Annual Foundational School I, 2014, Almora

Tutorial Problems in Group Theory

1 Week 1, Prof. Dinesh Khurana

Exercises for AFS-I; December 1-27, 2014, CEMS Almora

By $\circ(X)$ we will denote the number of elements in the finite set X. We will denote the the order of an element g of a group by $\circ(g)$. By $< S >$ we will denote the subgroup of a group G generated by $S \subseteq G$. If $S \subseteq X$, then we denote $X \backslash S$ by S^c.

1. Let F be a field with q elements. Find the order of $\mathrm{GL_n(F)}$ and $\mathrm{SL_n(F)}$.
2. Prove that for a finite group G the following conditions are equivalent:

   (i) G is cyclic.

   (ii) For every divisor d of $\circ(G)$ there exists a unique subgroup of G of order d.

   (iii) For every divisor d of $\circ(G)$ there are exactly d solutions of $x^d = e$ in G.

   (iv) For every divisor d of $\circ(G)$ the number of solutions of $x^d = e$ in G is at most d.
3. Deduce from previous exercise that every finite multiplicative group of a field is cyclic.
4. Let H_1, H_2 be two proper subgroups of a group G and $H =< (H_1 \cup H_2)^c >$. Then Prove that

   (i) If $< H_1 \cup H_2 > \neq G$, then $H = G$.

   (ii) If $< H_1 \cup H_2 >= G$, then $G =< (H_1 \cap H_2)^c >=< (H_1 \backslash H_2) \cup (H_2 \backslash H_1) >$.

   (iii) H has index 1 or 2.

   (iv) For a group G the following conditions are equivalent:

   (a) G is union of three proper subgroups.

   (b) The Klein's group V_4 is a homomorphic image of G.

   (c) G contains two subgroups of index 2.
5. Let $G = A_1 \cup A_2 \cup A_3$ where each A_i is a proper subgroup of G. Let $f : G \to V_4 = \{e, a_1, a_2, a_3\}$ be defined as

   $$f(g) = \begin{cases} a_i \text{ if } \ g \in A_i \backslash (A_j \cup A_k) \\ e \ \text{ otherwise} \end{cases}$$

   Prove that

   (i) $A_i \cap A_j \subseteq A_k$ where $\{i, j, k\} = \{1, 2, 3\}$.

   (ii) If $x, y \notin A_i$, then $xy \in A_i$ for any $i \in \{1, 2, 3\}$.

   (iii) f is a group epimorphism and so V_4 is a homomorphic image of G.
6. Let H be a subgroup of S_n and K be the set of all even permutations in H. Prove that K is a subgroup of H of index 1 or 2.
7. If in a group G, $(ab)^2 = (ba)^2$ for all $a, b \in G$ and $x^2 = e$ implies $x = e$, then prove that G is abelian.
8. Let G be a group of order $2m$ where m is odd. Prove that there exists a subgroup of order m in G.

9. Let G be a group of order $2^k m$ where m is odd. If there exists an element of order 2^k in G, then prove that there exists a normal subgroup of order m in G.

10. Let S and T be two subsets of a finite group G such that $\circ(S) + \circ(T) > \circ(G)$. Prove that $G = ST$. Use this to prove that in a finite field every element is a sum of two squares.

11. Find all group homomorphisms between two cyclic groups and then between any two finite abelian groups.

12. Let G be a finite abelian group of odd order. Prove that $\circ(\mathrm{Aut(G)})$ is even.

13. Write down all subgroups of order 8 of S_4 and S_5.

14. For a group G and $n \in \mathbb{N}$ let $G^{(n)} = \{g^n : g \in G\}$. Prove that if for some $n \in \mathbb{N}$ $G^{(n)} = G$ and $(ab)^n = a^n b^n$ for all $a, b \in G$, then $G^{(n-1)} \subseteq Z(G)$ and $(ab)^{n-1} = a^{n-1}b^{n-1}$ for all $a, b \in G$.

15. Let G be a finite group such that $3 \nmid \circ(G)$ and $(ab)^3 = a^3 b^3$ for all $a, b \in G$. Prove that G is abelian. Is the result true if we replace 3 with some other odd prime?

16. If for some $n \in \mathbb{N}$ $G^{(n)} = G^{(n-1)} = G$ and $(ab)^n = a^n b^n$ for all $a, b \in G$, then prove that G is abelian.

17. **Euler's Theorem**. Let a, n be two co-prime integers. Prove that $n \mid a^{\phi(n)} - 1$.

18. **Wilson's Theorem**. Prove that if p is a prime, then $p \mid (p-1)! + 1$.

