ICTP workshop, Higher Grothendieck-Witt groups

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December 12, 2012

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1 Disclaimer

These are preliminary notes of the talks given by Marco at the ICTP workshop on classical non-stable K-theory. I take full responsibility for all errors and typos.

2 Preliminaries

Throughout this talk we assume for simplicity that R is a commutative ring and $\epsilon \in \{\pm 1\}$.

Definition 2.1. An inner product space $(P, \langle -,- \rangle)$ (IPS) is a pair of a finitely generated projective R-module P and a non degenerate ϵ-symmetric bilinear form $\langle -,- \rangle$ on P. Here, a bilinear form is non-degenerate if the adjoint map $P \xrightarrow{\cong} P^* = Hom_R(P,R) : x \mapsto \langle x, - \rangle$ is an isomorphism, and the form is ϵ-symmetric if $\langle x, y \rangle = \epsilon \langle y, x \rangle$. Further let ${}_\epsilon IPS(R)$ be the set of isometry classes inner product spaces.
For two IPS V and W we denote by $V \perp W$ the orthogonal sum. Then the triple $({}_\epsilon IPS(R), \perp, 0)$ is an abelian monoid.

Remark 2.2. *If $\epsilon = 1$ we usually omit the index corresponding to ϵ in the notation.*

Recall 2.3. *Let $(M, +, 0)$ be an abelian monoid. Then*

$$K_0(M, +, 0) = \{(a,b) | a, b \in M\} / \sim$$

where $(a,b) \sim (a',b')$ if $\exists c \in M$, s.t $a + b' + c = a' + b + c$. K_0 is the left adjoint of the forgetful functor $abeliangroups \to abelianmonoids$. We often write $a - b$ or $[a] - [b]$ instead of (a,b) for an element of $K_0(M, +, 0)$.

Definition 2.4. The Grothendieck-Witt group of R is defined as

$$_\epsilon GW_0(R) = K_0({}_\epsilon IPS(R), \perp, 0) \tag{1}$$

Example 2.5.
- *$rk : GW_0(F) \xrightarrow{\cong} \mathbb{Z}$ for an algebraically closed field F*
- *$GW_0(\mathbb{Z}) \xrightarrow{\cong} GW_0(\mathbb{R}) \xrightarrow{\cong} \mathbb{Z} \oplus \mathbb{Z}$, where the second isomorphism is given by $P \mapsto (i_+(P), i_-(P))$. Here, $i_\pm(P) := \#\{i | \pm \langle b_i, b_i \rangle > 0\}$ for an orthogonal basis $\{b_1, \dots, b_n\}$ of P.*
- *$(rk, det) : GW_0(F) \xrightarrow{\cong} \mathbb{Z} \oplus F^*/F^{2*}$ for a finite field F.*
- *$GW_0(\mathbb{Q}) \xrightarrow{\cong} GW(\mathbb{R}) \oplus \bigoplus_{\substack{p \in \mathbb{Z} \\ prime}} W_0(\mathbb{F}_p)$, where W_0 is the Witt group of R as defined below.*

Definition 2.6. Let $\mathscr{P}(R)$ be the category of f.g. projective modules. Moreover let $i\mathscr{P}(R)$ denote the set of isomorphism classes. Define the Grothendieck group of R as

$$K_0(R) := K_0(i\mathscr{P}(R), \oplus, 0) \tag{2}$$

Definition 2.7. The hyperbolic map

$$K_0(R) \xrightarrow{{}_\epsilon H} {}_\epsilon GW_0(R) \tag{3}$$

is defined by

$$P \mapsto {}_\epsilon HP \tag{4}$$

where ${}_\epsilon HP$ is the inner product space $P \oplus P^*$ with ϵ-symmetric bilinear form $\langle x, f | y, g \rangle = g(x) + \epsilon f(y)$.
Moreover define the (ϵ-symmetric) Witt group of R as the cokernel

$$W_0(R) := coker({}_\epsilon H : K_0(R) \to {}_\epsilon GW_0(R)) \tag{5}$$

3 The aim of the talks

We want to define groups $GW_i(R)$ for all $i \in \mathbb{Z}$ and rings R and more generally groups $GW_i(X)$ for all $i \in \mathbb{Z}$ and schemes X. Moreover we show that these define cohomology theories.
For now we define $GW_i(R)$, $i \geq 1$.

4 Definition of the groups $GW_i(R)$ for $i \geq 1$

4.1 The classifying space of a group

Reference here is [Hus94, Definition 10.5 and Summary 12.5]. Let G be a topological group (which has the homotopy type of a CW complex and for which $1 \in G$ is retract of some neighbourhood), then the classifying space BG is a topological space (of the homotopy type of a CW complex) defined by the property that for all CW-complexes X we have

$$[X, BG] \cong \text{isomorphism classes of principal } G\text{-bundles} \tag{6}$$

Example 4.1. *Let G be a discrete group, then BG is a pointed CW space determined by the property*

$$\pi_i BG = \begin{cases} * & i \neq 1 \\ G & i = 1 \end{cases} \tag{7}$$

4.2 Plus construction

References here are [Lod76], [Ber82].

Proposition 4.2. *Let X be a connected CW-complex such that $\pi_1 X$ is quasi-perfect, i.e. the commutator subgroup $G = [\pi_1 X, \pi_1 X]$ is perfect, i.e. $[G, G] = G$. Then there exists a continuous map of CW spaces $X \to X^+$, unique up to homotopy, such that*

$$H_*(X, A) \xrightarrow{\cong} H_*(X^+, A) \tag{8}$$

for all local coefficient systems A on X^+ and

$$\pi_1 X \to \pi_1 X^+ = \pi_1 X / [\pi_1 X, \pi_1 X] = (\pi_1 X)^{ab} \tag{9}$$

X^+ obtained form X by attaching 2- and 3-cells.

4.3 The infinite orthogonal group

Definition 4.3. [Kar73, p. 8] Let $V \in IPS$ then $O(V) :=$ group of isometries of V.
Further we define

$$_\epsilon O_\infty(R) = \bigcup_{n \geq 0} O(_\epsilon H(R^n)), \tag{10}$$

where $O(_\epsilon H^n) \subset O(_\epsilon H^{n+1})$ via $g \mapsto \begin{pmatrix} g & \\ & 1_H \end{pmatrix}$

Lemma 4.4. *$O_\infty(R)$ is quasi-perfect.*

Proof. Set $E := [O_\infty, O_\infty]$, the commutator subgroup of O_∞. Recall that this subgroup is normal in O_∞. We need to show $[E, E] \supset E$. Thus let $g, h \in O_\infty \Rightarrow g, h \in O(H^n)$ for some n.
Clearly we have

$$\begin{pmatrix} ghg^{-1}h^{-1} & & \\ & 1 & \\ & & 1 \end{pmatrix} = \left[\begin{pmatrix} g & & \\ & g^{-1} & \\ & & 1 \end{pmatrix}, \begin{pmatrix} h & & \\ & 1 & \\ & & h^{-1} \end{pmatrix} \right] \tag{11}$$

Hence it suffices to show that $\begin{pmatrix} g & & \\ & g^{-1} & \\ & & 1 \end{pmatrix} \in E$.

The permutation matrices are clearly isometries: $\Sigma_3 \subset O((H^n)^3)$ and an easy computation shows:

$$(123) = [(23), (12))] \in E \tag{12}$$

Finally observe

$$(1 \oplus g^{-1} \oplus 1)(123)(g \oplus 1 \oplus 1) = (123) \tag{13}$$

Therefore

$$\begin{pmatrix} g & & \\ & g^{-1} & \\ & & 1 \end{pmatrix} = 1 \bmod E \tag{14}$$

□

Remark 4.5. *Similarly $Gl(R) := \bigcup Gl_n(R)$ is quasi-perfect.*

Definition 4.6. We define

- ${}_\epsilon GW_i(R) = \pi_i B_\epsilon O_\infty(R)^+$ for $i \geq 1$ and $2 \in R^*$ (Karoubi [Kar73])
- $K_i(R) = \pi_i BGl(R)^+$ for $i \geq 1$ (and R arbitrary) (Quillen '70 [Gra76]).

Corollary 4.7. *By construction we have*

- $K_1(R) = Gl(R)^{ab}$
- ${}_\epsilon GW_1(R) = {}_\epsilon O_\infty(R)^{ab}$

5 Karoubi's fundamental theorem and Bott periodicity

We will define spaces ${}_\epsilon GW(R)$, $K(R)$ such that

$${}_\epsilon GW_i(R) = \pi_{i\epsilon} GW(R) \tag{15}$$

and

$$K_i(R) = \pi_i K(R) \tag{16}$$

Warning 5.1. *As functors in R*

$$K(R) \ncong K_0(R) \times BGL(R)^+ \tag{17}$$

and

$$GW(R) \ncong GW_0(R) \times BO_\infty(R)^+ \tag{18}$$

For an explanation, see [Sch11, 2.2.9].

Proposition 5.2. *There exists a map*

$${}_\epsilon H : K(R) \to {}_\epsilon GW(R) \tag{19}$$

such that the induced map

$$K_0(R) \to {}_\epsilon GW_0(R) \tag{20}$$

is given by $P \mapsto {}_\epsilon HP$ and

$$BGL^+ \to BO_\infty^+ \tag{21}$$

is given level-wise by the maps

$$Aut(R^n) = Gl_n \to O(H^n) \tag{22}$$

which sends $g \mapsto \begin{pmatrix} g & \\ & (g^*)^{-1} \end{pmatrix}$.
Moreover there is the forgetful map

$$F : {}_\epsilon GW(R) \to K(R) \tag{23}$$

defined by $(P, \langle , \rangle) \mapsto P$.

