mathcal{O}_{X,x_p} - fl)$.

Moreover, $\mathbf{C}$ is endowed with a duality, obtained by sewing the dualities on $Ch^b(\mathcal{O}_{X,x_p}-fl)$ and $Ch^b(\mathcal{O}_{X,x_p})_{fl}$ (see [2, proof of Lemma 6.4]). So finally we see that $\mathbf{C}$ is a dg-catgeory with weak-equivalences and duality. The functor $p:\mathbf{C}\to Ch^b(\mathcal{O}_{X,x_p}-fl)$ preserves these structures. Moreover, taking the total complex associated to $P_{\bullet,\bullet}$ (without $M_\bullet$, see [2, loc. cit.]), we also get a dg functor $q:\mathbf{C}\to Ch^b(\mathcal{O}_{X,x_p})_{fl}$ preserving the weak-equivalences and dualities. Both q and p yield equivalences at the level of the associated triangulated categories and therefore we get isomorphisms for any m,n by [12, Theorem 6.5] (after twisting $n-p$ times):

$$GW^n_m(D^b_{fl}(\mathrm{Spec}(\mathcal{O}_{X,x_p})),L_{x_p})\simeq GW^{n-p}_m(D^b(\mathcal{O}_{X,x_p}-fl),\flat,\varpi_{ext}). \tag{5}$$

Let $\mathcal{V}(p)$ be the category of finite dimensional $k(x_p)$-vector spaces. There is a duality $(\_)^*:=\mathrm{Hom}_{k(x_p)}(\_,\omega^{\sharp_L}_{x_p})$ on $\mathcal{V}(p)$ where ω_{x_p} is the one dimensional vector space $\mathfrak{m}_{x_p}/\mathfrak{m}^2_{x_p}$ and $\omega^{\sharp_L}_{x_p}$ is $\mathrm{Hom}_{k(x_p)}(\omega_{x_p},L_{x_p}\otimes k(x_p))$. The usual canonical isomorphism $1\to(\_)^{**}$ is denoted by can. Now $\mathcal{V}(p)\subset\mathcal{O}_{X,x_p}-fl$ and the functor

$$(Ch^b(\mathcal{V}(p)),\mathrm{qis},(\_)^*,can)\to(Ch^b(\mathcal{O}_{X,x_p}-fl),\mathrm{qis},\flat,\varpi_{ext})$$

is a functor of dg-categories with weak-equivalences and duality (use the canonical isomorphism $\omega^{\sharp_L}_{x_p}\simeq\mathrm{Ext}^p_{\mathcal{O}_{X,x_p}}(k(x_p),L_{x_p})$). It induces an isomorphism in K-theory and an isomorphism of Witt groups. By Karoubi induction ([12, Lemma 6.4]), this functor induces an isomorphism

$$GW^{n-p}_m(D^b(k(x_p)),(\_)^*,can)\simeq GW^{n-p}_m(D^b(\mathcal{O}_{X,x_p}-fl),\flat,\varpi_{ext}). \tag{6}$$

According to our conventions, the group $GW^{n-p}_m(D^b(k(x_p)),(\_)^*,can)$ is denoted by $GW^{n-p}_m(k(x_p),\omega^{\sharp_L}_{x_p})$. Putting (4), (5) and (6), we get the result. □

6.0.6. *Some computations.* The goal of this section is to compute some low dimensional Grothendieck-Witt groups of fields whose characteristic is different from 2. We first recall from M. Schlichting's lectures that for any ring R such that $\frac{1}{2}\in R$ we have $GW^0_i(R)=K_iO(R)$ and $GW^2_i(R)=K_iSp(R)$.

Lemma 6.6. *Let F be a field with* $\mathrm{char}(F)\neq 2$. *Then the hyperbolic map $K_0(F)\to GW^2(F)$ is an isomorphism. Moreover, $GW^2_1(F)=K_1Sp(F)=0$.*

Proof. If V is an even-dimensional vector space and $\phi:V\to V^*$ is a skew-symmetric isomorphism, then there exists $n\in\mathbb{N}$ and $\alpha:F^{2n}\to V$ such that $\alpha^t\phi\alpha=\psi_{2n}$. It follows that the hyperbolic map is surjective. The map is also injective since two modules of different ranks cannot be conjugate under an invertible matrix.