19. Generalize Wilson's Theorem by proving that if p is a prime and $p \leq n < 2p$, then $p \mid \frac{n!}{p(n-p)!} + 1$.

20. If $a > 1$ is an integer, then prove that $n \mid \phi(a^n - 1)$ for all $n \in \mathbb{N}$.

21. Let H be a subgroup of a finite group G such that $\circ(G) \nmid i(H)!$, where $i(H)$ denotes the index of H i.e., $i(H) = \circ(G)/\circ(H)$. Prove that H contains a normal subgroup of G with more that one elements.

22. Use previous exercise and the fact that A_n is simple for every $n > 4$ to prove that A_n for any $n > 4$ does not have a subgroup of index k such that $1 < k < n$.

23. Without using the simplicity of A_n for $n > 4$ prove that A_n does not have a subgroup of index 2 for any $n > 3$.

24. Find all normal subgroups of S_n for every n.

25. Let s be an r-cycle in S_n. Find $N(s) = \{x \in S_n : xs = sx\}$. Also find $N(s)$ where $s = (1\,2\,3)(4\,5\,6)$ in S_6.

26. Find two elements in A_5 which are conjugate in S_5 but not in A_5.

27. Let $\sigma \in A_n$. Prove that either $C_{A_n}(\sigma) = C_{S_n}(\sigma)$ or $\circ(C_{A_n}(\sigma)) = \circ(C_{S_n}(\sigma))/2$, where $C_G(g)$ denotes the conjugacy class of g in G.

28. Write down the orders of every conjugacy classe of A_5 and prove that A_5 is simple.

29. Let G be a finite group, $T \in \mathrm{Aut(G)}$ and $S = \{g \in G : T(g) = g^{-1}\}$. Prove that
    1. If $\circ(S) > \frac{3}{4} \circ (G)$, then G is abelian and $S = G$.
    2. If $\circ(S) = \frac{3}{4} \circ (G)$, then G contains an abelian group of index 2.

    Also give an example of a non-abelian group G with $T \in \mathrm{Aut(G)}$ such that $\circ(S) = \frac{3}{4} \circ (G)$.

30. Let d be a a divisor of a finite abelian group G and $A_d = \{g \in G : g^d = e\}$. Prove that $d \mid \circ(A_d)$.

31. If $x = xyx$, then we will call y to be an inner inverse of x. Prove that a semigroup in which every element has a unique inner inverse is a group.

32. Find all conjugacy classes of the dihedral group D_n.

33. Let H be a proper subgroup of a finite group G. Prove that G cannot be union of all conjugates of H.

34. Let H be a subgroup of G such that $1 < i(H) < \infty$. Prove that G cannot be union of all conjugates of H.

35. Let G be a finite group such that for any x, $y \in G$ there exists $T \in \mathrm{Aut(G)}$ such that $T(x) = y$. Prove that there exists a prime p such that $\circ(a) = p$ for every $a \neq e$ in G.

36. Let p be the smallest prime divisor of the order of a finite group G. Prove that any subgroup of index p is normal in G.

37. For any group G prove that the commutator subgroup $G' = \{a_1 a_2 ... a_n a_1^{-1} a_2^{-1} ... a_n^{-1} : a_i \in G\}$.

38. Find the commutator subgroup of S_n, A_n, D_n, $GL_n(F)$, $SL_n(F)$, where F is a field.

2 Week 2, Prof. M. K. Yadav

Isometries: Problems for tutorials

1. Let A be an orthogonal 3×3 matrix with determinant 1. Show that A has an eigen value equal to 1.

2. Let A be a 3×3 matrix corresponding to a rotation of $\mathcal{R}^3$. Show that A is orthogonal with determinant 1.

3. Let A be an orthogonal 3×3 matrix with determinant 1. Show that it represents a rotation of $\mathcal{R}^3$.

4. Compute all complex eigenvalues of the matrix A that represents a rotation of $\mathcal{R}^3$ through the spin (θ, u).

5. For which values of n, O_n is isomorphic to $SO_n \times \{\pm I\}$?

6. Determine the matrices of the following rotations of $\mathcal{R}^3$:
(i) angle θ, the axis e_2;
(ii) angle $2\pi/3$, axis contains the vector $(1, 1, 1)^t$.

7. Give an example of an infinite group acting on a finite set.

8. Let a group G acts on a finite group H. Then $H^g = \{h \in H \mid h^g = h\}$, the fix of an element $g \in G$, is a subset of H. Frame minimal conditions on the acting element g so that H^g becomes a subgroup.

9. Show that a group G acts on the set of all left cosets of its subgroup H in it. Compute the stabilizer of a given element gH under this action.

