Module 1 (Prof. Sholapurkar)

Summary of lectures

Lecture 1. Complex differentiation, Cauchy Riemann Equations , Sufficient condition for complex differentiability, difference between real and complex differentiability

Lecture 2. Power Series, radius of convergence, behaviour on boundary, examples. Exponential function. Definition of complex line integral

Lecture 3. Cauchy Goursat Theorem for triangle, existence of primitive of a holomorphic function on a disc. Cauchy integral formula

Lecture 4. Consequences of Cauchy Integral Formula: Power series representation of holomorphic functions, Zeros of holomorphic functions, Cauchy's Estimate, Liouville's Theorem

Lecture 5. Fundamental theorem of Algebra, Morera's Theorem, Convergence of sequence of holomorphic funtions on compact sets, convergence of integrals of holomorphic functions.

Lecture 6. Runge's theorem and illustrations.

Tutorial problems

1. Treating $\mathbf{C}$ as a 2-dimensional real vector space, consider a linear trans- formation $T : \mathbf{C} \to \mathbf{C}$ given by $T(X) = AX$ where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Show that T is a complex linear map if and only if $a = d$ and $b = -c$.

2. Consider the function defined by $f(x + iy) = |xy|$ for $x, y \in \mathbf{R}$. Show that f satisfies the Cauchy-Riemann equations at the origin, yet f is not holomorphic at 0.

3. Consider the $n - 1$ diagonals of a regular n-gon inscribed in a unit circle obtained by connecting 1 to all others. Show that the product of their lengths is n.

4. Let $f(z) = c_{00} + c_{10}x + c_{01}y + + c_{nm}x^n y^m$ be analytic in a domian D. Show that f is a polynomial in z.

5. If $f(z) = u(x, y) + iv(x, y)$ is analytic in C, then show that $|\nabla u| = |\nabla v| = |f'|$.

6. If $f(z) = u(x, y) + iv(x, y)$ is analytic in $\mathbf{C}$, then show that$|\nabla u|$ and $|\nabla v|$ are orthogonal vectors.

7. Consider the function f defined on $\mathbf{R}$

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ e^{-1/x^2} & \text{if } x > 0. \end{cases}$$

Prove that f is indefinitely differentiable on $\mathbf{R}$, and $f^(n)(0) = 0$ for all $n \geq 1$.

8. Prove that for a fixed w in the unit disc $\mathbf{D}$, the mapping $F(z) = (w - z)/(1 - \overline{w}z)$ satisfies the following conditions:

(a) F maps the unit disc to itself and is holomorphic.
(b) F interchanges 0 and w
(c) $|F(z)| = 1$ if $|z| = 1$.

(d) F is bijective.

9. Determine the radius of convergence of the series ??$\sum a_n z^n$ when:

(a) $a_n = (\log n)^2$.
(b) $a_n = n!$.
(c) $a_n = n^2/(4^n + 3n)$.

10. Let f be a power series centered at the origin. Prove that f has a power series expansion around any point in its disc of convergence.

11. Prove the following:

(a) The power series $\sum nz^n$ does not converge on any point of the unit circle.
(b) The power series $\sum z^n/n^2$ converges at every point of the unit circle.
(c) The power series $\sum z^n/n$ converges at every point of the unit circle except at $z = 1$.

12. Show that for $|z| < 1$,

$$\frac{z}{1-z^2} + \frac{z^2}{1-z^4} + \cdots + \frac{z^{2^n}}{1-z^{2^n+1}} = \frac{z}{1-z}$$
$$\frac{z}{1+z} + \frac{2z^2}{1+z^2} + \cdots + \frac{2^n z^{2^n}}{1+z^{2^n}} = \frac{z}{1-z}$$

13. (a) Evaluate the integrals

$$\int_\gamma z^n dz$$

for all integers n. Here γ is any circle centered at the origin with the positive (counterclockwise) orientation.

(b) Same question as before, but with γ any circle not containing the origin.
(c) Show that if $|a| < r < |b|$, then

$$\int_\gamma \frac{1}{(z-a)(z-b)}\, dz = \frac{2\pi i}{a-b},$$

where γ denotes the circle centered at the origin, of radius r, with the positive orientation.

14. Suppose f is continuously complex differentiable on Ω and $T \subset \Omega$ is a triangle whose interior is also contained in Ω. Apply Green?s theorem to show that $\int_T f(z)\, dz = 0$.

15. Let Ω be an open subset of $\mathbf{C}$ and and $T \subset \Omega$ is a triangle whose interior is also contained in Ω. Suppose f is a function holomorphic in Ω except possibly at a point w inside T. Prove that if f is bounded near w, then $\int_T f(z) dz = 0$.

16. Suppose $f : D \to \mathbf{C}$ is holomorphic. Show that the diameter $d = \sup |f(z) - f(w)|$ of the image of f satisfies $2|f'(0)| \leq d$. Moreover, it can be shown that equality holds precisely when f is linear.

17. Let Ω be a bounded open subset of $\mathbf{C}$ and $f : \Omega \to \Omega$ a holomorphic function. Prove that if there exists a point $z_0 \in \Omega$ such that $f(z_0) = z_0$ and $f'(z_0) = 1$ then f is linear.

18. Can every continuous function on the closed unit disc be approximated uniformly by polynomials in the variable z?

19. Suppose f is an analytic function defined everywhere in $\mathbf{C}$ and such that for each z_0 in $\mathbf{C}$, at least one coefficient in the expansion

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

is equal to 0. Prove that f is a polynomial.

20. Suppose f is a non-vanishing continuous function on $\mathbf{D}$ that is holomorphic in $\mathbf{D}$. Prove that if $|f(z)| = 1$ whenever $|z| = 1$, then f is constant.