It follows from Vaserstein stability thm that $Sp_2(F)\to K_1Sp(F)$ is surjective. Moreover, we have $E_2(F)=SL_2(F)=Sp_2(F)$. Since $E_2(F)\subset ESp_2(F)$, this shows that $K_1Sp(F)=0$. □

Lemma 6.7. *The hyperbolic map $K_1(F)\to GW^3_1(F)$ is an isomorphism.*

Proof. The Bott sequence reads as

$$GW^2_1(F)\longrightarrow K_1(F)\longrightarrow GW^3_1(F)\longrightarrow GW^2(F)\longrightarrow K_0(F).$$

The above lemma shows that $GW^2_1(F)=0$ and that the hyperbolic map $K_0(F)=\mathbb{Z}\to GW^2(F)$ is an isomorphism. Moreover, it is easy to see that the composite

$$K_0(F)\xrightarrow{H}GW^2(F)\xrightarrow{f}K_0(F)$$

is equal to the multiplication by 2. Thus the hyperbolic map H is injective. The Bott sequence yields the result. $\square$

Remark 6.8. Alternatively, one can show by hand that $W_E(F) = 0$ and that the Pfaffian homomorphism $W'_E(F) \to F^\times$ is an isomorphism.

Consider now the Witt ring $W(F)$ and the fundamental ideal $I(F) \subset W(F)$. If $n \in \mathbb{N}$, we denote by $I^n(F)$ the n-th power of this ideal.

Lemma 6.9. *The forgetful homomorphism $GW_i^n(F) \to K_n(F)$ induces an exact sequence*

$$0 \longrightarrow I^{n+1}(F) \longrightarrow GW_n^n(F) \longrightarrow K_n(F) \longrightarrow 0$$

for $0 \leq n \leq 2$.

Proof. For $n = 0$, this is obvious. For $n = 1$, we refer to [3, Corollaire 4.5.1.5] and the case $n = 2$ follows from [13, Corollary 6.4]. $\square$

Corollary 6.10. *We have $GW_n^{n+1}(F) = 0$ for $0 \leq n \leq 2$.*

Proof. Using the Bott sequence, we obtain

$$GW_n^n(F) \to K_n(F) \to GW_n^{n+1}(F) \xrightarrow{\eta} GW_{n-1}^n(F) \to K_{n-1}(F) \to GW_{n-1}^{n+1}(F)$$

If we specialize the above sequence at $n = 1$, we obtain a sequence

$$GW_1^1(F) \to K_1(F) \to GW_1^2(F) \xrightarrow{\eta} GW_0^1(F) \to K_0(F) \to GW^2(F)$$

and Lemma 6.6 shows that $GW_0^1(F) = 0$. Specializing now at $n = 2$, we get

$$GW_2^2(F) \to K_2(F) \to GW_2^3(F) \xrightarrow{\eta} GW_1^2(F) \to K_1(F) \to GW_1^3(F)$$

Lemma 6.6 shows that $GW_1^2(F) = 0$, and Lemma 6.9 proves that $GW_2^2(F) \to K_2(F)$ is surjective. Therefore $GW_2^3(F) = 0$. $\square$

For an abelian group A, we denote by ${}_nA$ the elements of n-torsion in A.

Lemma 6.11. *The forgetful functor induces surjections $f : GW_i^{i-1}(F) \to {}_2K_i(F)$ for $i = 1, 2$. Moreover, if F is algebraically closed then $f : GW_1^0(F) \to \{\pm 1\}$ is an isomorphism.*

Proof. For any field F, the Bott sequence sequence

$$GW_i^{i-1}(F) \xrightarrow{f} K_i(F) \xrightarrow{H} GW_i^i(F)$$

and Lemma 6.9 yield surjective homomorphisms

$$GW_i^{i-1}(F) \to {}_2K_i(F)$$

for $i = 1, 2$. Suppose now that F is algebraically closed. The Bott exact sequence

$$GW_1^3(F) \xrightarrow{f} K_1(F) \xrightarrow{H} GW_1^0(F) \to GW^3(F) \xrightarrow{f} K_0(F) \xrightarrow{H} GW(F)$$

and Lemma 6.7 (together with the easy fact that $H : K_0(F) \to GW(F)$ is injective) give an exact sequence

$$0 \longrightarrow K_1(F)/2 \longrightarrow GW_1^0(F) \longrightarrow GW^3(F) \longrightarrow 0\ .$$

Since F is algebraically closed, the left term is trivial. Now $GW^3(F) = \mathbb{Z}/2$ by Lemma 6.6 and the Bott sequence once again. $\square$

We now have all the tools in hand to prove the main theorem of the section.