Definition 5.3. Let $f : X \to Y$ be a map of topological spaces and $y \in Y$. Then we define the fibre of f as

$$Fibre_y(f) = \{(\sigma, x) | \sigma : I \to Y,\ x \in X,\ \sigma(0) = y, \sigma(1) = f(x)\} \tag{24}$$

Proposition 5.4 ([Whi78]). *There exists a long exact sequence in homotopy*

$$\cdots \to \pi_n(Fibre_y(f)) \to \pi_n(X) \to \pi_n(Y) \to \pi_{n-1}(Fibre_y(f)) \to \cdots \tag{25}$$

Definition 5.5. [Kar73] Set

- ${}_\epsilon U(R) = Fibre({}_\epsilon H : K(R) \to {}_\epsilon GW(R))$
- ${}_\epsilon V(R) = Fibre(F : {}_\epsilon GW(R) \to K(R))$

Remark 5.6. *To remember which one is which it is convenient to know that "V=vergessen" is German for "to forget" though this is probably not the reason for why it is called V-theory...*

Theorem 5.7 (Fundamental Theorem, Karoubi '73, '80 [Kar73], [Kar80])**.** *If* $2 \in R^*$ *then*

$$_{-\epsilon}V(R) \sim \Omega_\epsilon U(R) \tag{26}$$

Theorem 5.8 (Bott Periodicity)**.** $\mathbb{Z} \times BO^{top} \sim \Omega^8(\mathbb{Z} \times BO^{top})$

The topological version of the Fundamental Theorem implies Bott periodicity:

Definition 5.9. Let A be a Banach algebra with involution. Then define

- ${}_\epsilon GW_0^{top}(A) = {}_\epsilon GW_0^{top}({}_\epsilon IPS(A))$
- ${}_\epsilon GW_i^{top}(A) = \pi_i BO_\infty^{top}(A)$

If we apply the topological version of the Fundamental Theorem to $A = \mathbb{R}, \mathbb{C}, \mathbb{H}$, where he first two come with the trivial involution, and $\mathbb{H}$ with the usual involution $i, j, k \mapsto -i, -j, -k$ we obtain the list where V and U denote V^{top} and U^{top}

$$\begin{array}{ccc}
V(\mathbb{R}) \sim \mathbb{Z} \times BO & V(\mathbb{C}) \sim U/O & V(\mathbb{H}) \sim \mathbb{Z} \times BSp \\
U(\mathbb{R}) \sim O & U(\mathbb{C}) \sim O/U & U(\mathbb{H}) \sim Sp \\
{}_-V(\mathbb{R}) \sim O/U & {}_-V(\mathbb{C}) \sim U/Sp & {}_-V(\mathbb{H}) \sim \mathbb{Z} \times Sp/U \\
{}_-U(\mathbb{R}) \sim U/O & {}_-U(\mathbb{C}) \sim Sp/U & {}_-U(\mathbb{H}) \sim U/Sp
\end{array}$$

Hence, the homotopy equivalence $_{-\epsilon}V(R) \sim \Omega_\epsilon U(R)$ together with the canonical homotopy equivalences $O \sim \Omega(BO)$ and $Sp \sim \Omega(BSp)$ imply

$$\mathbb{Z} \times BO \sim \Omega(U/O) \sim \Omega^2(Sp/U) \sim \Omega^3(Sp) \sim \Omega^4(\mathbb{Z} \times Sp) \sim \Omega^5(U/Sp)$$

$$\sim \Omega^6(O/U) \sim \Omega^7(O) \sim \Omega^8(\mathbb{Z} \times BO)$$

6 Addendum: On the order of $K_3(\mathbb{Z})$

One of the first applications of hermitian K-theory was to disprove a conjecture of Lichtenbaum predicting $K_3(\mathbb{Z})$ to be $\mathbb{Z}/24$. To explain the context, consider the string of maps

$$\mathbb{Z} \cong \pi_3 O \xrightarrow{J} \pi_3 \Omega^\infty S^\infty = \pi_3^s S^0 \to K_3(\mathbb{Z})$$

in which the isomorphism $\pi_3 O = \pi_4 BO = \mathbb{Z}$ is by Bott periodicity, the map J is Adams' J-homomorphism [Ada66], and the last map is the unit map $S^0 \to K(\mathbb{Z})$ of the ring spectrum $K(\mathbb{Z})$. Adams showed in [Ada66, Theorem 1.5] that the image of $\mathbb{Z}$ in $\pi_3^s S^0$ is $\mathbb{Z}/24$, and Quillen showed in [Qui76, p. 183] that the map $\pi_3^s S^0 \to K_3(\mathbb{Z})$ is injective on the image of J, that is, we have an injection $\mathbb{Z}/24 \subset K_3(\mathbb{Z})$. Lichtenbaum predicted that this inclusion is in fact an isomorphism (compare [Lic73, 2.6]

Proposition 6.1 (Karoubi '74 [Kar74]). *The order of $K_3(\mathbb{Z})$ is divisible by* 48.

Proof. For a finite abelian group A, write $A_{(2)}$ for the 2-primary torsion subgroup. Also, write $\mathbb{Z}'$ for $\mathbb{Z}[1/2]$. Quillen showed that the map $K_3(\mathbb{Z}') \to K_3(\mathbb{Z})$ is an isomorphism on 2-primary subgroups (since kernel and cokernel are quotient and subgroup of $K_3(\mathbb{F}_2)$ and $K_2(\mathbb{F}_2)$ both of which are finite groups without 2-primary torsion). The map $\pi_3^s S^0 \to K_3(\mathbb{Z})$ factors through $GW_3(\mathbb{Z}') \xrightarrow{F} K_3(\mathbb{Z}') \to K_3(\mathbb{Z})$ (simply because the maps $GW(\mathbb{Z}') \xrightarrow{F} K(\mathbb{Z}') \to K(\mathbb{Z})$ are maps or ring spectra).
By Quillen's result that $\pi_3^s S^0 \to K_3(\mathbb{Z})$ is injective on the image of J, the same has to be true for the map $\pi_3^s S^0 \to GW_3(\mathbb{Z}')$. In particular, $\mathbb{Z}/8 \subset GW_3(\mathbb{Z}')_{(2)}$. Now, the map $H : K_3(\mathbb{Z}') \to GW_3(\mathbb{Z}')$ is surjective (we will learn later how to prove this, see Lemma **??** below). In particular, $|K_3(\mathbb{Z})_{(2)}| = |K_3(\mathbb{Z}')_{(2)}| \geq |GW_3(\mathbb{Z}')_{(2)}$. Therefore, if $\mathbb{Z}/24 \cong K_3(\mathbb{Z})$, then $\mathbb{Z}/8 \cong K_3(\mathbb{Z})_{(2)}$, and $8 \leq GW_3(\mathbb{Z}')_{(2)} \leq |K_3(\mathbb{Z}')_{(2)}| = 8$. Hence $H : K_3(\mathbb{Z}')_{(2)} \to GW_3(\mathbb{Z}')_{(2)}$ and $F : GW_3(\mathbb{Z}')_{(2)} \to K_3(\mathbb{Z}')_{(2)}$ have to be isomorphisms. The composition

$$K_3(\mathbb{Z}')_{(2)} \xrightarrow{H} GW_3(\mathbb{Z}')_{(2)} \xrightarrow{F} K_3(\mathbb{Z}')_{(2)}$$

is of the form $1 + *$ with $* : P \mapsto P^*$ a map inducing an isomorphism on K-groups. But a map of the form $1 + *$ can never be an isomorphism between finite 2-primary torsion groups. □

Later Lee and Szczarba proved in [LS76] that $K_3(\mathbb{Z}) \cong \mathbb{Z}/48$. Nowadays, we know all groups $K_{2n+1}(\mathbb{Z})$, we know the orders of the groups $K_{4n+2}(\mathbb{Z})$ which are predicted to be cyclic, and the groups $K_{4n}(\mathbb{Z})$ are conjectured to be 0 except for $n = 0$ (where $K_0(\mathbb{Z}) = \mathbb{Z}$) and $n = 1$ (where it is known that $K_4(\mathbb{Z}) = 0$ due to Soulé and Rognes). The last conjecture is equivalent to Vandiver's conjecture and implies the previous conjecture on the structure of the groups $K_{4n+2}(\mathbb{Z})$. For a survey about these statements, see [Wei05].

7 Grothendieck-Witt groups of exact categories

7.1 Motivation

Even if we are only interested in the K-groups of rings (and possibly schemes) it is necessary to work in a more general framework: at least in the framework of exact categories. Here are two reasons why.

- Consider the localization map $R \to S^{-1}R$, where $S \subset R$ is a multiplicative set of non-zero-divisors. Then there exist induced maps $K_n(R) \to K_n(S^{-1}R)$. These maps fit into a long exact sequence [Gra76, Theorem, p. 229]

$$\cdots \to K_n(\mathscr{E}) \to K_n(R) \to K_n(S^{-1}R) \to K_{n-1}(\mathscr{E}) \to \cdots \tag{27}$$

  where the additional terms are not defined as the K-theory of a ring but rather of some exact category $\mathscr{E}$.

- Let X be a scheme and let $Vect(X)$ denote the category of vector bundles over X. Then $Vect(X)$ is an exact category and the K-theory $K(X)$ is defined using this structure. This will be the example to keep in mind in what follows.

8 Exact categories

Definition 8.1 (Quillen [Qui73]). An exact category is an additive category $\mathscr{E}$ together with a choice of a class of sequences

$$A \rightarrowtail B \twoheadrightarrow C \tag{28}$$

called admissible exact sequences or conflations. Maps that occur as the first map in a conflation are called admissible monomorphisms or inflations and are depicted as $\rightarrowtail$, maps that occur as the second map in a conflation are called admissible epimorphisms or deflations and are depicted as $\twoheadrightarrow$. The class of conflations is subject to some axioms. We omit listing these axioms in favour of the characterisation below.

Lemma 8.2 (Appendix A in [TT90], Appendix in [Kel90]). *A small additive category $\mathscr{E}$ together with a set of sequences $A \rightarrowtail B \twoheadrightarrow C$ is an exact category if and only if there exists a full and faithful embedding $\mathscr{E} \subset \mathscr{A}$ into an abelian category $\mathscr{A}$ such that a sequence in $\mathscr{E}$ is admissible exact if and only if its image in $\mathscr{A}$ is exact (in the sense of abelian categories) and $\mathscr{E}$ is closed under extensions in $\mathscr{A}$.*

Definition 8.3. Let $\mathscr{E}$ be an exact category. Then $K_0(\mathscr{E})$ is the abelian group generated by symbols $[E]$, one for each $E \in Ob\mathscr{E}$subject to the relations

- $[E] = [F]$ if $E \cong F$
- $[B] = [A] + [C]$ for each admissible exact sequence $A \rightarrowtail B \twoheadrightarrow C$.

