Module 2 (Dr. Diganta Borah)

Summary of lectures

Lecture 1. Stereographic projection and distance function on the extended complex plane, Riemann sphere, Möbius transformations.

Lecture 2. Preservation of circles, Cross ratios, Mapping properties of Möbius transformations.

Lecture 3. Conformal maps and connection with holomorphy, harmonic functions.

Lecture 4. Harmonic conjugates, existence and construction of harmonic conjugates on convex domains, mean value theorem, maximum principle, characterisation of harmonic functions.

Lecture 5. Dirichlet problem on discs, Poisson kernel, Poisson integral formula, Geometric interpretation of Poisson integrals

Lecture 6. Schwarz reflection principle, symmetric points with respect to a circle, reflection principle for circles.

Tutorial problems

1. For each of the following points in $\mathbf{C}$, give the corresponding point of S^2: 0, $1+i$, $3+2i$.

2. If $z, w \in \mathbf{C}$, prove that

$$d_\infty(z,w) = \frac{2|z-w|}{\sqrt{(|z|^2+1)(|w|^2+1)}}.$$

Conclude that $z \to w$ in $\mathbf{C}$ if and only if $z \to w$ on $\hat{\mathbf{C}}$.

3. If $z \in \mathbf{C}$, then prove that

$$d_\infty(z,\infty) = \frac{2}{\sqrt{1+|z|^2}}.$$

4. If $R \geq 0$, prove that the set $\{z \in \mathbf{C} : |z| > R\} \cup \{\infty\}$ is an open ball in the Riemann sphere with center ∞. What is the radius of this ball?

5. Let Λ be a circle lying on S^2 (i.e., the intersection of a plane and S^2). If Λ contains the north pole, prove that its projection on $\mathbf{C}$ is a straight line. Otherwise, Λ projects onto a circle in $\mathbf{C}$.

6. If

$$f(z) = \frac{z+2}{z+3} \quad \text{and} \quad g(z) = \frac{z}{z+1}$$

find $f \circ g$, $g \circ f$ and $f^{-1} \circ g$.

7. Show that any Möbius transformation which maps the real axis into itself can be written with real coefficients.

8. Prove that $\mathrm{Aut}(\hat{\mathbf{C}}) \cong PGL(2,\mathbf{C})$.

9. Prove that (z_0, z_1, z_2, z_3) is real if and only if $z_0, \ldots, z_3$ lie on a circle.

10. Prove that cross ratios are invariant under Möbius transformations.

11. Let
$$f(z) = \frac{z-i}{z+i}.$$
Determine the image of real axis and the upper half plane.

12. Find the Möbius transformation which maps $1+i, 2, 0$ to $0, \infty, i-1$ respectively. Determine the image of the circle $|z-1| = 1$ and the disc $|z-1| < 1$.

13. Fix a complex number ζ with $|\zeta| < 1$. Prove that

(a) $\left|\dfrac{z-\zeta}{1-\bar{\zeta}z}\right| = 1$ if $|z| = 1$, (b) $\left|\dfrac{z-\zeta}{1-\bar{\zeta}z}\right| < 1$ if $|z| < 1$.

14. Find a formula for symmetric point with respect to a circle with center at a and radius r.

15. Find a formula for symmetric point with respect to the line $ax + by + c = 0$.

16. Prove that Möbius transformations preserve symmetric points.

17. Prove that any Möbius transformation which maps the open unit disc onto itself is of the form
$$e^{i\theta}\frac{z-\zeta}{1-\bar{\zeta}z}$$
for some $|\zeta| < 1$ and $\theta \in \mathbf{R}$.

18. Find all the Möbius transformation that fix the points 1 and -1. Is the group of those transformations abelian?

19. Let α, β be real numbers such that $\beta - \alpha < 2\pi$.
(a) Find the image of the horizontal strip $\{z : \alpha < \Im z < \beta\}$ under e^z.
(b) Find the image of the wedge $\{z : \alpha < \arg z < \beta\}$ under the $\log z$.

20. Find a conformal map
(a) of the wedge $\{0 < \arg z < \beta\}$, $0 < \beta \leq 2\pi$, onto the upper half plane $\{\Im z > 0\}$.
(b) of the upper half plane $\{\Im z > 0\}$ onto the unit disc $\mathbf{D}$.
(c) of the wedge $\{0 < \arg z < \beta\}$, $0 < \beta \leq 2\pi$ onto the unit disc $\mathbf{D}$.

21. Let D be a domain in $\mathbf{C}$ and $f : D \to \mathbf{C}$ be a C^1 function. Suppose the (real) Jacobian matrix of f does not vanish at any point of D. Also, suppose f maps orthogonal curves to orthogonal curves. Prove that either f or $\overline{f}$ is complex differentiable on D.

22. Show that the Cayley transform $h_C : \mathbf{H} \to \mathbf{D}$, $z \to \frac{z-i}{z+i}$, where $\mathbf{H}$ is the upper half space and $\mathbf{D}$ is the unit disc, maps the first quadrant
$$Q_1 = \{z \in \mathbf{H} : \Re z > 0\}$$
bijectively onto $\{w \in \mathbf{D} : \Im w < 0\}$.

23. Supply complex differentiable, bijective and angle-preserving mappings of Q_1 onto $\mathbf{D} \setminus (-1, 0]$ and onto $\mathbf{D}$.

24. If f is a harmonic function on an open subset Ω of $\mathbf{C}$, prove that $\partial f/\partial z$ is holomorphic on Ω.

25. If f is a nonvanishing holomorphic function on an open subset Ω of $\mathbf{C}$, prove that $\log|f|$ is harmonic on Ω.

26. Find where in the complex plane the following functions are harmonic and express them as real parts of holomorphic functions if possible.

(a) $xy + 3x^2y - y^3$ (b) $e^{x^2-y^2}\cos(2xy)$ (c) $\dfrac{x}{x^2+y^2}$, $(x, y) \neq 0$ (d) $\tan^{-1}\dfrac{y}{x}$, $x > 0$

27. Let $v(z) = \text{Im}\,(1/z^2)$ if $z \neq 0$ and $v(0) = 0$. Prove that v_{xx} and v_{yy} exist at all points in $\mathbf{C}$ and satisfy
$$v_{xx} + v_{yy} = 0,$$
but v is not harmonic in $\mathbf{C}$.

28. Show that if $h(z)$ is a harmonic function on a domain D such that $zh(z)$ is also harmonic, then prove that $h(z)$ is holomorphic on D.

29. Show that if v is a harmonic conjugate of u, then u is also a harmonic conjugate of $-v$.

30. Give an example of a smooth function $u(z)$ on the unit disc $\mathbf{D}$ which attains its maximum at an interior point.

31. Prove that the series

$$P_r(\theta) = \sum_{k=-\infty}^{\infty} r^{|k|} e^{ik\theta}$$

converges uniformly if $re^{i\theta}$ lies on a compact subset of $\mathbf{D}$.