Theorem 6.12. *Let X be a smooth affine threefold over an algebraically closed field k. Then the Gersten-Grothendieck-Witt spectral sequence $E(3)^{p,q}$ yields an isomorphism*

$$W_E(X) \simeq H^2(X, K_3).$$

Proof. First observe that the line $q = 1$ is trivial by Corollary 6.10. The line $q = 2$ reads as follows:

$$GW_1^3(k(X), \omega_X) \longrightarrow \bigoplus_{x \in X^{(1)}} GW^2(k(x), \omega_x) \longrightarrow 0$$

Lemmas 6.6 and 6.7 show that this is isomorphic, via the hyperbolic homomorphism H, to

$$K_1(k(X)) \longrightarrow \bigoplus_{x \in X^{(1)}} K_0(k(x)) \longrightarrow 0$$

whose homology at degree 0 is just $\mathcal{O}_X(X)^\times$. Now the Pfaffian homomorphism $GW_1^3(X) \to \mathcal{O}_X(X)^\times$ is clearly split, and we see that the kernel of the edge homomorphism $GW_1^3(X) \to E(3)_\infty^{0,2}$ is precisely $W_E(X)$.

We now show that $E(3)_\infty^{2,0} \simeq H^2(X, K_3)$. By definition, $E(3)_\infty^{2,0}$ is the homology of the complex

$$\bigoplus_{x \in X^{(1)}} GW_2^2(k(x), \omega_x) \to \bigoplus_{x \in X^{(2)}} GW_1^1(k(x), \omega_x) \to \bigoplus_{x \in X^{(3)}} GW(k(x), \omega_x).$$

We use the forgetful functor to compare this sequence with the corresponding sequence

$$\bigoplus_{x \in X^{(1)}} K_2(k(x)) \to \bigoplus_{x \in X^{(2)}} K_1(k(x)) \to \bigoplus_{x \in X^{(3)}} K_0(k(x))$$

in K-theory. Now the choice of a generator of ω_x induces isomorphisms $GW_m^n(k(x)) \to GW_m^n(k(x), \omega_x)$ such that the following diagram commutes for any $m, n \in \mathbb{N}$

$$\begin{tikzcd}
GW_m^n(k(x)) \arrow[r, "f"] \arrow[d] & K_m(k(x)) \arrow[d, equal] \\
GW_m^n(k(x), \omega_x) \arrow[r, "f"'] & K_m(k(x)).
\end{tikzcd}$$

Lemma 6.9 now yields exact sequences

$$0 \longrightarrow I^{n+1}(k(x)) \longrightarrow GW_n^n(k(x)) \xrightarrow{f} K_n(k(x)) \longrightarrow 0$$

for any $0 \leq n \leq 2$ and any $x \in X^{(n)}$. If $x \in X^{(n)}$, then $cd(k(x)) \leq 3 - n$ by Corollary 5.11. Hence $H^{4-n}(k(x), \mu_2) = 0$ and the latter is isomorphic to $I^{4-n}(k(x))/I^{5-n}(k(x))$ by [10, Theorem 4.1] and [16, Theorem 7.4]. The Arason-Pfister Hauptsatz [1] then shows that $I^{4-n}(k(x)) = 0$. The forgetful homomorphism therefore induces an isomorphism of complexes between

$$\bigoplus_{x \in X^{(1)}} GW_2^2(k(x), \omega_x) \to \bigoplus_{x \in X^{(2)}} GW_1^1(k(x), \omega_x) \to \bigoplus_{x \in X^{(3)}} GW(k(x), \omega_x)$$

and

$$\bigoplus_{x\in X^{(1)}} K_2(k(x)) \to \bigoplus_{x\in X^{(2)}} K_1(k(x)) \to \bigoplus_{x\in X^{(3)}} K_0(k(x)).$$

