

① Throughout this course, by a ring we shall mean a commutative ring with identity 1, such that $1 \neq 0$.

② An integral domain is a ring A such that if $x, y \in A$ and $xy = 0$, then either $x = 0$ or $y = 0$.

③ An ideal $p \subset A$ is called prime if

① $p \subsetneq A$, and

② $x, y \notin p \Rightarrow xy \notin p$

or equivalently, A/p is a non-zero integral domain.

④ An ideal $m \subset A$ is called maximal if

① $m \subsetneq A$, and

② if $x \in A \setminus m$, then $m + (x) = A$

or equivalently, A/m is a non-zero field.

⑤ The first important result we want to prove is that every ring has maximal ideals. The idea of the proof is the following, start with the 0 ideal and keep adding elements until we get a maximal ideal. The formal proof follows.

Definition: A subset $S \subset A$ is called multiplicatively closed if $x, y \in S \Rightarrow xy \in S$, $1 \in S$ and $0 \notin S$.

Lemma: Let S be a multiplicative set. Let Σ be the collection of ideals of A such that $I \cap S = \emptyset$. There is a partial ordering by inclusion on Σ . Then Σ has maximal elements and every maximal element is a prime ideal of A .

Pf: Observe that Σ is non-empty as $0 \in \Sigma$. That Σ has maximal elements follows from Zorn's lemma. Let Λ be a totally ordered set, that is, for $\alpha, \beta \in \Lambda$, either $\alpha \leq \beta$ or $\beta \leq \alpha$. Suppose for each $\alpha \in \Lambda$, we are given $I_\alpha \in \Sigma$ such that if $\alpha \leq \beta$, then $I_\alpha \subset I_\beta$. Consider the ideal $J = \bigcup_{\alpha \in \Lambda} I_\alpha$.

$$J \in \Sigma \text{ as } J \cap S = \left(\bigcup_{\alpha \in \Lambda} I_\alpha \right) \cap S = \bigcup_{\alpha \in \Lambda} I_\alpha \cap S = \emptyset.$$

Moreover, J is an upper bound for the chain $\{I_\alpha\}_{\alpha \in \Lambda}$. Thus, by Zorn's lemma, Σ has maximal elements. Let $p \in \Sigma$ be a maximal element. We claim that p is prime. If not, $\exists x, y \notin p$ but $xy \in p$. Since $p \neq p + (x), p + (y) \Rightarrow (p + (x)) \cap S \neq \emptyset \Rightarrow \exists a + bx \in S$ and similarly, $a' + b'y \in S \Rightarrow (a + bx)(a' + b'y) \in S$. But as $xy \in p$, this element is also in p , $\Rightarrow p \cap S \neq \emptyset$, contradicting $p \in \Sigma$. This completes the proof of the lemma.

Thm: Let A be a ring in which $1 \neq 0$. Then A has maximal ideals.

Pf: Apply the above lemma by taking $S = \{1\}$. Let m be a maximal element such that $m \cap S = \emptyset$. Then if $x \in A \setminus m$, $(m + (x)) \cap S \neq \emptyset, \Rightarrow m + (x) = A$.

Cor: If $x \in A$ is not a unit, then there is a maximal ideal containing x .

Pf: Let S be the set of units in A . Then S is a multiplicative set. Let Σ be the collection of ideals I such that $x \in I$ and $I \cap S = \emptyset$. The ideal $(x) \in \Sigma$ and so $\Sigma \neq \emptyset$. Let m be a

maximal element of Σ . We claim m is maximal, and we omit the simple proof.

UPSHOT : Every ring has prime ideals.

⑥ Let $f: A \rightarrow B$ be a ring homomorphism. If $\mathfrak{q} \subset B$ is a prime ideal, then $f^{-1}(\mathfrak{q}) \subset A$ is a prime ideal. To see this, note that the natural map \bar{f} is an inclusion. Since B/\mathfrak{q}

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A/f^{-1}(\mathfrak{q}) & \xrightarrow{\bar{f}} & B/\mathfrak{q} \end{array}$$

is a domain, $\Rightarrow A/f^{-1}(\mathfrak{q})$ is a domain, $\Rightarrow f^{-1}(\mathfrak{q})$ is a prime ideal.