Remark 8.4. *In fact the first relation is a special case of the second one.*

Definition 8.5. An exact category with duality is a triple $(\mathscr{E}, *, can)$ where $\mathscr{E}$ is an exact category, $* : \mathscr{E}^{op} \to \mathscr{E}$ an exact functor and $can : 1 \xrightarrow{\cong} **$ a natural isomorphism such that for all $A \in Ob\mathscr{E}$ we have $can_A^* \circ can_{A^*} = 1_{A^*}$

$$\begin{tikzcd} & A^* \arrow[dl] \arrow[dr, "can_{A^*}"] & \\ A^* & & A^{***} \arrow[ll, "can_A^*"] \end{tikzcd} \tag{29}$$

Example 8.6. *The triple $(Vect(X), Hom_{O_X}(-, L), can)$ is an exact category with duality. Here L is a line bundle and $can_V : V \to V^{**}$ maps $x \mapsto (f \mapsto f(x))$*

Definition 8.7. Let $(\mathscr{E}, *, can)$ be an exact cat with duality. An inner product space (IPS) in $\mathscr{E}$ is a pair (V, ϕ), $V \in Ob\mathscr{E}$ and $\phi : V \xrightarrow{\cong} V^*$ such that $\phi^* \circ can_V = \phi$.

Definition 8.8 (Knebusch, I§5 in [Kne77]). Let $(\mathscr{E}, *, can)$ be an exact category with duality. Its Grothendieck-Witt group is the abelian group $GW_0(\mathscr{E}, *, can)$ generated by symbols $[V, \phi]$, one for each IPS (V, ϕ), subject to the relations

- $[V, \phi] = [W, \psi]$ if $(V, \phi) \cong (W, \psi)$ are isometric.
- $[(V, \phi)\perp(W, \psi)] = [V, \phi] + [W, \psi]$
- $[V, \phi] = [U \oplus W, \begin{pmatrix} 0 & \phi_W \\ \phi_U & 0 \end{pmatrix}]$ for any IPS in the the exact category (with duality) of admissible exact sequences in $\mathscr{E}$. Here a non-degenerate symmetric bilinear form in the category of exact sequences is a triple of isomorphisms $\phi = (\phi_A, \phi_B, \phi_C)$ making the diagram commute

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, "\cong"', "\phi_A"] & B \arrow[r, two heads] \arrow[d, "\cong"', "\phi_B"] & C \arrow[d, "\cong"', "\phi_C"] \\ C^* \arrow[r, tail] & B^* \arrow[r, two heads] & A^* \end{tikzcd} \tag{30}$$

  such that $\phi^* \circ can = \phi$, that is, $\phi_A^* \circ can_C = \phi_C$, $\phi_B^* \circ can_B = \phi_B$ and $\phi_C^* \circ can_A = \phi_A$.

Definition 8.9. As usual we define the Witt group as the cokernel of the hyperbolic map

$$W_0(X) = coker(H : K_0(X) \to GW_0(X)) \tag{31}$$

Lemma 8.10.

$$GW_0(\mathscr{P}(R), Hom(-, R), \epsilon \cdot can) \cong {}_\epsilon GW_0(R) \tag{32}$$

Recall 8.11. *The construction of $K_i(\mathscr{E})$ for an exact category $\mathscr{E}$ is done in several steps:*

- *To the exact category $\mathscr{E}$ we associate another category $Q\mathscr{E}$,*
- *to the category $Q\mathscr{E}$ we associate a topological space $BQ\mathscr{E}$, the classifying space of $Q\mathscr{E}$,*
- *the K-theory space $K(\mathscr{E})$ is defined as the loop space $K(\mathscr{E}) = \Omega BQ\mathscr{E}$, and*
- *the K-groups are defined as the homotopy groups $K_i(\mathscr{E}) = \pi_i K(\mathscr{E})$*

8.1 From categories to topological spaces

To any small category $\mathscr{C}$ we associate a topological space, called the classifying space $B\mathscr{C}$ of $\mathscr{C}$. It has the following properties

- $B\mathscr{C}$ is a CW complex
- The 0-cells are the objects of $\mathscr{C}$.
- The 1-cells are the non-identity arrows $A_0 \xrightarrow{f} A_1$ in $\mathscr{C}$ glued in by attaching source and target of f to the corresponding 0-cells.
- The 2-cells are the pairs of composable arrows $A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2$ in $\mathscr{C}$ with $f_1, f_2 \neq id$. For each 2-cell $A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2$ we glue in a triangle with edges f_0, f_1 and $f_1 f_0$ attached to the corresponding arrows in the 1-skeleton.
- For arbitrary n the n-cells are given by the sequences $A_0 \xrightarrow{f_0} \cdots \xrightarrow{f_{n-1}} A_n$ of n-composable arrows in $\mathscr{C}$ such that none of the f_i's is an identity map. They are glued in appropriately as above.

For the precise definition, see Definition 12.3 and [Qui73, §1].

Example 8.12. *Let G be a (discrete) group. If we understand G as the category with one object and G as the set of morphisms, then BG is the usual classifying space of G.*

8.2 From exact categories $\mathscr{E}$ to $Q\mathscr{E}$

Definition 8.13 (§2 in [Qui73])**.** Given an exact category $\mathscr{E}$ we define a category $Q\mathscr{E}$ with the same objects as $\mathscr{E}$ where a morphism $A \to B$ is an equivalence class of triples (U, p, i)

$$A \overset{p}{\twoheadleftarrow} U \overset{i}{\rightarrowtail} B \tag{33}$$

where $(U, p, i) \sim (U', p', i')$ if there exists an isomorphism $g : U \xrightarrow{\cong} U'$ such that $p = p'g$ and $i = i'g$. The composition of $A \xrightarrow{(U,p,i)} B \xrightarrow{(W,q,j)} C$ is defined as $(U \times_B W, pq', ji')$, where $U \times_B W$, q' and i' are defined as a pull-back:

$$\begin{tikzcd}
 & & C \\
 & B & W \arrow[u, tail, "j"] \arrow[l, two heads, "q"] \\
A & U \arrow[l, two heads, "p"] \arrow[u, tail, "i"] & U \times_B W \arrow[l, two heads, "p'"] \arrow[u, tail, "i'"]
\end{tikzcd} \tag{34}$$

8.3 The hermitian Q-construction

Definition 8.14 (Karoubi, Giffen, Uridia [Uri90], Charney-Lee [CL86])**.** Let $(\mathscr{E}, *, can)$ be an exact category with duality. Define $Q^h\mathscr{E}$ to be the category with inner product spaces (A, ϕ) (in $\mathscr{E}$) as objects. A morphisms $(A, \phi) \to (B\psi)$ is given by an equivalence class of triples (U, p, i)

$$A \overset{p}{\twoheadleftarrow} U \overset{i}{\rightarrowtail} B \tag{35}$$

as in Quillen's Q-construction such that $\phi|_U = \psi|_U$, i.e. $p^*\phi p = i^*\psi i$ and $Ker\ i^* \circ \psi =: U^\perp = Ker\ p$. The composition is defined as in Quillen's Q-construction $Q\mathscr{E}$.

Lemma 8.15 ([Uri90], Proposition 4.8 in [Sch10a])**.** *There exists an isomorphism*

$$\pi_0 BQ^h\mathscr{E} \xrightarrow{\cong} W_0(\mathscr{E}) \tag{36}$$
$$(V, \phi) \mapsto [V, \phi] \tag{37}$$

Definition 8.16. Let $(\mathscr{E}, *, can)$ be an exact category with duality. The Grothendieck-Witt theory space is defined as the fibre

$$GW(\mathscr{E}) = Fibre(BQ^h\mathscr{E} \to BQ\mathscr{E}) \tag{38}$$
$$(V, \phi) \mapsto V \tag{39}$$

Then the Grothendieck-Witt groups are given by

$$GW_i(\mathscr{E}) = \pi_i GW(\mathscr{E}). \tag{40}$$

We write $GW(X, L)$ for the Grothendieck-Witt theory space of the exact category with duality $(Vect(X), Hom(-, L), {}_\epsilon can$

Lemma 8.17 (Proposition 4.11 in [Sch10a])**.**

$$\pi_0 GW(\mathscr{E}) = GW_0(\mathscr{E}) \tag{41}$$

Remark 8.18. *Compare the above result to the classical [Qui73, §2 Theorem 1]*

$$\pi_0 \Omega BQ\mathscr{E} = K_0(\mathscr{E}) \tag{42}$$

for $K(\mathscr{E}) := \Omega BQ\mathscr{E}$.

9 The Grothendieck-Witt group of formations

Definition 9.1. A formation in an exact category with duality $\mathscr{E} = (\mathscr{E}, *, can)$ is a tuple (X, ϕ, L_1, L_2), where (X, ϕ) is an inner product space in $\mathscr{E}$ and $L_i \rightarrowtail X$, $j = 1, 2$ are two Lagrangians. A Lagrangian in X is an object L together with an admissible monomorphism $L \rightarrowtail X$ such that $L = L^\perp = Ker\ i^*\phi$.
Two formations (X, ϕ, L_1, L_2) and (X', ϕ', L_1', L_2') are isomorphic, if there exists an isometry $f : (X, \phi) \to (X', \phi')$ such that $f(L_i) = L_i'$ for $i = 1, 2$.