To conclude, we prove that $E(3)_\infty^{3,-1}=0$. It suffices to show that the cokernel of the homomorphism

$$\bigoplus_{x_2\in X^{(2)}} GW_2^1(k(x_2),\omega_{x_2}) \longrightarrow \bigoplus_{x_3\in X^{(3)}} GW_1^0(k(x_3),\omega_{x_3})$$

is trivial. Lemma 6.11 yields a commutative diagram

$$\begin{tikzcd}
\bigoplus_{x_2\in X^{(2)}} GW_2^1(k(x_2),\omega_{x_2}) \arrow[r] \arrow[d, "f"'] & \bigoplus_{x_3\in X^{(3)}} GW_1^0(k(x_3),\omega_{x_3}) \arrow[d, "f"] \\
\bigoplus_{x_2\in X^{(2)}} {}_2K_2(k(x_2)) \arrow[r] & \bigoplus_{x_3\in X^{(3)}} {}_2K_1(k(x_3))
\end{tikzcd}$$

in which the left vertical map is surjective and the right vertical map is an isomorphism. Hence both sequences have the same cokernel.

For any field F and any integer $n\in\mathbb{N}$, define a homomorphism $g_n: K_n(F)/2\to {}_2K_{n+1}(F)$ by $\alpha\mapsto\{-1\}\cdot\alpha$. It is clear that g_0 is an isomorphism, and g_1 is surjective by [13].

Using the definition of the residue homomorphisms, it is straightforward to check that the diagram

$$\begin{tikzcd}
\bigoplus_{x_2\in X^{(2)}} K_1(k(x_2))/2 \arrow[r] \arrow[d, "\sum g_1"'] & \bigoplus_{x_3\in X^{(3)}} K_0(k(x_3))/2 \arrow[d, "\sum g_0"] \\
\bigoplus_{x_2\in X^{(2)}} {}_2K_2(k(x_2)) \arrow[r] & \bigoplus_{x_3\in X^{(3)}} {}_2K_1(k(x_3))
\end{tikzcd}$$

commutes and therefore the cokernels of the rows are isomorphic. The cokernel of the top homomorphism is $CH^3(X)/2$ which is trivial by [6, Lemma 1.2]. The result follows. □

7. Lecture 7: Proof of the main theorem

Theorem 7.1. *Let R be a d-dimensional normal affine algebra over an algebraically closed field k such that* $\gcd((d-1)!,\mathrm{char}(k))=1$. *If $d=3$, suppose moreover that R is smooth. Then every stably free R-module P of rank $d-1$ is free.*

Proof. Let P be a stably free module of rank $d-1$. Since the result is clear when $d\le 2$, we assume that $d\ge 3$. Using Suslin's cancellation theorem 2.5, we can suppose that there is an isomorphism $P\oplus R\simeq R^d$, and therefore that P is given by a unimodular row $(a_1,\dots,a_d)$. In view of Corollary 2.4, to prove that P is free it suffices to show that there exists a unimodular row $(b_1,\dots,b_d)$ such that $(a_1,\dots,a_d)=(b_1^{(d-1)!},\dots,b_d)$ in $Um_d(R)/E_d(R)$.

Suppose that $d \geq 4$. Let J be the ideal of the singular locus of R. Since R is normal, J has height at least 2 and $\dim(R/J) \leq d-2$. It follows from [4, Theorem 3.5] that $Um_d(R/J) = e_1E_d(R/J)$ and we can therefore assume, performing elementary operations if necessary, that $a_d \equiv 1 \pmod J$ and $a_1, \ldots, a_{d-1} \in J$. Using now Swan's Bertini theorem 0.2, we can perform elementary operations on $(a_1, \ldots, a_d)$ such that $B := R/(a_4, \ldots, a_d)$ is either empty, either a non-singular threefold outside the singular locus of R. In the first case, the row $(a_4, \ldots, a_d)$ is unimodular, and therefore the row $(a_1, \ldots, a_d)$ is completable in an elementary matrix. Thus we can restrict to the second case. In this situation, we see that B is actually smooth since $a_d \equiv 1 \pmod J$.

