PROBLEM SHEET-I

IST, Functional Analysis, Bhaskaracharya Pratishthana, Pune (Oct.23-Nov.4, 2017)

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1. Given a finite family of Banach spaces $\{X_i; 1 \leq i \leq n\}$, show that the product

$$X = \prod_{i=1}^{n} X_i = \left\{x = (x_i)_{i=1}^{n};\ x_i \in X_{i,} 1 \leq i \leq n\right\}$$

is a Banach space when equipped with the norm:

$$\|x\| = \sum_{i=1}^{n} \|x_i\|\,, x = (x_i)_{i=1}^{n} \in X.$$

2. Show that a normed space X is complete if and only if absolutely convergent series in X are convergent.

3. Compute $\|f + g\|$ in C[0, 1] where

$$f(x) = sin\pi x, g(x) = cos\pi x.$$

4. Show that it is possible that two metrics on a set are equivalent whereas one of them is complete and that the other is incomplete.

5. Show that the closure of a subspace/convex subset of a normed space is a subspace/convex subset. How about the convex hull of a closed set?

6. Prove that a linear functional on a normed space is continuous if and only Kerf is a closed subspace of X.

7. For $1 \leq p < q < \infty$, show that $\ell_p \subsetneq \ell_q$.

PROBLEM SHEET-II

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1. Give an example of a discontinuous linear map between Banach spaces.

2. Prove that two equivalent norms are complete or incomplete together.

3. Show that a normed space is finite dimensional if all norms on it are equivalent.

4. Given that a normed space X admits a finite total subset of X*, can you say that it is finite dimensional?

5. What can you say about a normed space if it is topologically homeomorphic to a a finite dimensional normed space.

6. Show that the uniform limit of a sequence of polynomials of degree at most n is a polynomial of degree at most n. Can you say the same for polynomials of the same degree?

7. Let $C^1[0,1]$ be the space of all continuously differentiable functions on [0, 1] and determine if the set S is closed where S is given by

$$S = \{f \in C^1[0,1] : f'(1) = 0\}.$$

PROBLEM SHEET-III

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1. Show that there exist discontinuous linear functional on each infinite dimensional normed linear space.

2. Show that c_0 is closed as a subspace of ℓ_∞.

3. Prove that the sum of a closed subspace and a finite dimensional subspace of a normed space is always closed.

4. What happens if in Ex.3 above, the assumption of finite dimensionality is dropped?

5. Show that the conclusion of Ex.3 still holds for closed *subsets*, with one of these assumed to be compact?

6. Prove that a finite co-dimensional subspace of a normed space is always complemented.

7. Derive the analytic form of the Hahn Banach theorem from the geometric form.

PROBLEM SHEET-IV

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1. Check if the 'sup'-norm $\| \quad \|_\infty$ on P[a, b], the space of all polynomials on [a, b] is complete. Does there exist a complete norm on this space?

2. Let $T_{\mathfrak{J}}$ be the projective topology on X induced by a family $\mathfrak{J}$ of mappings $f_i: X \to Y_i, i \in \Lambda$. Given a topological space Z and a map $g: Z \to (X, T_{\mathfrak{J}})$, show that g is continuous if and only if $f_i o g: Z \to Y_i$ is continuous for each $i \in \Lambda$.

3. Prove that the projective topology is the weakest topology rendering each $f_i: X \to Y_i$ continuous for each $i \in \Lambda$.

4. For a normed linear space X, show that a subset A ⊊ X is weakly bounded if and only if f(A) is bounded for each $f \in X^*$.

5. Let X be a vector space and $f, f_i, 1 \leq i \leq n$ be linear functional on X. Show that

$$f = \sum_{i=1}^{n} \alpha_i \, f_i \text{ if and only if } \bigcap_{i=1}^{n} Ker(f_i) \subsetneq Ker(f).$$

6. Show that for a normed space X, $(X, weak)^* = X^*$.

7. Prove that a weakly open set in a normed space cannot be bounded.

8. Show that the weak topology on an infinite dimensional normed space can never be metrizable. Show that the same is true for the weak*- topology.

PROBLEM SHEET-V

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1. $Let \; \| \quad \|_1$ and $\| \quad \|_2$ be complete norms defined on a vector space X such that $\|x\|_1 \leq \|x\|_2, for\ all\ x \in X$. Show that $\| \quad \|_1$ and $\| \quad \|_2$ are equivalent.

2. Show that the space $C^1[0,1]$ consisting of continuously differentiable functions on [0,1] is incomplete with respect to the sup-norm:

$$\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|.$$

   Further check whether the norm: $\|f\|_0 = \|f\|_\infty + \|f'\|_\infty$ is equivalent to the above norm.

3. Show that a closed subspace Y of a Banach space X is complemented if and only if there exists a closed subspace Z of X such that $X = Y \oplus Z$.

4. Let X, Y and Z Banach spaces, $T: X \to Y\ and\ S : Y \to Z$ be an injective continuous linear map. Does it follow that T is continuous if ST is continuous.

5. Assuming the open mapping theorem, deduce the closed graph theorem.

6. Given a closed subspace Y of a normed space X, show that the quotient map from X onto the quotient space of X by Y is an open map, i.e., maps open sets onto open sets.