Chapter 3 : Localization

① Recall the construction of rational numbers as ratios. Formally, $\mathbb{Q}$ consists of equivalence classes $\left[\frac{a}{b}\right]$ where $a, b \in \mathbb{Z}$, $b \neq 0$ and the equivalence is given by $\frac{a}{b} \sim \frac{a'}{b'}$ if $ab' - a'b = 0$ in $\mathbb{Z}$.

We want to generalize this construction to general rings.

Let $S \subset A$ be a subset such that

- $1 \in S$
- $x, y \in S \Rightarrow xy \in S$
- $0 \notin S$

Consider the ring $S^{-1}A$ whose elements are equivalence classes of the type $\frac{a}{s}$, where $a \in A$ and $s \in S$. The equivalence relation is given by $\frac{a}{s} \sim \frac{a'}{s'}$ if $(as' - a's)s'' = 0$ in A for some $s'' \in S$. We remark that if A is a domain, then this is same as saying $(as' - a's) = 0$ in A.

- It is routine to check that there is a well-defined addition and multiplication on these equivalence classes, which makes $S^{-1}A$ into a commutative ring. Let us check that $1 \neq 0$ in this ring. By definition of $S^{-1}A$, $\frac{1}{1} = \frac{0}{1} \iff \exists\, s \in S$ such that $s = 0$, but $0 \notin S$.
- There is a ring homomorphism $A \xrightarrow{f} S^{-1}A$, given by $a \mapsto \frac{a}{1}$ and this homomorphism has the property that $f(s)$ is a unit in $S^{-1}A$, since $s \cdot \frac{1}{s} = 1$.
- The above homomorphism is universal for this property, that is, if $g: A \to B$ is any ring hom such that $g(S) \subset$ units of B, then there is a unique arrow $\tilde{g}$ making the diagram commute. The definition of $\tilde{g}$ is forced, we only need to define $\tilde{g}\left(\frac{1}{s}\right) := \frac{1}{g(s)}$.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & B \\ S^{-1}A \arrow[ur, dashed, "\exists!\,\tilde{g}"'] & \end{tikzcd}$$

In other words, define $\tilde{g}\left(\frac{a}{s}\right) := \frac{g(a)}{g(s)} = g(a)g(s)^{-1}$. This can be done since $g(s)$ is a unit in B. It is once again routine to check that $\tilde{g}$ is well defined on equivalence classes.

- Let M be an A-module. In a similar manner define the module $S^{-1}M$. This has a natural structure as an $S^{-1}A$ module, that is, $\frac{a}{s} \cdot \frac{m}{s'} := \frac{am}{ss'}$.

- If $f: M \to M'$ is an A-mod hom, then $S^{-1}f: S^{-1}M \to S^{-1}M'$ is defined to be $S^{-1}f\left(\frac{m}{s}\right) := \frac{f(m)}{s}$. We will abuse notation and write f for $S^{-1}f$.

<u>Propn.</u>: (Localization is exact): Let $M' \xrightarrow{f} M \xrightarrow{g} M''$ be an exact seq of A-modules. Then $S^{-1}M' \xrightarrow{f} S^{-1}M \xrightarrow{g} S^{-1}M''$ is exact.

<u>Pf</u>: We need to show if $g\left(\frac{m}{s}\right) = 0$, then $\frac{m}{s} \in \mathrm{Im}(f)$.

$g\left(\frac{m}{s}\right) = \frac{g(m)}{s} = 0 \Rightarrow \exists\, s' \in S$ such that $g(m)s' = 0$.

$\Rightarrow g(s'm) = 0 \Rightarrow \exists\, m' \in M'$ such that $f(m') = s'm$

$\Rightarrow \frac{m}{s} = \frac{s'm}{ss'} = \frac{f(m')}{ss'} = f\left(\frac{m'}{ss'}\right)$.

② Localization has several nice and useful properties, like the proposition above. Another very important property is the description of prime ideals of $S^{-1}A$ in terms of the prime ideals of A.

<u>Definition</u>: Let $A \to B$ be a ring homomorphism. If $I \subset A$ is an ideal, then we denote by I^e the ideal generated by $f(I)$ in B.

<u>Propn</u>: Consider the natural map $A \to S^{-1}A$. Every ideal in $S^{-1}A$ is an extended ideal. In fact, for $I \subset S^{-1}A$, $I^{ce} = I$.