Given a unimodular row $(\overline{a}, \overline{b}, \overline{c})$ on B, we can choose lifts $a, b, c \in R$ and consider the unimodular row $(a, b, c, a_4, \ldots, a_d)$ on R. It is straightforward to check that this gives a well-defined map

$$Um_3(B)/E_3(B) \to Um_d(R)/E_d(R),$$

showing that $(a_1, \ldots, a_d)$ comes from the unimodular row $(\overline{a}_1, \overline{a}_2, \overline{a}_3)$ on B. We are thus reduced to the case where R is the affine algebra of a smooth threefold. By Theorem 4.4, the set $Um_3(R)/E_3(R)$ is in bijection with $W_E(R)$ and is thus endowed with the structure of an abelian group. Since -1 is a square in k, Lemma 4.11 shows that $n \cdot (a_1, a_2, a_3) = (a_1^n, a_2, a_3)$ in $Um_3(R)/E_3(R)$ for any $n \in \mathbb{N}$. Now Theorems 6.12 and 5.25 show that $Um_3(R)/E_3(R)$ is a divisible group prime to the characteristic of k. Since $\gcd((d-1)!, \mathrm{char}(k)) = 1$, there exists a unimodular row $(b_1, b_2, b_3) \in Um_3(R)$ such that

$$(a_1, a_2, a_3) = (d-1)! \cdot (b_1, b_2, b_3) = (b_1^{(d-1)!}, b_2, b_3)$$

in $Um_3(R)/E_3(R)$. The result follows. □

References

[1] Jón Kristinn Arason and Albrecht Pfister. Beweis des Krullschen Durchschnittsatzes für den Wittring. *Invent. Math.*, 12:173–176, 1971.

[2] Paul Balmer and Charles Walter. A Gersten-Witt spectral sequence for regular schemes. *Ann. Sci. École Norm. Sup. (4)*, 35(1):127–152, 2002.

[3] Jean Barge and Jean Lannes. *Suites de Sturm, indice de Maslov et périodicité de Bott*, volume 267 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2008.

[4] H. Bass. *Algebraic K-theory*. W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[5] J.-L. Colliot-Thélène, R. Hoobler, and B. Kahn. The Bloch-Ogus-Gabber theorem. In R. Jardine and V. Snaith, editors, *Proceedings of the Great Lakes K-theory Conference (Toronto 1996)*, volume 16 of *The Fields Institute for Research in Mathematical Sciences Communications Series*, pages 31–94, Providence, RI, 1997. Amer. Math. Soc.

[6] J.-L. Colliot-Thélène and C. Scheiderer. Zero-cycles and cohomology on real algebraic varieties. *Topology*, 35(2):533–559, 1996.

[7] Jean Fasel. Stably free modules over smooth affine threefolds. *Duke Math. J.*, 156(1):33–49, 2011.

[8] James S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.

[9] John Milnor. Algebraic K-theory and quadratic forms. *Invent. Math.*, 9:318–344, 1969/1970.

[10] Dmitry Orlov, Alexander Vishik, and Vladimir Voevodsky. An exact sequence for $K_*^M/2$ with applications to quadratic forms. *Ann. of Math. (2)*, 165(1):1–13, 2007.

[11] Ravi A. Rao and Wilberd van der Kallen. Improved stability for SK_1 and WMS_d of a non-singular affine algebra. *Astérisque*, 226:11, 411–420, 1994. K-theory (Strasbourg, 1992).

[12] Marco Schlichting. Hermitian K-theory, derived equivalences and Karoubi's fundamental theorem. arXiv:1209.0848, 2012.

[13] A. A. Suslin. Torsion in K_2 of fields. *K-Theory*, 1(1):5–29, 1987.

[14] A.A. Suslin. On stably free modules. *Math. U.S.S.R. Sbornik*, pages 479–491, 1977.
[15] L. N. Vaseršteĭn and A. A. Suslin. Serre's problem on projective modules over polynomial rings, and algebraic K-theory. *Izv. Akad. Nauk SSSR Ser. Mat.*, 40(5):993–1054, 1199, 1976.
[16] Vladimir Voevodsky. Motivic cohomology with $\mathbf{Z}/2$-coefficients. *Publ. Math. Inst. Hautes Études Sci.*, 98:59–104, 2003.
[17] Ton Vorst. The general linear group of polynomial rings over regular rings. *Comm. Algebra*, 9(5):499–509, 1981.
[18] Charles Weibel. *An introduction to homological algebra.* Cambridge University Press, 1994.

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