Definition 9.2. The GW group of formations is the abelian group $GW_{form}(\mathscr{E})$ generated by isomorphism classes $[X, \phi, L_1, L_2]$ of formations, subject to the relations

- $[X, \phi, L_1, L_2] + [X', \phi', L_1', L_2'] = [X \oplus X', \phi \oplus \phi', L_1 \oplus L_1', L_2 \oplus L_2']$
- $[X, \phi, L_1, L_2] + [X, \phi, L_2, L_3] = [X, \phi, L_1, L_3]$
- If (X, ϕ, L_1, L_2) is a formation and $U \rightarrowtail X$ with $U \subset L_1, L_2$ (hence $U \subset U^\perp$) then $[X, \phi, L_1, L_2] = [U^\perp/U, \bar{\phi}, L_1/U, L_2/U]$.

Remark 9.3. *If* $L \subset (X, \phi)$ *is a Lagrangian, then it defines an arrow* $0 \to X$ *in* $Q^h\mathscr{E}$ *via*

$$0 \overset{p}{\twoheadleftarrow} L \overset{i}{\rightarrowtail} X, \phi \tag{43}$$

and therefore a path $[L]$ *from* 0 *to* (X, ϕ) *in* $BQ^h\mathscr{E}$. *If* (X, ϕ, L_1, L_2) *is a formation then* $[L_2]^{-1}[L_1]$ *is a loop in* $BQ^h\mathscr{E}$ *based at* 0.

Lemma 9.4. *[Sch10a, Proposition 4.9]*

$$GW_{from}(\mathscr{E}) \xrightarrow{\cong} \pi_1(BQ^h\mathscr{E}) \tag{44}$$
$$[X, \phi, L_1, L_2] \mapsto [L_2]^{-1}[L_1] \tag{45}$$

10 The proof of Lemma 8.17

The sequence of functors $iIPS(\mathscr{E}) \to Q^h\mathscr{E} \to Q\mathscr{E}$ induces a map on the classifying spaces $BiIPS(\mathscr{E}) \hookrightarrow BQ^h\mathscr{E} \to BQ\mathscr{E}$. Observe that the composition is homotopic to the trivial map and thus we obtain a map into the fibre

$$BiIPS(\mathscr{E}) \to GW(\mathscr{E}) \tag{46}$$

and in particular a map

$$\pi_0 BiIPS(\mathscr{E}) \to \pi_0 GW(\mathscr{E}) \tag{47}$$

where the left hand side is the abelian monoid of isometry classes of inner product spaces in $\mathscr{E}$. This induces the map

$$GW_0(\mathscr{E}) \to \pi_0 GW(\mathscr{E}) \tag{48}$$

Now recall the formulation of Lemma 8.17:

Lemma 10.1. *[Sch10a, Proposition 4.11] The map (48) is an isomorphism.*

Proof. The rows in the following commutative diagram are exact

$$\begin{tikzcd} GW_{form}(\mathscr{E}) \arrow[r] \arrow[d, "\cong"] & K_0(\mathscr{E}) \arrow[r, "H"] \arrow[d, "\cong"] & GW_0(\mathscr{E}) \arrow[r] \arrow[d] & W_0(\mathscr{E}) \arrow[r] \arrow[d, "\cong"] & 0 \arrow[d, "="] \\ \pi_1 BQ^h\mathscr{E} \arrow[r] & \pi_1 BQ\mathscr{E} \arrow[r] & \pi_0 GW\mathscr{E} \arrow[r] & \pi_0 BQ^h\mathscr{E} \arrow[r] & \pi_0 BQ \end{tikzcd} \tag{49}$$

where the upper left horizontal arrow is $[X, \varphi, L_1, L_2] \mapsto [L_1] - [L_2]$. We already know that all but the middle vertical arrow are isomorphisms. by the five-lemma we are done. □

Proposition 10.2. *[Sch12, Appenxix A] For $i \geq 1$ and $2 \in R^*$ we have an isomorphism*

$$GW_i(\mathscr{P}(R), Hom(-, R), \epsilon \cdot can) \cong \pi_i B_\epsilon O_\infty(R)^+ \tag{50}$$

Remark 10.3. *The statement of Proposition 10.2 was claimed in [CL86] but the proof in that paper is wrong as explained in [Sch04] which also gives an alternative proof.*

11 Grothendieck-Witt groups of exact categories with weak equivalences

Quillen proves in [Qui73] a collection of powerful theorems for the K-theory of exact categories (Resolution, Localisation, Additivity, Dévissage). They imply most of what was known about the K-theory $K(X)$ of a regular noetherian separated scheme X before the introduction of motivic cohomology. The GW-analogs of Resolution, Localisation, Additivity and Dévissage hold ([Sch10b, Lemma 9], [Sch10a, Introduction], [Sch10a, Theorems 7.1 and 7.2] and [Sch10a, Theorem 6.1]) but they don't imply anything interesting about GW(X), not even when X is regular noetherian separated. One of the reasons is explained in the following example and remark.

Example 11.1. *Let $Z \hookrightarrow X$ be a closed embedding of smooth schemes over some field k. Denote by U the open complement $U = X - Z$. Quillen shows [Qui73] that there exists a long exact sequence of the form*

$$\cdots \to K_i(Z) \to K_i(X) \to K_i(U) \to K_{i-1}(Z) \to \cdots \tag{51}$$

where as usual $K(X) = K(Vect(X))$.

A summary of the proof is as follows.

Proof.

- By the Resolution Theorem we have $K_i Vect(X) \to K_i Coh(X)$ for regular X where $Coh(X)$ is the category of coherent sheaves,
- by the Localisation Theorem the short exact sequence $Coh_Z(X) \hookrightarrow Coh(X) \to Coh(U)$ induces a long exact sequence of K groups, and
- by Dévissage we have $K_i Coh(Z) \xrightarrow{\cong} K_i Coh_Z(X)$.

□

Remark 11.2. *The above does not work for GW groups because the duality on $Vect(X)$ does not extend to a duality on $Coh(X)$ (unless X has dimension 0). Hence we need a new framework, namely categories of chain complexes. The motivation hereby comes from the work of Thomason-Trobaugh [TT90] and Balmer's triangulated Witt groups [Bal05].*

Example 11.3. *In the following, the example to keep in mind is the tuple*

$$(Ch^bVect(X), quis, Hom(-, L[n]), \epsilon \cdot can), \tag{52}$$

where $Ch^bVect(X)$ is the exact category of bounded complexes in $Vect(X)$, quis is the set of quasi-isomorphisms and $L[n]$ is a line bundle l shifted by n, i.e. the chain complex $L[n]$ with L concentrated in degree $-n$.

Definition 11.4. A small exact category with weak equivalences and duality (ExCatWD) is a tuple $(\mathscr{E}, \omega, *, can)$, where $\mathscr{E}$ is a (small) exact category, $\omega \subset Mor(\mathscr{E})$ a set of morphisms called weak equivalences, which is closed under composition, isomorphism and retracts, which contains all identites, and which satisfies the "two out of three" property, i.e. if two out of f, g, fg are weak equivalences then so is the third. $* : (\mathscr{E}^{op}, \omega) \to (\mathscr{E}, \omega)$ is an exact functor which respects weak equivalences $*(\omega) \subset \omega$ and $can : 1 \to **$ natural transformation such that $can_V : V \to V^{**}$ is a weak equivanlence for all $V \in \mathscr{E}$ and $can_V^* \circ can_{V^*} = 1_{V^*}$.

Definition 11.5. [Sch10b, Definition 1] The Grothendieck-Witt group of $(\mathscr{E}, \omega, *, can) \in ExtCatWD$ is the abelian group $GW_0(\mathscr{E}) = GW_0(\mathscr{E}, \omega, *, can)$ generated by inner product spaces $[X, \phi]$ in $(\mathscr{E}, \omega, *, can)$, i.e. objects $X \in \mathscr{E}$ together with weak equivalences $\phi : X \to X^*$ such that $\phi^* can_X = \phi$, subject to the relations

- $[X, \phi] = [W, \psi]$ if there is a weal equivalence $f : V \xrightarrow{\sim} W$ such that $\phi = f^*\psi f$
- $[X \oplus W, \phi \oplus \psi] = [X, \phi] + [W, \psi]$
- $[B, \phi_B] = [A \oplus C, \begin{pmatrix} 0 & \phi_C \\ \phi_A & 0 \end{pmatrix}]$ for any given inner product space in the category of exact sequences in $\mathscr{E}$, that is, a commutative diagram of the form

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, "\phi_A", "\sim"'] & B \arrow[r, two heads] \arrow[d, "\phi_B", "\sim"'] & C \arrow[d, "\phi_C", "\sim"'] \\ C^* \arrow[r, tail] & B^* \arrow[r, two heads] & A^* \end{tikzcd} \tag{53}$$

  where $\phi^* can = \phi$ is a (tuple of) weak equivalences $\phi = (\phi_A, \phi_B, \phi_C)$.

The following remark is clear from the definition.

Remark 11.6. *If $(\mathscr{E}, *, can)$ is an exact category with duality then $(\mathscr{E}, iso, *, can) \in ExCatWD$ and*

$$GW_0(\mathscr{E}, *, can) \xrightarrow{\cong} GW_0(\mathscr{E}, iso, *, can) \tag{54}$$

is an isomorphism.

The following lemma is a special case of Theorem 14.7.

Lemma 11.7. *Let $(\mathscr{E}, *, can)$ be an exact category with duality. Then $(Ch^b, quis, \mathscr{E}, *, can) \in ExCatWD$ and the functor which sends any inner product space to the chain complex concentrated in degree* 0 *induces an isomorphism*

$$GW_0(\mathscr{E}, *, can) \xrightarrow{\cong} GW_0(Ch^b\mathscr{E}, quis, *, can) \tag{55}$$

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12 Simplicial objects

Standard references for simplicial homotopy theory are [GJ09], [FP90], [May67].

Definition 12.1. Let Δ be the category with the ordered sets $[n] = \{0 < \cdots < n\}$ for $n \in \mathbb{N}$ as objects and order preserving maps (of sets) as morphisms. A different way to understand Δ is the following: Let $[n]$ be the category with objects $0, \dots, n$ and for $0 \leq i, j \leq n$ there is a unique morphism $i \to j$ if $i \leq j$ (necessarily the identity when $i = j$) and no morphism from i to j otherwise.
A simplicial set (space, category,...) is a functor

$$X : \Delta^{op} \to Sets\ (Top,\ Cat,\ \dots) \tag{56}$$

Similarly a cosimplicial set (space, category,...) is a functor

$$Y : \Delta \to Sets\ (Top,\ Cat,\ \dots). \tag{57}$$

Example 12.2.
- *The functor*

$$\Delta \to Cat \tag{58}$$
$$n \mapsto [n] \tag{59}$$

is a cosimplicial category.