<u>Pf</u>: The proof is based on the trivial observation that all elements of S are units in $S^{-1}A$. In general, for a hom $A \to B$, $I^{ce} \subset I$. Now suppose $\frac{a}{s} \in I$, then $a \in I$ $\Rightarrow a \in I^c \Rightarrow \frac{a}{1} \in I^{ce} \Rightarrow \frac{1}{s}\frac{a}{1} \in I^{ce} \Rightarrow \frac{a}{s} \in I^{ce}$.

<u>Propn</u>: There is 1-1 correspondence between prime ideals of $S^{-1}A$ and prime ideals of A which do not meet S, given by $\mathfrak{q} \to \mathfrak{q}^c$

<u>Pf</u>: Let $\mathfrak{q} \subset S^{-1}A$ be a prime. Then we know that $\mathfrak{q} = \mathfrak{q}^{ce}$ from the earlier propn. Moreover, since contraction of a prime is always a prime, $\mathfrak{q}^c \subset A$ is a prime. If $\mathfrak{q}^c \cap S \neq \phi$, then $\mathfrak{q}^{ce} = (1)$ which is a contradiction since $\mathfrak{q} \subsetneq S^{-1}A$.

If $\mathfrak{p} \subset A$ is a prime such that $\mathfrak{p} \cap S = \phi$, then $\mathfrak{p}^e$ is a prime. Suppose $\frac{a}{s}\frac{b}{t} \in \mathfrak{p}^e$, $\Rightarrow \frac{ab}{st} = \frac{c}{s'}$ where $c \in \mathfrak{p}$ $\Rightarrow (abs' - cst)s'' = 0 \Rightarrow abs's'' \in \mathfrak{p}$. Since $\mathfrak{p} \cap S = \phi$ and $s's'' \in S \Rightarrow ab \in \mathfrak{p} \Rightarrow a \in \mathfrak{p}$ or $b \in \mathfrak{p} \Rightarrow \frac{a}{s}$ or $\frac{b}{t} \in \mathfrak{p}^e$ $\Rightarrow \mathfrak{p}^e$ is prime. Next we show $\mathfrak{p}^{ec} = \mathfrak{p}$. It is trivial to check that $\mathfrak{p} \subset \mathfrak{p}^{ec}$. For the other inclusion, if $a \in \mathfrak{p}^{ec}$ $\Rightarrow \frac{a}{1} \in \mathfrak{p}^e \Rightarrow \frac{a}{1} = \frac{b}{s}$ for some $b \in \mathfrak{p} \Rightarrow at \in \mathfrak{p}$ for some $t \in S$. Since $t \notin \mathfrak{p}$ and $\mathfrak{p}$ is prime $\Rightarrow a \in \mathfrak{p}$.

From the above, the proposition follows trivially.

③ The most important examples of multiplicatively closed sets are $S = A \setminus \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal. For this S, we shall denote the ring $S^{-1}A$ by $A_{\mathfrak{p}}$.

Because of the previous discussion, we know that the prime ideals in $A_{\mathfrak{p}}$ are in 1-1 correspondence with prime ideals $\mathfrak{p}' \subset A$ such that $\mathfrak{p}' \cap (A \setminus \mathfrak{p}) = \phi \iff \mathfrak{p}' \subset \mathfrak{p}$. Thus, the primes of $A_{\mathfrak{p}}$ are exactly of the form $\mathfrak{q}^e$ or $\mathfrak{q}A_{\mathfrak{p}}$, where $\mathfrak{q} \subset \mathfrak{p} \subset A$ is a prime. In particular, we get that $A_{\mathfrak{p}}$ has exactly 1 maximal ideal, which is $\mathfrak{p}A_{\mathfrak{p}}$.

Propn: Let M be an A-module. Then the following are equivalent

① $M = 0$

② $M_{\mathfrak{p}} = 0$ $\forall$ prime ideals of A

③ $M_{\mathfrak{m}} = 0$ $\forall$ maximal ideals of A

Pf: ① $\Rightarrow$ ② $\Rightarrow$ ③ is obvious. Suppose $M_{\mathfrak{m}} = 0$ $\forall$ maximal ideals $\mathfrak{m} \subset A$. If $M \neq 0$, then $\exists\, x \in M$, $x \neq 0$, thus the ideal $\mathrm{Ann}(x) = \{a \in A \mid ax = 0\} \neq (1)$. In particular, there is a maximal ideal $\mathfrak{m}$ such that $\mathrm{Ann}(x) \subset \mathfrak{m}$. Since $M_{\mathfrak{m}} = 0$, $\Rightarrow \frac{x}{1} = 0$ in $M_{\mathfrak{m}} \Rightarrow xs = 0$ for some $s \in A \setminus \mathfrak{m}$. But this means that $s \in \mathrm{Ann}(x) \subset \mathfrak{m}$, a contradiction.