10. Let a group G have a subgroup H of finite index. Show that G has a normal subgroup of finite index contained in H.

11. Consider the natural action of the symmetric group $Sym(n)$ on the set $\{1, 2, \ldots, n\}$. Let H be the stabilizer of 1. Describe the left cosets of H in $Sym(n)$.

12. Compute the orbits of the plane under the action (on the points of the plane) of the group of all isometries of the plane.

13. Show that there are only five types of Platonic solids (regular polyhedra), which are Tetrahedron, Octahedron, Cube, Dodecahedron and Icosahedron.

14. Compute the groups of symmetries of a tetrahedron and a cube.

3 Week 3, Prof. D. Surya Ramana

AFS I, Almora December 2014

Tutorial I (Group Theory)

Date : 16 December 2014
Time : 6:30-8:30 PM

In this tutorial we will classify all groups of order 12 up to isomorphism. Throughout, G will denote a group of order 12. Further, H will denote a Sylow 2-subgroup of G and K a Sylow 3-subgroup of G.

PROBLEM 1. Check that H has either 1 or 3 conjugates and that K has either 1 or 4 conjugates in G.

PROBLEM 2. Show either H or K is a normal subgroup of G.

Hint.— Suppose K is not normal in G. Then K has 4 conjugates in G. What is the number of elements of G that do not lie in the union of conjugates of K in G ?

PROBLEM 3. Suppose both H and K are normal in G. Then show that the canonical map from $H \times K$ into G given by $(x, y) \mapsto xy$ is an isomorphism of groups.

Hint.— $[H, K] \subseteq H \cap K = \{e\}$. Thus the map is an injective homomorphism of groups. Now compare cardinalities.

PROBLEM 4. Conlcude from Problem 3 that when H and K are both normal in G, we have that G is isomorphic either to $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z}$ or to $\mathbf{Z}/4\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z}$.

PROBLEM 5. Suppose that K is not normal in G. Then show that the action of G by conjugation on the conjugates of K provides an injective homomorphism of G into S_4. Conclude that G is isomorphic to A_4 and hence that H is isomorphic to $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$.

PROBLEM 6. Suppose now that H is not normal in G and that it is isomorphic to $\mathbf{Z}/4\mathbf{Z}$. Then let x be a generator of H and y of K. Show that we have $x^4 = 1$, $y^3 = 1$ and $xy = y^2x$. Show that these relations determine G up to isomorphism. Realize G as a group of matrices.

Hint.— To show $xy = y^2x$ or, what is the same thing, $xyx^{-1} = y^2$, let H act by conjugation on K and note that $\mathrm{Aut}(K)$ has a unique non-trivial element, one that exchanges y with y^2.

PROBLEM 7. Finally suppose that H is not normal in G and that it is isomorphic to $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$. Then show that G is isomorphic to D_6.

END

AFS I, Almora December 2014

Tutorial II (Group Theory)

Date : 20 December 2014

Time : 3:30-5:30 PM

PROBLEM 1. Let $\mathbf{F}_q$ be the finite field of order q for an integer $q \geq 2$ and let $n \geq 1$ be an integer. Then show that the number of solutions to the equation $X^n = 1$ in the multiplicative group of $\mathbf{F}_q$ is $\gcd(n, q-1)$. Apply this to obtain the formula

$$|\mathrm{PSL}_n(\mathbf{F}_q)| = \frac{1}{(q-1)\gcd(n, q-1)} \prod_{0 \leq i \leq n-1} (q^n - q^i)$$

from the formula for the cardinality of $\mathrm{GL}_n(\mathbf{F}_q)$ given in the lectures.

PROBLEM 2. The purpose of this problem is to obtain a criterion of K. Iwasawa, given by the theorem below, which provides a method for showing that a group is simple.

THEOREM . — *Let G be a group acting on a set E and suppose that the following conditions are met.*

(a) The group G is equal to its commutator subgroup $[G, G]$.

(b) G acts doubly transitively on E.

(c) There is an element of E whose stabiliser in G contains a commutative normal subgroup A whose conjugates in G generate G.

Then every normal subgroup of G that is distinct from G acts trivially on E.

Prove this theorem by working out the following exercises.

(i) Recall from the lectures that since G acts doubly transitively on E, any normal subgroup N of G either acts transitively on E or acts trivially on E. Suppose that N is a normal sugroup of G that acts transitively on E. Then show that for any x in E we have $G = N.\,\mathrm{Stab}(x)$.

(ii) Suppose $x \in E$ satisfies the condition (c) of the theorem and let N be as in (i). Then show using (i) that $G = N.\,A$.