REMARK : The contraction of a maximal ideal need not be a maximal ideal. Consider the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$. Then (0) is maximal in \mathbb{Q} and $i^{-1}(0) = (0)$ is not maximal in \mathbb{Z} .

⑦ Let A be a ring and let $I \subsetneq A$ be an ideal. Let $\bar{\Phi}: A \rightarrow A/I$ be the natural map. There is 1-1 correspondence between the prime ideals of A/I and the prime ideals of A which contain I , explicitly given by

$$A \xrightarrow{\bar{\Phi}} A/I$$

$$\bar{\Phi}^{-1}(\mathfrak{p}) \longleftrightarrow \mathfrak{p}$$

We remark that under this correspondence, maximal ideals do contract to maximal ideals. This is because if $\mathfrak{m} \subset A/I$ is maximal, then $A/\bar{\Phi}^{-1}(\mathfrak{m}) \hookrightarrow (A/I)/\mathfrak{m}$ and this is also a surjection. Thus, it is an isomorphism. Since $(A/I)/\mathfrak{m}$ is a field $\Rightarrow A/\bar{\Phi}^{-1}(\mathfrak{m})$ is a field, $\Rightarrow \bar{\Phi}^{-1}(\mathfrak{m})$ is maximal.

⑧ Nilradical: Define $\text{Nil}(A) = \{x \in A \mid \exists n > 0 \text{ such that } x^n = 0\}$

Propn: $\text{Nil}(A) = \bigcap_{\text{primes}} \mathfrak{p}$

Pf: It is clear that $\text{Nil}(A) \subset \bigcap_{\text{primes}} \mathfrak{p}$, since if $x \in \text{Nil}(A)$,

$\Rightarrow \exists n > 0$ such that $x^n = 0 \in \mathfrak{p} \Rightarrow x \in \mathfrak{p}$.

Suppose $x \notin \text{Nil}(A)$. Let $S = \{1, x, x^2, \dots\}$, then $0 \notin S$ and S is multiplicative. Let Σ be the collection of ideals which do not meet S . Then $(0) \notin \Sigma$ and so $\Sigma \neq \emptyset$, and so there is a maximal element in Σ which is prime, say \mathfrak{p} . \Rightarrow

$\mathfrak{p} \cap S = \emptyset, \Rightarrow x \notin \mathfrak{p}$, thus $x \notin \bigcap_{\text{primes}} \mathfrak{p}$. This completes the proof.

⑨ Jacobson Radical: Define $\text{Jac}(A) = \bigcap_{\text{maximal ideals}} \mathfrak{m}$.

Propn: $x \in \text{Jac}(A) \Leftrightarrow 1 - xy$ is a unit in $A \ \forall y \in A$.

Pf: If $x \in \text{Jac}(A)$ and $y \in A$, then $xy \in \mathfrak{m}$ for every maximal ideal, $\Rightarrow 1 - xy \notin \mathfrak{m}$ for every maximal ideal, or else the maximal ideal would contain 1. $\Rightarrow 1 - xy$ is a unit.

$\Rightarrow \text{Jac}(A) \subset \{x \in A \mid 1 - xy \text{ is a unit } \forall y \in A\}$

Now suppose $x \notin \text{Jac}(A)$, $\Rightarrow \exists$ maximal ideal \mathfrak{m} such that

$x \notin \mathfrak{m}, \Rightarrow \mathfrak{m} + (x) = A, \Rightarrow 1 = a + xy \Rightarrow a = 1 - xy,$

$\Rightarrow \exists y$ such that $1 - xy$ is not a unit.

This completes the proof.

Chapter 2

① Nakayama's lemma: Let M be a finitely generated A -module, and assume that $M = JM$ for some ideal $J \in \text{Jac}(A)$. Then $M = 0$.

Pf: Assume that m_1, m_2, \dots, m_n are generators for M as an A -module.

Since $M = JM$, $m_i = \sum_{j=1}^n a_{ji} m_j$ for $a_{ji} \in J$. If $A = (a_{ij})$, then we may write the above equations in the form $(A - I) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0$.

Multiplying by the adjoint of $A - I$ on the left, we get

$\det(A - I) m_i = 0 \quad \forall i$. But note $\det(A - I)$ is an element of the form $\pm(1 + a_1 + a_2 + \dots + a_n)$ where $a_i \in J^2$. In particular, it is a unit. Thus, each $m_i = 0$, $\Rightarrow M = 0$.