Remark: We see that a module being 0 (or $\neq 0$) is a local property.

Propn: Let $M' \xrightarrow{f} M \xrightarrow{g} M''$ be a complex. The following are equivalent

① $M' \to M \to M''$ is exact

② $M'_{\mathfrak{p}} \to M_{\mathfrak{p}} \to M''_{\mathfrak{p}}$ is exact for all primes $\mathfrak{p} \subset A$

③ $M'_{\mathfrak{m}} \to M_{\mathfrak{m}} \to M''_{\mathfrak{m}}$ is exact $\forall$ maximal ideals of A.

Pf: We have already seen ① ⇒ ② for a general S.

② ⇒ ③ is obvious.

③ ⇒ ①. Suppose $x \in M$ and $g(x) = 0$, we need to show there is $y \in M'$ such that $f(y) = x$. Since $f(M')$ is a submodule of M, consider the ideal $I = \{a \in A \mid ax \in f(M')\}$. We need to show $I = (1)$. If not, $\exists$ maximal ideal $\mathfrak{m}$ such that $I \subset \mathfrak{m}$. Since $M'_{\mathfrak{m}} \to M_{\mathfrak{m}} \to M''_{\mathfrak{m}}$ is exact and $g(\frac{x}{1}) = 0$, $\Rightarrow \exists\ \frac{y}{s} \in M'_{\mathfrak{m}}$ such that $f(\frac{y}{s}) = \frac{f(y)}{s} = x$

$\Rightarrow \exists\ t \in A \setminus \mathfrak{m}$ such that $(f(y) - xs)t = 0$ in M.

$\Rightarrow f(ty) = xs \Rightarrow s \in I \subset \mathfrak{m}$, but $s \in A \setminus \mathfrak{m}$, a contradiction.

Remark: Checking a complex is exact is a local property.

④ Localization and tensor products

Propn: Let M be an A-module. Then there is a natural homomorphism of $S^{-1}A$-modules, $S^{-1}A \otimes_A M \longrightarrow S^{-1}M$ given by $\frac{a}{s} \otimes_A m \longmapsto \frac{am}{s}$. This is an isomorphism.

Pf: First we note that there is an A-bilinear map $S^{-1}A \times M \longrightarrow S^{-1}M$ given by $(\frac{a}{s}, m) \longmapsto \frac{am}{s}$. By the universal property of the tensor product we get an A-module homomorphism $S^{-1}A \otimes_A M \to S^{-1}M$. We want to show this is an isomorphism. Surjectivity is clear since $\frac{1}{s} \otimes_A m \longmapsto \frac{m}{s}$. To prove injectivity, suppose $\sum_{i=1}^{n} \frac{a_i}{s_i} \otimes_A m_i \longmapsto 0$, $\Rightarrow \sum_{i=1}^{n} \frac{a_i m_i}{s_i} = 0$ in $S^{-1}M$

$\Rightarrow$ Let $s = s_1 s_2 \cdots s_n$ and let $t_i = \frac{s}{s_i}$. Then we may write

$$\sum_{i=1}^{n} \frac{a_i m_i}{s_i} = \sum_{i=1}^{n} \frac{a_i t_i m_i}{s} = 0 \quad \text{in } S^{-1}M$$

$\Rightarrow \exists\, s' \in S$ such that $\left(\sum_{i=1}^{n} a_i t_i m_i\right) s' = 0$ in M

$$\therefore \sum_{i=1}^{n} \frac{a_i}{s_i} \otimes_A m_i = \sum_{i=1}^{n} \frac{a_i t_i s'}{s s'} \otimes_A m_i = \sum_{i=1}^{n} \frac{1}{s s'} \otimes_A a_i t_i s' m$$

$$= \frac{1}{s s'} \otimes_A \left(\sum_{i=1}^{n} a_i t_i s' m_i \right) = 0.$$ Thus, the map is also injective.