- *Let $\mathscr{C}$ be a small category, then*

$$\Delta^{op} \to Sets \tag{60}$$
$$n \mapsto Fun([n], \mathscr{C}) =: \mathscr{N}_n(\mathscr{C}) \tag{61}$$

is a simplicial set, called the nerve of $\mathscr{C}$. We may understand a functor $A : [n] \to \mathscr{C}$ as a string of maps $A_0 \xrightarrow{f_0} \cdots \xrightarrow{f_{n-1}} A_n$. As an example to illustrate what happens to morphisms consider the map

$$\partial_i : [n] \to [n+1] \tag{62}$$
$$j \mapsto \begin{cases} j & j < i \\ j+1 & j \geq i \end{cases} \tag{63}$$

Then for A as above we have

$$\partial_i^*(A) = A \circ \partial_i = A_0 \xrightarrow{f_0} \to \cdots \to A_{i-1} \xrightarrow{f_i \circ f_{i-1}} A_{i+1} \to \cdots \xrightarrow{f_{n-1}} A_n \tag{64}$$

- *The topological n-simplex is given as*

$$\Delta^n_{top} := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} | \sum_i x_i = 1, x_i \geq 0\} \tag{65}$$

endowed with the topology as a subspace of $\mathbb{R}^{n+1}$. The functor

$$\Delta \to Top \tag{66}$$
$$n \mapsto \Delta^n_{top} \tag{67}$$

is a cosimplicial topological space. Here a map $\theta : [n] \to [m]$ induces a map $\Delta^n_{top} \to \Delta^m_{top} : x \mapsto y$ via $y_i = \sum_{\theta(j)=i} x_j$.

- *Let X be a topological space. Then the functor*

$$\Delta^{op} \to Sets \tag{68}$$
$$n \mapsto Sing_n X = Top(\Delta^n_{top}, X) \tag{69}$$

defines a simplicial set.

Definition 12.3. Let $X : \Delta^{op} \to Top$ $(Sets)$ be a simplicial topological space (or a simplicial set). Then its topological realisation is the topological space

$$|X| = \coprod_{n \geq 0} X_n \times \Delta^n_{top} / \sim \; = X_\bullet \otimes_\Delta \Delta^\bullet \tag{70}$$

where $(\theta^* x, t) \sim (x, \theta_* t)$ for all $x \in X_m$, $t \in \Delta^n_{top}$ and $\theta : [n] \to [m]$.

Remark 12.4. *If $X \in \Delta^{op} Set$ is a simplicial set, then the geometric realisation $|X|$ is a CW complex.*

Definition 12.5. Let $\mathscr{C}$ be a small category. Define $B\mathscr{C} = |\mathscr{C}| = |\mathscr{N}_* C|$. For a simplicial category $\mathscr{C}_\bullet \in \Delta^{op} Cat$ define $|\mathscr{C}_\bullet| = |q \mapsto |p \mapsto \mathscr{N}_p \mathscr{C}_q|| = |p \mapsto |q \mapsto \mathscr{N}_p \mathscr{C}_q|| = |n \mapsto \mathscr{N}_n \mathscr{C}_n|$.

13 Waldhausen's $S_\bullet$-construction

The reference here is [Wal85].

Definition 13.1. Let $n \in \mathbb{N}$ be a positive integer. Define the arrow category by $\mathscr{A}r[n] := Fun([1], [n])$. Explicitly an object is of the form $a \leq b$ for $0 \leq a, b \leq n$ and there is exactly one morphism $(a \leq b) \to (a' \leq b')$ if and only if $a \leq a'$ and $b \leq b'$, i.e. if we have a diagram in $[n]$ of the form

$$\begin{tikzcd} a \arrow[r] \arrow[d] & b \arrow[d] \\ a' \arrow[r] & b' \end{tikzcd} \tag{71}$$

The functor

$$\Delta \to Cat \tag{72}$$
$$n \mapsto \mathscr{A}r[n] \tag{73}$$

is a cosimplicial category. If $A : \mathscr{A}r[n] \to \mathscr{C}$ is a functor for some category $\mathscr{C}$ we write $A_{p,q} := A(p \leq q)$.

Definition 13.2. Let $(\mathscr{E}, \omega)$ be an exact category with weak equivalences. We have a simplicial exact category with weak equivalences

$$\Delta^{op} \to Cat \tag{74}$$
$$n \mapsto Fun(\mathscr{A}r[n], \mathscr{E}) \tag{75}$$

Here a sequence $A \to B \to C$ is exact if for all $p \leq q$ the sequence $A_{p,q} \to B_{p,q} \to C_{p,q}$ is exact in $\mathscr{E}$ and a morphism $A \to B$ is a weak equivalence if for all $p \leq q$ the morphism $A_{p,q} \to B_{pq}$ is a weak equivalence.

Definition 13.3. Let $S_n\mathscr{E} \subset Fun(Ar[n], \mathscr{E})$ be the full subcategory of those functors $A : Ar[n] \to \mathscr{E}$ such that for all $0 \leq p \leq q \leq r \leq n$ the sequence

$$A_{p,q} \rightarrowtail A_{p,r} \twoheadrightarrow A_{q,r} \tag{76}$$

is an admissible short exact sequence in $\mathscr{E}$ and $A_{p,p} = 0$. Explicitly a functor $A \in S_n\mathscr{E}$ can be depicted as a diagram of the form

$$\begin{tikzcd} A_{01} \arrow[r, tail] & A_{02} \arrow[r, tail] \arrow[d] & A_{03} \arrow[r, tail] \arrow[d] & \cdots \arrow[r, tail] & A_{0n} \arrow[d] \\ & A_{12} \arrow[r, tail] & A_{13} \arrow[r, tail] \arrow[d] & \cdots \arrow[r, tail] & A_{1n} \arrow[d] \\ & & \vdots & & \vdots \arrow[d] \\ & & & & A_{n-2,n-1} \end{tikzcd} \tag{77}$$

Definition 13.4. Let $(\mathscr{C},\omega)$ be an exact category with weak equivalences. Write $\omega\mathscr{C}$ for the category with the same objects as $\mathscr{C}$ and weak equivalences as morphisms.

Definition 13.5. Let $(\mathscr{E},\omega)$ be an exact category with weak equivalences. Then

$$\omega S_\bullet\mathscr{E} : \Delta^{op} \to Cat \tag{78}$$
$$n \mapsto \omega S_n\mathscr{E} \tag{79}$$

is a simplicial excact category. Define $K(\mathscr{E},\omega) = \Omega|\omega S_\bullet\mathscr{E}|$.

Remark 13.6. *We can think of $S_\bullet\mathscr{E}$ as a bar construction of the K-theory of $\mathscr{E}$.*

Proposition 13.7. *[Wal85, 1.9 Appendix] Let $\mathscr{E}$ be an exact category considered as an exact category with weak equivalences $(\mathscr{E},i)$, where i is the class of isomorphisms in $\mathscr{E}$. Then*

$$|Q\mathscr{E}| \sim |iS_\bullet\mathscr{E}|. \tag{80}$$

14 The hermitian $S_\bullet$-construction

Let $(\mathscr{E},\omega,*,can) \in ExCatWD$ then we also have $(S_n\mathscr{E},\omega,*,can) \in ExCatWD$. Here we set $(A^*)_{p,q} = A^*_{n-q,n-p}$ for a functor $A : Ar[n] \to \mathscr{E} \in S_n\mathscr{E}$. Unfortunately $n \mapsto (S_n\mathscr{E},\omega,*,can)$ does not respect the simplicial identities:

Example 14.1. *Consider the functor $\partial_2 : [1] \to [2]$ then the diagram*

$$\begin{tikzcd}
(S_2\mathscr{E})^{op} \arrow[r, "\partial_2^*"] \arrow[d, "*"] & (S_1\mathscr{E})^{op} \arrow[d, "*"] \\
(S_2\mathscr{E}) \arrow[r, "\partial_2^*"] & (S_1\mathscr{E})
\end{tikzcd} \tag{81}$$

doesn't commute, since

$$(\partial_2^*(A_{01} \rightarrowtail A_{0,2} \twoheadrightarrow A_{1,2}))^* = A^*_{0,1} \neq \tag{82}$$
$$A^*_{1,2} = \partial_2^*(A^*_{1,2} \rightarrowtail A^*_{0,2} \twoheadrightarrow A^*_{0,1}) = \partial_2^*(A_{01} \rightarrowtail A_{0,2} \twoheadrightarrow A_{1,2})^*) \tag{83}$$

Definition 14.2. Let A,B be ordered sets. Then we write AB for the concatenation, i.e. the ordered set $A \sqcup B$ with $a < b$ for all $a \in A$ and $b \in B$. In particular write $[n]^{op}[n] = [2n+1] = \{n' < ... < 1' < 0' < 0 < 1 < ... < n\}$. If we interpret $[n]^{op}[n]$ as a category then it has the duality $p \mapsto p'$ and $p' \mapsto p$.

Definition 14.3. Let $(\mathscr{E},\omega,*,can) \in ExCatWD$. Define the simplicial category with duality

$$\Delta^{op} \to Cat \tag{84}$$
$$n \mapsto \mathscr{R}_n\mathscr{E} \tag{85}$$

where $\mathscr{R}_n\mathscr{E} = S^e_n\mathscr{E} = S_{[n]^{op}[n]}\mathscr{E}$ and $(A*)_{p,q} = A^*_{p',q'}$ for functors $A : Ar([n]^{op}[n]) \to \mathscr{E}$. We refer to this process as edge-wise subdivision.

