Equivariant coherent sheaf cohomology

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The concept of space has undergone throughout the history of mathematics significant developments; from affine space as a simple commutative space to non-commutative spaces and ∞-topoi. This modern concept is able to incorporate a space together with its symmetries. Equivariant sheaf cohomology is a powerful tool to compute invariants of such a non-commutative space which simultaneously generalizes sheaf and group cohomology. I will demonstrate a constructive approach to these invariants.

Modeling Abelian categories of coherent sheaves on non-affine schemes is more involved than modeling those of modules. I want to discuss recent advances allowing a constructive description of Abelian categories of coherent sheaves on spaces with a finitely generated Cox ring S as a Serre quotient category of the category of finitely presented graded S-modules. This includes the constructive treatment of (local and) global Ext's of which sheaf cohomology is a particular case.

One basic notion in homological algebra is that of an Abelian category. The standard way to express mathematical notions constructively requires algorithms for all disjunctions and all existential quantifiers appearing in the defining axioms. This led to the notion of a computable Abelian category. This does not only streamline classical approaches to computability in module categories but introduces a systematic way for future development of constructive homological algebra that we want to promote during the workshop, especially its applicability to other Abelian categories, in particular to categories of sheaves. Complicated bookkeeping structures like spectral sequences of filtered complexes can be constructively treated along these lines in a unified way.

The Abelian category of coherent sheaves on varieties X admitting a finitely generated Cox ring S can be constructively described as a Serre quotient category A/C of the category $A := S - grmod$ of finitely presented (multi)graded S-modules modulo the thick subcategory $C := S - grmod^0$ of all S-modules representing the zero sheaf on X. In general, a Serre quotient category A/C is a computable Abelian category as soon as A is computable and the membership of the thick subcategory C is decidable. This was implicitly proved in Gabriel's thesis. In particular, the Abelian category of coherent sheaves on a smooth toric variety is computable. In this workshop, we want to discuss possible generalizations to other classes of schemes.

Global extension groups Ext^i in Serre quotient categories are computable as a direct limit under additional assumptions. In the case of coherent sheaves on projective schemes the finiteness of the limit follows from the Castelnuovo-Mumford regularity. The BGG-correspondence would lead to more efficient algorithms. I want to discuss generalizing the finiteness argument in the case of smooth complete toric varieties using a multigraded version of the Castelnuovo-Mumford regularity. The same abstract tools can be used to treat the equivariant setup on an equal footing.