② Cayley-Hamilton Theorem: Let M be a finitely generated A -mod and suppose $\phi \in \text{End}_A(M)$ is such that $\phi(M) \subset JM$. Then ϕ satisfies a polynomial equation of the type $\phi^n + a_1 \phi^{n-1} + \dots + a_n = 0$ where $a_i \in J^i$.

Pf: The ideal is to modify the proof of Nakayama's lemma slightly. Let $R \subset \text{End}_A(M)$ be the commutative subring generated by $\langle A, I, \phi \rangle$. Then M is an R -module in a natural way.

Moreover, M is finitely generated as an R -module, since it is finitely generated as an A -module. Let m_1, m_2, \dots, m_n be its generators. Then $\phi(m_i) = \sum_{j=1}^n a_{ji} m_j$, which we rewrite as

$$(A - \phi) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0. \text{ Now } A - \phi \text{ is a matrix with coefficients in } R.$$

Taking adjoint, we get $\det(A - \phi) m_i = 0 \quad \forall i = 1, 2, \dots, n$.

But $\det(A - \phi) = \pm(\phi^n + a_1 \phi^{n-1} + \dots + a_n)$ $a_i \in I^i$. This completes

the proof of the Cayley-Hamilton theorem.

- ③ Tensor Products : Let M and N be A -modules. We want to define a pair $(\varphi, M \otimes_A N)$ consisting of
- An A -module which we denote $M \otimes_A N$
 - An A -bilinear map $\varphi: M \times N \rightarrow M \otimes_A N$

which is universal in the following sense

- For any A -module P and an A -bilinear map $f: M \times N \rightarrow P$, this factors uniquely as follows

$$\begin{array}{ccc} M \times N & \xrightarrow{f} & P \\ \varphi \searrow & \exists! \tilde{f} \nearrow & \\ & M \otimes_A N & \end{array} \quad \exists! \tilde{f} \rightarrow A\text{-linear}$$

Remark : The above is an example of a universal property. Here is a simpler example of a universal property - let G be a group and let $K \subset G$ be a normal subgroup. Consider the pair $(\pi, G/K)$, where $\pi: G \rightarrow G/K$ is the natural map. This pair is universal among all pairs (f, H) where $f: G \rightarrow H$ is a group homomorphism such that $K \subset \ker f$. In other words, any homomorphism $f: G \rightarrow H$ such that $K \subset \ker f$

factors as

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ \pi \searrow & \exists! \tilde{f} \nearrow & \\ & G/K & \end{array}$$

- let us denote by $\text{Bil}_A(M \times N, P)$ the space of all A -bilinear maps $M \times N \rightarrow P$, and by $\text{Hom}_A(M, N)$ the space of A -linear maps $M \rightarrow N$. Both these have natural A -module structures and there is a natural isomorphism $\text{Hom}_A(M, \text{Hom}_A(N, P)) \rightarrow \text{Bil}_A(M \times N, P)$ given by
- $$\varphi: M \rightarrow \text{Hom}_A(N, P) \mapsto (m, n) \mapsto \varphi(m)(n)$$

The universal property of tensor products is equivalent to saying, there is an isomorphism $\text{Bil}_A(M \times N, P) \rightarrow \text{Hom}_A(M \otimes_A N, P)$.

Thus, we have natural isomorphisms

$$\text{Hom}_A(M, \text{Hom}_A(N, P)) \xrightarrow{\sim} \text{Bil}_A(M \times N, P) \xrightarrow{\sim} \text{Hom}_A(M \otimes_A N, P).$$

Existence of the pair $(\varphi, M \otimes_A N)$ having the required universal property.

We use similar notation as in Atiyah & MacDonald.

Consider the free A -module with basis $(m, n) \in M \times N$. In other words, we mean the free module $C = \bigoplus_{(m, n) \in M \times N} A \cdot (m, n)$.