Definition 14.4. Let $(\mathscr{C},*,can)$ be a category with duality. Write $\mathscr{C}_h$ for category with objects (X,ϕ), with $X \in \mathscr{E}$ and $\phi : X \to X^*$ such that $\phi^* can_X = \phi$. A morphism $(X,\phi) \to (Y,\psi)$ is an $f : X \to Y$ such that $\phi = f^*\psi f$.

Proposition 14.5. *[Sch10b, Proposition 2] Let $(\mathscr{E},*,can)$ be an exact category with duality, then*

$$|(iR_\bullet\mathscr{E})_h| \sim |Q^h\mathscr{E}|. \tag{86}$$

Definition 14.6. [Sch10b, Definition 3] Let $(\mathscr{E},\omega,*,can) \in ExCatWD$. Then we define the Grothendieck-Witt space of $\mathscr{E}$ as

$$GW(\mathscr{E}) = GW(\mathscr{E},\omega,*,can) = Fibre(|(\omega R_\bullet\mathscr{E})_h| \to |\omega S_\bullet\mathscr{E}|) \tag{87}$$
$$(A,\phi) \mapsto A \circ i \tag{88}$$

where $i : [n] \to [n]^{op}[n]$ is the map with $p \mapsto p$.

Theorem 14.7. *[Sch10b, Proposition 6] . Let* $(\mathscr{E}, *, can)$ *be an exact category with duality. Then the functor* $(\mathscr{E}, iso, *, can) \to (Ch^b, quis, \mathscr{E}, *, can) \in ExCatWD$ *which sends an object* E *to the chain complex* E *concentrated in degree* 0 *induces an isomorphisms for all* $i \geq 0$

$$GW_i(\mathscr{E}, iso, *, can) \xrightarrow{\cong} GW_i(Ch^b\mathscr{E}, quis, *, can) \tag{89}$$

.

Definition 14.8. Let X be a scheme, L a line bundle over X. Then we write

$${}_{\epsilon}GW^n(X, L) = GW(Ch^bVectX, quis, Hom(-, L[n]), \epsilon can) \tag{90}$$

for the Grothendieck-Witt space and set ${}_{\epsilon}GW^n_i(X, L) = \pi_{i\epsilon}GW^n(X, L)$.

Lemma 14.9.

$${}_{\epsilon}GW^n_i(X, L) \cong {}_{-\epsilon}GW^{n+2}_i(X, L) \tag{91}$$

In particular

$$GW^n_i(X, L) \cong GW^{n+4}_i(X, L) \tag{92}$$

Proof. By the Koszul sign rule, the multiplication map $\mu : O_X[1] \otimes O_X[1] \to O_X[2]$ is -1-symmetric (and non-degenerate bilinear). Therefore, tensor product with the -1-symmetric inner product space $(\mathcal{O}_X[1], \mu)$ defines a functor

$$(Ch^bVect(X), Hom(-, L[n], \epsilon \cdot can)) \longrightarrow (Ch^bVect(X), Hom(-, L[n+2], -\epsilon \cdot can)) \tag{93}$$

which is an equivalence of categories. $\square$

Definition 14.10. Let $Ch^b_ZVect(X) \subset Ch^bVect(X)$ be the full dg subcategory of chain complexes with (cohomological) support in Z, i.e. the category of those complexes which are acyclic outside of Z. We define the Grothendieck-Witt groups with support in Z as

$$GW^n(X \text{ on } Z) = GW(Ch^b_ZVect(X), quis, Hom(-, L[n]), can) \tag{94}$$

Theorem 14.11. *[Sch10b, Theorems 10 and 14] Let* $Z \hookrightarrow X$ *be a closed subscheme of* X *and* $U = X - Z$ *the open complement. Assume further that* X *has an ample family of linebundles and that* X *and* U *are quasi-compact. Then the sequence*

$$Ch^b_ZVect(X) \to Ch^bVect(X) \to Ch^bVect(U) \tag{95}$$

induces a homotopy fibration of the form

$$GW^n(X \text{ on } Z, L) \to GW^n(X, L) \to GW^n(U, L). \tag{96}$$

In particular there is a long exact sequence

$$\cdots \to GW^n_i(X \text{ on } Z, L) \to GW^n_i(X, L) \to GW^n_i(U, L) \to GW^n_{i-1}(X \text{ on } Z, L) \to \cdots \tag{97}$$

Theorem 14.12. *[Sch10b, Theorems 11 and 15] Let* $Z \hookrightarrow X$ *be a closed subscheme of* X *and* $V \hookrightarrow X$ *an open subscheme such that* $Z \subset V$*. Moreover assume that* X *has an ample family of linebundles and that* X*,* $X - Z$ *and* V *are quasi-compact. Then the morphism of categories*

$$CH^b_ZVect(X) \to CH^b_ZVect(V) \tag{98}$$

induces isomorphisms

$$GW^n_i(X \text{ on } Z, L) \xrightarrow{\cong} GW^n_i(V \text{ on } Z, L) \tag{99}$$

Corollary 14.13. *[Sch10b, Theorem 16] Let* $X = U \cup V$ *be an open cover of a scheme* X*, such that* X *has an ample family of line bundles and such that* X*,* U *and* V *are quasi-compact. Then there exists a long exact sequence*

$$\cdots \to GW^n_i(X, L) \to GW^n_i(U, L) \oplus GW^n_i(V, L) \to GW^n_i(U \cap V, L) \to GW^n_{i-1}(X, L) \to \cdots \tag{100}$$

As explainced in [CTHK97], Corollary 14.13 implies the following.

Corollary 14.14. *Let X be a noetherian scheme, such that X has an ample family of line bundles. Then there exists a spectral sequence, called the Brown-Gersten-Quillen spectral sequence, of the form*

$$E_1^{p,q} = \bigoplus_{\substack{x \in X \\ dim\mathcal{O}_{X,x}=p}} GW^n_{-p-q}(\mathcal{O}_{X,x} \text{ on } x) \Rightarrow GW^n_{-q}(X) \tag{101}$$

If $\frac{1}{2} \in O_{X,x}$ and $O_{X,x}$ is regular local with residue field $k(x)$, then there are isomorphisms $GW^n_i(\mathcal{O}_{X,x} \text{ on } x) \cong GW^{n-d}_i(k(x))$ where d is the dimension of $O_{X,x}$. See Proposition 16.8 below.

15 Higher Grothendieck-Witt groups of DG categories

Throughout this section let k be a commutative ring.

Definition 15.1. A differentially graded k-module (dg-module) is a chain complex of k-modules $(M^\bullet, d)$ together with a (k-linear) differential $d : M^i \to M^{i+1}$ (d is a differential if $d \circ d = 0$). The tensor product of two dg-modules $M^\bullet$ and $N^\bullet$ is defined degree-wise by

$$(M^\bullet \otimes N^\bullet)^n = \bigoplus_{p+q=n} M^p \otimes N^q \tag{102}$$

with differential $d(x \otimes y) = dx \otimes y + (-1)^{|x|} x \otimes dy$ where $|x|$ denotes the degree of a homogeneous element x. The internal homomorphism complex is defined as

$$[M^\bullet, N^\bullet]^n = \prod_{-p+q=n} Hom_k(M^p, N^q) \tag{103}$$

with differential $df = d \circ f - (-1)^{|f|} f \circ d$. There are three distinct maps of dg-modules,

- Evaluation

$$e : [M^\bullet, N^\bullet] \otimes M^\bullet \to N^\bullet \tag{104}$$
$$f \otimes x \mapsto f(x) \tag{105}$$

- Covaluation

$$\nabla : M^\bullet \to [(N^\bullet, M^\bullet \otimes N^\bullet] \tag{106}$$
$$x \mapsto (y \mapsto x \otimes y) \tag{107}$$

- Symmetry

$$c : M^\bullet \otimes N^\bullet \to N^\bullet \otimes M^\bullet \tag{108}$$
$$x \otimes y \mapsto (-1)^{|x||y|} y \otimes x \tag{109}$$

They are all chain maps, that is, they commute with the differentials. The tuple $(DGMod_k, c, e, \nabla, 1_k)$ is a closed symmetric monoidal category. Here 1_k is the chain complex with k considered as a module over itself, concentrated in degree 0.

Remark 15.2. *The functor $[M^\bullet, -] : DGMod_k \to DGMod_k$ is the left adjoint of $M^\bullet \otimes - : DGMod_k \to DGMod_k$. The sign convention for the differential on $[M^\bullet, N^\bullet]$ is uniquely determined if we require the differentials to commute with the evaluation map.*

Definition 15.3. A DG-category $\mathscr{A}$ is given by

- A set $Ob\mathscr{A}$ of objects,
- for all $A, B \in Ob\mathscr{A}$ a dg k-module $A^\bullet(A, B)$ with a distinct element $1_A \in \mathscr{A}(A, A)$ for all A, and

- for all $A, B, C \in Ob\mathscr{A}$ a composition map $A^\bullet(A,B) \otimes A^\bullet(B,C) \to A^\bullet(A,C)$ of dg k-modules (and thus commuting with differentials) which is associative and unital.

Example 15.4. *The category $Ch^bVect(X)$ has the structure of a DG category, where the objects are bounded chain complexes of vector bundles over X and for any two objects $E^\bullet, F^\bullet$ the homomorphism chain complex is given by the dg-module $[E,F]^n = \prod_{p-q=n} Hom_{\mathcal{O}_X}(E^q, F^p)$ with differential $df = d \circ f - (-1)^{|f|} f \circ d$.*

Definition 15.5. Let $M \in DGMod_k$ be a differentially graded module. We associate the k-modules $Z^0M = Ker(M^0 \to M^1)$, $B^0M = Ker(M^{-1} \to M^0)$ and $H^0M = Z^0M/B^0M$. Similarly, for a DG-category $\mathscr{A}$, we define the categories $Z^0\mathscr{A}$, $B^0\mathscr{A}$ and $H^0\mathscr{A}$, which all have the same objects as $\mathscr{A}$ and morphisms $(Z^0\mathscr{A})(A,B) = Z^0(\mathscr{A}(A,B))$, $(B^0\mathscr{A})(A,B) = B^0(\mathscr{A}(A,B))$ and $(H^0\mathscr{A})(A,B) = H^0(\mathscr{A}(A,B))$, respectively.