Hint.— Let g be an element of G. Then since the conjugates of A generate G, there is an integer $k \geq 1$ such that

$$g = \prod_{1 \leq i \leq k} g_i a_i g_i^{-1}$$

for some g_i in G and a_i in A. By (i) we have that each $g_i = n_i s_i$, for some n_i in N and s_i in $\mathrm{Stab}(x)$. Use the normality of N in G to now check that there is an n in N such that $g = n \prod_{1 \leq i \leq k} s_i a_i s_i^{-1}$. Deduce from this that $G = N.A$ by remarking that A is normal in $\mathrm{Stab}(x)$.

(iii) Using (ii) above show that $G = [G, G] = [N.A, N.A] \subseteq N$. Conclude the theorem.

The theorem of Jordan-Dickson announced in the lectures states that $\mathrm{PSL}_n(K)$ is a simple group except when $n = 2$ and K is either $\mathbf{F}_2$ or $\mathbf{F}_3$. In the following problems we shall verify this theorem for $n > 2$ with the aid Iwasawa's criterion given in Problem 2. As we saw in the lectures, $\mathrm{PSL}_2(\mathbf{F}_2)$ is isomorphic to S_3 and $\mathrm{PSL}_2(\mathbf{F}_3)$ is isomorphic to A_4. While the proof of the theorem for $n = 2$ and $|K| > 3$ is similar to the case $n > 2$ given below, a certain number of modifications become necessary.

PROBLEM 3.— Let $n > 2$ be an integer and K be a field. Further, let V be the vector space K^n. For any i, j with $1 \leq i, j \leq n$ with $i \neq j$ we write E_{ij} to denote the square matrix of order n with the (i, j)-th entry 1 and all other entries 0. Further, for any λ in K, we write $B_{ij}(\lambda)$ to denote the square matrix $I_n + \lambda E_{ij}$.

(i) Check that for any λ in K and each (i, j) as above, the matrix $B_{ij}(\lambda)$ is an element of $\mathrm{SL}_n(K)$.

A *transvection* of V is a K-linear automorphism $T \neq \mathrm{id}$ of V such that there is a hyperplane H of V so that $T(v) = v$ for all v in H and $T(v) - v \in H$ for all v in V.

(ii) Verify that each $B_{ij}(\lambda)$ with $\lambda \neq 0$ represents a transvection with respect to the canonical basis of V. Further, show that for each transvection of V there is a basis of V such that T is represented by the matrix $B_{12}(1)$ with respect to this basis. Conclude that each $B_{ij}(\lambda)$ is conjugate to $B_{12}(1)$ in $\mathrm{GL}_n(K)$, for any $n \geq 2$.

(iii) Show that when $n > 2$, each $B_{ij}(\lambda)$ is in fact conjugate to $B_{12}(1)$ in $\mathrm{SL}_n(K)$.

(iv) Cite an appropriate theorem from linear algebra that implies that $\mathrm{SL}_n(K)$ is generated by the matrices $B_{ij}(\lambda)$.

(v) With $n = 3$, let $g = B_{13}(1)$ and $h = B_{32}(1)$. Then show that $B_{12}(1)$ is the commutator of g and h in $\mathrm{SL}_3(K)$. Use this to express $B_{12}(1)$ as a commutator in $\mathrm{SL}_n(K)$, for all $n > 2$.

(vi) Conclude from the above parts that $[\mathrm{SL}_n(K), \mathrm{SL}_n(K)] = \mathrm{SL}_n(K)$, when $n > 2$.

PROBLEM 4.— With $V = K^n$, let $P(V)$ denote the set of one dimensional subspaces of V. This is generally denoted by $P^{n-1}(K)$ and is called the Projective Space of dimension $n-1$ over K.

(i) Show that the natural action of $\mathrm{SL}_n(K)$ on $P(V)$ is doubly transitive.

(ii) Show that the subgroup of $\mathrm{SL}_n(K)$ that acts trivially on $P(V)$ is the same as the center of $\mathrm{SL}_n(K)$.

(iii) Let $[e_1]$ denote the one dimensional subspace of V generated by e_1. Determine the stabiliser of $[e_1]$ in $\mathrm{SL}_n(K)$.

(iv) Let $n > 2$ and let A be the subgroup of $\mathrm{SL}_n(K)$ consisting of square matrices of order n with the top row of the form $[1\, a_1\, a_2\, \ldots\, a_{n-1}]$, where each a_i in K, and the remaining rows equal to $e_2^t, e_3^t, \ldots, e_n^t$, in that order. Then show that A is a commutative normal subgroup of the stabiliser of $[e_1]$ satisfying the condition (c) of Iwasawa's criterion.