Let $D \subset C$ be the submodule generated by elements of the form

- $a(m, n) - (am, n)$
- $a(m, n) - (m, an)$
- $(m_1 + m_2, n) - (m_1, n) - (m_2, n)$
- $(m, n_1 + n_2) - (m, n_1) - (m, n_2)$

Then there is a natural map $C \xrightarrow{\pi} C/D$. Further, consider the set map $M \times N \rightarrow C$ given by $(m, n) \mapsto (m, n)$ (this denotes the basis vector). Call the composite map $M \times N \xrightarrow{\varphi} C/D$.

Claim 1: φ is A -bilinear.

This is straight forward check which follows from the definition of D .

Claim 2: φ is universal among such bilinear maps.

Suppose $f: M \times N \rightarrow P$ is an A -bilinear map. Then we want to define an A -linear map $\tilde{f}: C/D \rightarrow P$ which makes the following diagram commute

$$\begin{array}{ccc} M \times N & \xrightarrow{f} & P \\ \varphi \searrow & & \nearrow \tilde{f} \\ & C/D & \end{array} \quad \tilde{f} \rightarrow A\text{-linear}$$

First let us define a map $C \rightarrow P$. Since C is a free A -module, to define an A -linear hom $C \rightarrow P$, it is enough to specify where the basis elements go. Define a map by sending

the basis element $(m, n) \mapsto f(m, n)$. We extend this hom to the whole of C A -linearly. Next we check that this hom is 0 on D . It is enough to check this is 0 on the generators of D . For example, $(m_1 + m_2, n) - (m_1, n) - (m_2, n) \mapsto$

$$f(m_1 + m_2, n) - f(m_1, n) - f(m_2, n) = 0 \text{ as } f \text{ is } A\text{-bilinear.}$$

Similarly, $a(m, n) - (am, n) \mapsto af(m, n) - f(am, n) = 0$.

The other checks are equally simple. Thus, we get a

$$\begin{array}{ccc} M \times N & \xrightarrow{f} & P \\ \varphi \searrow & \nearrow \tilde{f} & \\ & C/D & \end{array} \quad \tilde{f} \rightarrow A\text{-linear}$$

Moreover, for any \tilde{f} which makes the above diagram commute, it is forced that $\tilde{f}(\varphi(m, n)) = f(m, n)$. Since as an A -module C/D is generated by $\varphi(m, n)$, we get that \tilde{f} is unique. For two modules M and N , we shall denote the map $M \times N \xrightarrow{\varphi} C/D$ by $M \times N \rightarrow M \otimes_A N$, and given by $(m, n) \mapsto m \otimes_A n$. We remark that, by construction one has

- $(am) \otimes_A n = a(m \otimes_A n) = m \otimes_A (an)$
- $(m_1 + m_2) \otimes_A n = m_1 \otimes_A n + m_2 \otimes_A n$
- $m \otimes_A (n_1 + n_2) = m \otimes_A n_1 + m \otimes_A n_2$

Suppose $f: M \rightarrow M'$ and $g: N \rightarrow N'$ are A -module homomorphisms.

We want to construct an A -module homomorphism

$f \otimes g: M \otimes_A M' \rightarrow N \otimes_A N'$. Recall that we have only one way of constructing A -module homomorphisms from $M \otimes_A M'$, which is

to give an A -bilinear map $M \times M' \rightarrow N \otimes_A N'$. Define such a map by $(m, m') \mapsto f(m) \otimes_A g(m')$. It is trivial to check this map is A -bilinear, and so it gives an A -linear map $M \otimes_A M' \rightarrow N \otimes_A N'$, given by $m \otimes_A m' \mapsto f(m) \otimes_A g(m')$. Using the same method as above, one can prove the following

- $(M \otimes_A N) \otimes_A P \xrightarrow{\sim} M \otimes_A (N \otimes_A P)$
- $(M \oplus N) \otimes_A P \xrightarrow{\sim} M \otimes_A P \oplus N \otimes_A P$

Propn : There is a natural map $A \otimes_A M \rightarrow M$ which is an isomorphism.

Pf: Since the map $A \times M \rightarrow M$ given by $(a, m) \mapsto am$ is A -bilinear, we get a A -linear map $A \otimes_A M \rightarrow M$.

This is surjective since $1 \otimes_A m \mapsto m$. To see this is injective, suppose $\sum_{i=1}^n a_i \otimes_A m_i \mapsto 0$, $\Rightarrow \sum_{i=1}^n a_i m_i = 0$ in M .

$$\sum_{i=1}^n a_i \otimes_A m_i = \sum_{i=1}^n 1 \otimes_A a_i m_i = 1 \otimes_A \left(\sum_{i=1}^n a_i m_i \right) = 0.$$

\Rightarrow Injective.