Example 15.6. *For the DG-category $\mathscr{A} = Ch^bVect(X)$, the category $Z^0\mathscr{A}$ is the category of bounded chain complexes with chain maps as morphisms. $H^0\mathscr{A} =: \mathscr{K}^hVect(X)$ is the category of bounded chain complexes with chain homotopy classes of chain maps as morphisms.*

Definition 15.7. Let $\mathscr{A}$ be a DG-category. A sequence $A \xrightarrow{f} B \xrightarrow{g} C \in Z^0(A)$ is called exact if and only if there exist $r \in A^0(B,A)$ and $s \in A^0(C,B)$ such that $rf = 1, gs = 1, fr + sg = 1$. The DG-category $\mathscr{A}$ is called exact if these sequences make Z^0A into an exact category.

Example 15.8. *The category $Ch^bVect(X)$ is an exact DG-category, where the exact sequences are precisely the degree-wise split exact sequences.*

Definition 15.9. Let $\mathscr{A}$ and $\mathscr{B}$ be DG-categories. We define the DG-category $\mathscr{A} \otimes \mathscr{B}$ via

$$Ob(\mathscr{A} \otimes \mathscr{B}) := Ob(\mathscr{A}) \times Ob(\mathscr{B}) \tag{110}$$

$$Hom^\bullet_{\mathscr{A}\otimes\mathscr{B}}(A_0 \otimes B_0, A_1 \otimes B_1) := Hom^\bullet_{\mathscr{A}}(A_0, A_1) \otimes Hom^\bullet_{\mathscr{B}}(B_0, B_1) \tag{111}$$

The composition is given as $(f_0 \otimes g_0) \otimes (f_1 \otimes g_1) = (-1)^{|g_0||f_1|} f_0 \circ f_1 \otimes g_0 \circ g_1$

Definition 15.10. A DG-category $\mathscr{A}$ is called (strongly) pretriangulated if $\mathscr{A}$ is exact and the functor

$$\mathscr{A} \to Ch^b(k) \otimes \mathscr{A} \tag{112}$$

$$\mathscr{A} \mapsto 1 \otimes \mathscr{A} \tag{113}$$

is an equivalence of categories. Here $Ch^b(k)$ is the category of bounded chain complexes of finitely generated free k-modules.

Lemma 15.11. *The DG-category $Ch^bVect(X)$ is pretriangulated.*

Proof. Write $\mathscr{A} = Ch^bVect(X)$. Then the functors

$$\mathscr{A} \to Ch^b(k) \otimes \mathscr{A} \xrightarrow{\otimes} \mathscr{A} \tag{114}$$

are inverse to each other. □

Proposition 15.12. *If $\mathscr{A}$ is a pretriangulated DG-category, then $H^0\mathscr{A}$ is triangulated.*

Definition 15.13. Let $\mathscr{A}$ be a pretriangulated DG-category and $\omega \subset MorZ^0\mathscr{A}$ a set of morphisms. Denote by A^ω the full subcategory of $\mathscr{A}$ of all objects $A \in \mathscr{A}$ such that $0 \to A$ lies in ω. Then the pair $(\mathscr{A}, \omega)$ is called a pretriangulated category with weak equivalences (ptrDGCatW) if A^ω is also pretriangulated and if a map $f \in MorZ^0\mathscr{A}$ lies in ω if and only if it induces an isomorphism in the Verdier quotient of triangulated categories $\mathscr{T}(\mathscr{A}, \omega) := H^0\mathscr{A}/H^0\mathscr{A}^\omega$. The category

$$\mathscr{T}(\mathscr{A}, \omega)$$

is called the triangulated category associated with $(\mathscr{A}, \omega)$.

Example 15.14. *The category $(Ch^bVect(X), quis)$ is a pretriangulated DG category with weak equivalences. In particular $Ch^bVect(X)^{quis}$ is the category of acyclic chain complexes and $\mathscr{T}(Ch^bVect(X), quis) = \mathscr{D}^bVect(X)$ is the usual (bounded) derived category of $Vect(X)$.*

Definition 15.15. Let $(\mathscr{A}, \omega)$ be a pretriangulated DG-category with weak equivalences. Then $(Z^0\mathscr{A}, \omega)$ is an exact category with weak equivalences, and we define the K-theory space of $(\mathscr{A}, \omega)$ by

$$K(\mathscr{A}, \omega) = K(Z^0\mathscr{A}, \omega) = \Omega|\omega S_\bullet Z^0\mathscr{A}| \tag{115}$$

Theorem 15.16. *[TT90, Theorem 1.9.8] Let $F : (\mathscr{A}, \omega) \to (\mathscr{B}, \omega)$ be a map of pretriangulated DG-categories with weak equivalences which induces an equivalence $F : \mathscr{T}(\mathscr{A}, \omega) \cong \mathscr{T}(\mathscr{B}, \omega)$ of associated triangulated categories. Then the induced map*

$$K_i(A, \omega) \xrightarrow{\cong} K_i(B, \omega) \tag{116}$$

is an isomorphism for all i.

Definition 15.17. Let $\mathscr{A}$ be a pretriangulated DG-category. Then the dual pretriangulated DG-category $\mathscr{A}^{op}$ has the same objects as $\mathscr{A}$ and morphism complexes

$$\mathscr{A}^{op}(A, B)) := \mathscr{A}(B, A), \tag{117}$$

where composition is defined by $f \circ g = -1^{|f||g|} g \circ f$.

Definition 15.18. A pretriangulated DG-category with weak equivalences and duality is a tuple $(\mathscr{A}, \omega, *, can)$ where $(\mathscr{A}, \omega)$ is a pretriangulated DG-category with weak equivalences, $* : \mathscr{A}^{op} \to \mathscr{A}$ is a dg functor and $can : 1 \to ** \in \omega$ is a natural weak equivalence with $can_A^* \circ can_{A^*} = 1$. Define the Grothendieck-Witt space of $(\mathscr{A}, \omega, *, can)$ by

$$GW(\mathscr{A}, \omega, *, can) = GW(Z^0\mathscr{A}, \omega, *, can). \tag{118}$$

Example 15.19. *The tuple $(Ch^bVectX, quis, Hom(-, L[n]), can)$ is a pretriangulated DG-category with weak equivalences and duality.*

Remark 15.20. *[Sch12, Proposition 6.3] The above definition gives Grothendieck-Witt groups $GW_i(\mathscr{A})$ for $i \geq 0$. One can extend this definition to all $i \in \mathbb{Z}$ by setting for $i < 0$*

$$GW_i(\mathscr{A}) = W^{-i}(\mathscr{T}\mathscr{A}), \tag{119}$$

where the latter are Balmer-Witt groups. In particular we have

$$GW_i^n(X, L) = W^{n-i}(X, L) = W_0(Ch^bVectX, quis, Hom(-, L[n-i]), can), \tag{120}$$

for $i < 0$.

Theorem 15.21. *[Sch12, Theorem 6.5] Let $(\mathscr{A}, \omega, *, can) \to (\mathscr{B}, \omega, *, can)$ be a map of pretriangulated DG-categories with weak equivalences and duality, such that $\mathscr{T}(\mathscr{A}, \omega) \xrightarrow{\cong} T(\mathscr{B}, \omega)$ and $\frac{1}{2} \in \mathscr{A}, \mathscr{B}$. Then*

$$GW(\mathscr{A}, \omega) \xrightarrow{\cong} GW(\mathscr{B}, \omega) \tag{121}$$

Remark 15.22. *If $\frac{1}{2} \notin \mathscr{A}, \mathscr{B}$ then there exist counterexamples to the above theorem; see [Sch12, Proposition 2.1].*

Theorem 15.23. *[Sch12, Theorem 6.6] Let $(\mathscr{A}, \omega) \to (\mathscr{B}, \omega) \to (\mathscr{C}, \omega)$ be a sequence of pretriangulated DG-categories with weak equivalences and duality such that $\mathscr{T}(\mathscr{A}, \omega) \to \mathscr{T}(\mathscr{B}, \omega) \to \mathscr{T}(\mathscr{C}, \omega)$ is an exact sequence of triangulated categories, i.e. a sequence of functors such that $\mathscr{T}(\mathscr{A}, \omega) \subset \mathscr{T}(\mathscr{B}, \omega)$ is the full subcategory of those objects in $\mathscr{T}(\mathscr{B}, \omega)$ which are zero in $\mathscr{T}(\mathscr{C}, \omega)$ and such that $\mathscr{T}(\mathscr{B}, \omega) \to \mathscr{T}(\mathscr{C}, \omega)$ induces an equivalence $\mathscr{T}(\mathscr{B}, \omega)/\mathscr{T}(\mathscr{A}, \omega) \to \mathscr{T}(\mathscr{C}, \omega)$. Then there exists a homotopy fibration*

$$GW\mathscr{A} \to GW\mathscr{B} \to GW\mathscr{C} \tag{122}$$

16 Higher Grothendieck-Witt groups of schemes

Lemma 16.1. *Write* $(\mathscr{E}, {}_\epsilon\#_L^n) = (Ch^bVect(X), quis, Hom(-, L[n]), \epsilon can)$*. Then the sequence of functors*

$$(\mathscr{E}, {}_\epsilon\#_L^n) \to (Mor\mathscr{E}, {}_\epsilon\#_L^n) \xrightarrow{cone} (\mathscr{E}, {}_\epsilon\#_L^{n+1}) \tag{123}$$
$$E^\bullet \mapsto 1_{E^\bullet} \tag{124}$$

induces an exact sequence of associated triangulated categories and therefore a homotopy fibration

$$\begin{tikzcd} {}_\epsilon GW^n(X,L) \arrow[r] \arrow[dr, "F"'] & {}_\epsilon GW^n(Mor\mathscr{E}) \arrow[r] \arrow[d, "\cong"] & {}_\epsilon GW^{n+1}(X,L) \\ & {}_\epsilon GW^n(\mathscr{E}\times\mathscr{E}^{op}) = K(X) \arrow[ur, "-H"'] & \end{tikzcd} \tag{125}$$