Hint.— To show that the conjugates of A in $\mathrm{SL}_n(K)$ generate $\mathrm{SL}_n(K)$, remark that A contains $B_{12}(1)$ and use appropriate parts of Problem 3.

(v) By means of Iwasawa's criterion conclude that for any $n > 2$, any normal subgroup of $\mathrm{SL}_n(K)$ is either equal to it or is contained in its center. Thus obtain the Jordan-Dickson Theorem for $n > 2$.

END

4 Week 4, Prof. Chanchal Kumar

Tutorial - II

Q1. Consider the standard action of $SL_2(\mathbb{C})$ on rays $\left\{ r\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} : r \geq 0 \right\}$ in $\mathbb{C}^2$. Prove that $SL_2(\mathbb{C})$ is homeomorphic to $SU_2 \times H$, where $H = \mathrm{Stab}\left(\left\{\begin{bmatrix} r \\ 0 \end{bmatrix} : r \geq 0\right\}\right)$ is the stabilizer subgp of the ray $\left\{\begin{bmatrix} r \\ 0 \end{bmatrix} : r \geq 0\right\}$.

Q2. Let $\mathcal{H}$ be the space of all Hermitian 2×2 matrices. Then show that $SL_2(\mathbb{C})$ acts on $\mathcal{H}$ by $(P, A) \longmapsto PAP^*$, $P \in SL_2(\mathbb{C})$ and $A \in \mathcal{H}$. Prove that the function

$$\langle A, B \rangle = \det(A+B) - \det(A) - \det(B) \quad ; \quad A, B \in \mathcal{H}$$

is a bilinear form on the real vector space $\mathcal{H}$. Use it to construct a homomorphism $\eta : SL_2(\mathbb{C}) \longrightarrow O_{3,1}$ with $\operatorname{Ker}\eta = \{\pm I\}$.

Q3. a) Describe all one-parameter subgp of $\mathbb{C}^*$.

b) Determine the differentiable homomorphisms from $\mathbb{C}^+$ to $SL_2(\mathbb{C})$.

Q4. Let $\varphi : (\mathbb{R}, +) \to GL_2(\mathbb{R})$ be a one-parameter subgp. Show that $\operatorname{Ker}\varphi$ is either $\{0\}$, or $\mathbb{R}$ or else an infinite cyclic subgp.

Q5. Find conditions on a matrix $A \in M_n(\mathbb{C})$ so that e^{tA} is a one-parameter subgp of SU_n. Compute $\dim(SU_n)$.

Q6. Let $G = \left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} : x, y \in \mathbb{R} \text{ with } x > 0 \right\}$. Determine $A \in M_n(\mathbb{R})$ such that e^{tA} is a one-parameter subgp of G. Also compute e^A for such matrices A.

Q7. Determine the one-parameter subgps of U_2.

Q8. Compute $(A+B\varepsilon)^{-1}$, assuming that A is invertible.

Q9. Let $G = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x, y \in \mathbb{R} \text{ with } xy = 1 \right\}$. Compute Lie algebra $\mathfrak{g}$ of G.

Q10. Determine the Lie algebra of the unitary group U_n.

(D)

Q11. Prove that $\det(I+A\varepsilon) = 1 + (\mathrm{Tr}(A))\varepsilon$. For an invertible matrix E, compute $\det(E+A\varepsilon)$.

Q12. Show that O_2 operates by conjugation on its Lie algebra. Further, the conjugation operation preserves the bilinear form

$$\langle A, B\rangle = \frac{1}{2}\mathrm{Tr}(AB), \quad A, B \in \mathcal{O}_2 = \text{Lie algebra of } O_2.$$

Use it, to define a homomorphism $\varphi : O_2 \longrightarrow O_2$ and describe φ explicitly.

Q13. Compute Lie algebra of the following groups.

a) U_n b) SU_n c) $O_{3,1}$ d) $SO_n(\mathbb{C})$.

14. Show that $\mathbb{R}^3$ becomes a 3-dimensional Lie algebra if bracket is defined by the cross product of vectors in $\mathbb{R}^3$. Let $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ & $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$, then

$$[v, w] = v \times w = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix} \in \mathbb{R}^3.$$

Further show that this Lie algebra is isomorphic to the Lie algebra of SO_3.

Q15. The (real) symplectic gp $SP_{2n}(\mathbb{R}) = \{P \in GL_{2n}(\mathbb{R}) : {}^tPJP = J\}$.

Determine the Lie algebra of $SP_{2n}(\mathbb{R})$. Also compute $\dim(SP_{2n}(\mathbb{R}))$.