④ Right exactness of Tensor Products.

Chain complexes : A complex of A -modules is a sequence

$$\dots \rightarrow M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \rightarrow \dots \quad \text{where}$$

- Each M_i is an A -module
- $d_i : M_i \rightarrow M_{i-1}$ is an A -module hom
- $d_i \circ d_{i+1} = 0$, that is, $\text{Im } d_{i+1} \subset \text{Ker } d_i$

We say the complex is exact at i if $\text{Im } d_{i+1} = \text{Ker } d_i$

We define the homology groups of the complex M_\bullet by

$$H_i(M_\bullet) = \frac{\ker d_i}{\operatorname{Im} d_{i+1}}$$

Remark: $H_i(M_\bullet) = 0 \Leftrightarrow$ exact at i .

A short exact sequence is an exact complex of the type $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$, that is, $M' \subset M$ and $M'' \cong M/M'$.

Propn: Suppose $M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$ is exact, then the complex $M' \otimes_A N \xrightarrow{f \otimes 1} M \otimes_A N \xrightarrow{g \otimes 1} M'' \otimes_A N \rightarrow 0$ is exact.

Pf: The proof is based on the following result

- ① Let $C' \xrightarrow{h'} C \xrightarrow{h} C'' \rightarrow 0$ be a complex of A -modules. Then this complex is exact $\Leftrightarrow \forall$ A -modules P , the resulting complex $0 \rightarrow \operatorname{Hom}_A(C'', P) \rightarrow \operatorname{Hom}_A(C, P) \rightarrow \operatorname{Hom}_A(C', P)$ is exact.

We leave the proof of this as an exercise. Applying $\operatorname{Hom}_A(-, \operatorname{Hom}_A(N, P))$ to the exact sequence $M' \rightarrow M \rightarrow M'' \rightarrow 0$, we get the following sequence is exact

$$\begin{array}{ccccccc} 0 \rightarrow \operatorname{Hom}_A(M'', \operatorname{Hom}_A(N, P)) & \rightarrow & \operatorname{Hom}_A(M, \operatorname{Hom}_A(N, P)) & \rightarrow & \operatorname{Hom}_A(M', \operatorname{Hom}_A(N, P)) \\ & \downarrow s & & \downarrow s & & \downarrow s \\ (*) \quad 0 \rightarrow \operatorname{Hom}_A(M'' \otimes_A N, P) & \rightarrow & \operatorname{Hom}_A(M \otimes_A N, P) & \rightarrow & \operatorname{Hom}_A(M' \otimes_A N, P) \\ \Rightarrow \forall P, \text{ the sequence } (*) \text{ is exact} & \Rightarrow & \text{using lemma} \end{array}$$

$$M' \otimes_A N \rightarrow M \otimes_A N \rightarrow M'' \otimes_A N \rightarrow 0 \text{ is exact.}$$

Remark: Tensor product is not left exact, as the following example shows. Consider the short exact sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$$

Tensoring this with $\mathbb{Z}/2\mathbb{Z}$ yields the complex

$$0 \rightarrow \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \rightarrow 0$$

One of the exercises asks to prove that $A/I \otimes_A M \cong M/IM$.
Using this exercise we get a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} & \xrightarrow{2} & \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} & \rightarrow & \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \rightarrow 0 \\ & & \downarrow s & & \downarrow s & & \downarrow s \\ 0 & \rightarrow & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{2} & \mathbb{Z}/2\mathbb{Z} & \rightarrow & \mathbb{Z}/2\mathbb{Z} \rightarrow 0 \end{array}$$

The bottom row is not exact since $\mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/2\mathbb{Z}$ is not an inclusion.

Definition: An A -module M is called flat if for every short exact sequence $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$, the resulting complex $0 \rightarrow N' \otimes_A M \rightarrow N \otimes_A M \rightarrow N'' \otimes_A M \rightarrow 0$ is exact.

We remark that in the above, the only check which needs to be done is that $0 \rightarrow N' \otimes_A M \rightarrow N \otimes_A M$ is exact.