Corollary 16.2. *[Sch12, Theorem 6.1 and Remark 6.7] The sequence*

$${}_\epsilon GW^n(X,L) \xrightarrow{F} K(X) \xrightarrow{H} {}_\epsilon GW^{n+1}(X,L) \tag{126}$$

is a homotopy fibration. Therefore we have the identifications ${}_\epsilon U = {}_\epsilon GW^{-1} = {}_{-\epsilon}GW^1$*,* ${}_\epsilon V = \Omega_\epsilon GW^1$ *and combined we obtain Karoubi's fundamental theorem* ${}_{-\epsilon}V = \Omega_\epsilon U$*.*

Remark 16.3. *[Sch12, Proposition 8.7] The so called Karoubi-Grothendieck-Witt groups take on the form*

$$\mathbb{G}W_i(X,L) = \begin{cases} GW_i(X,L) & i \geq 0 \\ something & i < 0 \end{cases} \tag{127}$$

They are the analog of the non-connective K-theory groups $\mathbb{K}_i(X)$*. If* $\mathbb{K}_i(X) = 0$ *for all* $i < 0$*, e.g if* X *is regular noetherian separated, we have [Sch12, Proposition 9.3]*

$$GW_i(X,L) \xrightarrow{\cong} \mathbb{G}W_i(X,L) \tag{128}$$

for all $i \in \mathbb{Z}$*.*

Theorem 16.4. *[Sch10b, Theorem 16] Let* $X = U \cup V$ *be an open cover of a scheme* X*, such that* $\frac{1}{2} \in X$ *and* X *has an ample family of line bundles. Then there exists a long exact sequence of the form*

$$\cdots \to \mathbb{G}W_i^n(X) \to \mathbb{G}W_i^n(U) \oplus \mathbb{G}W_i^n(V) \to \mathbb{G}W_i^n(U\cap V) \to \mathbb{G}W_{i-1}^n(X) \to \cdots \tag{129}$$

Theorem 16.5. *[Sch12, Theorem 9.9] Consider the pull-back square of schemes*

$$\begin{tikzcd} Y' \arrow[r, hook] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, hook] & X \end{tikzcd} \tag{130}$$

where the map $Y \to X$ *is a regular embedding and* X' *is the blow-up of* X *along* Y*. Assume that* $\frac{1}{2} \in X$ *and that* X *has an ample family of line bundles. Then there exists a long exact sequence of the form*

$$\cdots \to \mathbb{G}W_i^n(X) \to \mathbb{G}W_i^n(Y) \oplus \mathbb{G}W_i^n(X') \to \mathbb{G}W_i^n(Y') \to \mathbb{G}W_{i-1}^n(X) \to \cdots \tag{131}$$

Theorem 16.6 (Walter, Schlichting Remark 9.11 in [Sch12])**.** *Assume that* $\frac{1}{2} \in X$*. Then we have*

$$\mathbb{G}W_i^n(\mathbb{P}_X^r) = \begin{cases} \mathbb{G}W_i^n(X) \oplus K_i(X)^{\frac{r-1}{2}} \oplus GW_i^{n-\frac{r+1}{2}}(X) & r \text{ odd} \\ \mathbb{G}W_i^n(X) \oplus K_i(X)^{\frac{r}{2}} & r \text{ even} \end{cases} \tag{132}$$

Theorem 16.7 (Bass' fundamental theorem for GW, Theorem 9.13 in [Sch12])**.** *Let X be a scheme such that $\frac{1}{2} \in X$ and such that X has an ample family of line bundles. Then there exists a split exact sequence*

$$0 \to \mathbb{G}W_i^n(X) \to \mathbb{G}W_i^n(X[T]) \oplus \mathbb{G}W_i^n(X[T^{-1}]) \to \mathbb{G}W_i^n(X[T, T^{-1}]) \to \mathbb{G}W_{i-1}^{n-1}(X) \to 0 \tag{133}$$

This defines $\mathbb{G}W_i^n$ for $i < 0$ inductively.

Recall the Brown-Gersten-Quillen spectral sequence

$$E_1^{p,q} = \bigoplus_{\substack{x \in X \\ dim \mathcal{O}_{X,x} = p}} GW_{-p-q}^n(\mathcal{O}_{X,x} \text{ on } x) \Rightarrow GW_{-q}^n(X) \tag{134}$$

The next proposition identifies the E_1-term with the Grothendieck-Witt groups of the residue fields.

Proposition 16.8. *Let X be a regular local scheme of dimension d such that $\frac{1}{2} \in X$ and let $x \in X$ be the closed point. Then there exists a homotopy equivalence*

$$GW^{n-d}(k(x)) \xrightarrow{\sim} GW^n(X \text{ on } x). \tag{135}$$

This equivalence depends on a choice of a system of parameters of R.

Proof. Let $X = Spec\, R$ for a regular local ring $(R, \mathfrak{m}, k)$ choose a regular system of parameters $(f_1, \ldots, f_d) = \mathfrak{m}$. Further let $R \xrightarrow{f_i} R$ be the differentially graded algebra concentrated in degrees 0 and -1. Then the Koczul complex gives a quasi-isomorphism of differentially graded algebras

$$K(f_1, \ldots, f_d) := \bigotimes_{i=1}^{d} (R \xrightarrow{f_i} R) \to k \tag{136}$$

Hence we have isomorphisms

$$GW_i^n(k) \xleftarrow{\cong} GW_i^n(K(f_1, \ldots, f_n)) \to GW_i^n(R \text{ on } \mathfrak{m}, Hom(-, R[d])) = GW^{n+d}(R \text{ on } \mathfrak{m}) \tag{137}$$

where the first one holds by invariance under derived equivalences (Theorem **??**). The second map is defined by $(E, \varphi) \mapsto (E, \pi\varphi)$ where $\pi : K(f_1, \ldots, f_d) \to R[d]$ is the projection onto the component of degree $-d$. This map induces isomorphism of Grothendieck-Witt groups, by devissage. □

We finish this section with a proof of the surjectivity of the map $H : K_3(\mathbb{Z}') \to GW_3(\mathbb{Z}')$ used in the proof of Lemma 6.1.

Lemma 16.9. *Let $\mathbb{Z}' = \mathbb{Z}[\frac{1}{2}]$. Then the group $GW_2^3(\mathbb{Z}') = 0$, in particular, the map $H : K_3(\mathbb{Z}') \to GW_3^0(\mathbb{Z}')$ is surjective.*

Proof. By Corollary 16.2, we have an exact sequence

$$K_3(\mathbb{Z}') \xrightarrow{H} GW_3^0(\mathbb{Z}') \to GW_2^3(\mathbb{Z}').$$

Hence, the vanishing of $GW_2^3(\mathbb{Z}') = 0$ implies the surjectivity of $H : K_3(\mathbb{Z}') \to GW_3^0(\mathbb{Z}')$. By the same corollary, we have an exact sequence

$$GW_2^2(\mathbb{Z}') \xrightarrow{F} K_2(\mathbb{Z}') \xrightarrow{H} GW_2^3(\mathbb{Z}') \to GW_1^2(\mathbb{Z}').$$

We have $GW_1^2(\mathbb{Z}') = Sp(\mathbb{Z}')^{ab} = 0$ because $[Sp(\mathbb{Z}'), Sp(\mathbb{Z}')] = Sp(\mathbb{Z}')$ as $\mathbb{Z}'$ is Euclidean. Moreover, the map $GW_2^2(\mathbb{Z}') \xrightarrow{F} K_2(\mathbb{Z}')$ is surjective because $K_2(\mathbb{Z}') \cong K_2(\mathbb{Z}) = \mathbb{Z}/2$ is generated by the symbol $\{-1, -1\}$ [Mil71, §10] which lifts to an element in $GW_2^2(\mathbb{Z}')$ (as the cup product of $[-1] \in GW_1^1(\mathbb{Z}')$ with itself). □

17 Coherent GW-groups

In this section we explain the GW-analog of Example 11.1.

Definition 17.1. Let X be a noetherian scheme. Denote by $QCoh^b_c(X)$ the category of bounded chain complexes of quasi-coherend $\mathcal{O}_X$-modules with coherent cohomology; it has the structure of a DG-category with the same definitions as in Example 15.4. A dualising complex on X is a bounded chain complex $I^\bullet$ of injective quasi-coherent $\mathcal{O}_X$-moduls such that

$$can : E \to [[E, I], I] \tag{138}$$

is a quasi-isomorphism for all $E \in QCoh^b_c(X)$ (this only needs to be checked for $E = O_X$). The map can_E is defined as the composition

$$E \xrightarrow{\nabla} [[E, I], E \otimes [E, I]] \xrightarrow{[1,c]} [[E, I], [E, I] \otimes E] \xrightarrow{[1,e]} [[E, I], I].$$

Define the Grothendieck-Witt space

$$GW(X, I) = GW(QCoh^b_c(X), quis, \#_I, can) \tag{139}$$

where $\#_I = [-, I]$.

Lemma 17.2. *Let $Z \hookrightarrow X$ be a closed subscheme of a noetherian scheme X, and $U = X - Z$ the open complement. Moreover let I be a dualising complex on X. Then $i^b I = \mathscr{H}om^\bullet_{\mathcal{O}_X}(i_*\mathcal{O}_Z, I)$ is a dualising complex on Z.*

Theorem 17.3. *[Sch12, Theorem 9.19] Let $Z \hookrightarrow X$ be a closed subscheme of a noetherian scheme X, and $U = X - Z$ the open complement. Moreover let I be a dualising complex on X. Then there exists a homotopy fibration*

$$GW(Z, i^b I) \to GW(X, I) \to GW(U, I) \tag{140}$$

Theorem 17.4. *[Sch12, Theorem 9.18] Let X be a noetherian regular separated scheme. Then an injective resolution $\mathcal{O}_X \to I^\bullet$ is a dualising complex for X and induces a homotopy equivalence*

$$GW(X) \xrightarrow{\cong} GW(X, I^\bullet) \tag{141}$$

where the lefthand side is defined in terms of vector bundles, and the right hand side in terms of complexes with coherent cohomology.

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