IST on Fourier analysis :Problem session - 01

1. Let f be a 2π-periodic Riemann integrable function on R.
(a) Show that the Fourier series of the function f can be written as
$$f(x) \sim \widehat{f}(0) + \sum_{n=1}^{\infty} (\widehat{f}(n) + \widehat{f}(-n)) \cos nx + \sum_{n=1}^{\infty} i(\widehat{f}(n) - \widehat{f}(-n)) \sin nx.$$
(b) Prove that if f is even, then $\widehat{f}(n) = \widehat{f}(-n))$ and we get a cosine series.
(c) Prove that if f is odd,$\widehat{f}(n) = -\widehat{f}(-n))$ then and we get a sine series.
(d) Suppose that $f(\pi + x) = f(x)$ for all $x \in \mathbb{R}$. Show that $\widehat{f}(n) = 0$ for all odd n.
(e) Show that f is real-valued if and only if $\overline{\widehat{f}(n)} = \widehat{f}(-n)$.

2. Suppose f is 2π-periodic and integrable on any fnite interval.Prove that if $a, b \in \mathbb{R}$, then show that $\int_a^b f(x)dx = \int_{a+2\pi}^{b+2\pi} f(x)dx = \int_{a-2\pi}^{b-2\pi} f(x)dx.$
Also prove that $\int_{-\pi}^{\pi} f(x+a)dx = \int_{-\pi}^{\pi} f(x)dx = \int_{a-\pi}^{a+\pi} f(x)dx.$

3. Consider the 2π-periodic odd function defined on $[0, \pi]$ by $f(x) = f(\pi - x)$.
(a) Draw the graph of f .
(b) Compute the Fourier coefficients of f , and show that $f(x) = \frac{8}{\pi} \sum_{k \geq 1, odd} \frac{\sin kx}{k^3}$

4. Let f be the function defined on $[-\pi, \pi]$ by $f(x) = |x|$.
(a) Draw the graph of f .
(b) Calculate the Fourier cofficients of f , and show that $\widehat{f}(n) = O(1/n^2)$.
(c) What is the Fourier series of f in terms of sines and cosines?
(d) Taking $x = 0$, prove that $\sum_{n \geq 1, odd} \frac{1}{n^2} = \frac{\pi^2}{8}, \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$

5. Suppose f is a periodic function of period 2π which belongs to the class C^k. Show that $\widehat{f}(n) = O(1/|n|^k)$ as $|n| \to \infty$.

6. Give an example on discontinous function whoes Fourier series converges for all x.

7. Give example of a function whoes Fourier series converges for all x ,but does not converge absolutely.

8. Prove that the Fejer kernel is given by $F_N(x) = \frac{\sin^2(Nx/2)}{N sin^2(x/2)}$.

IST on Fourier analysis :Problem session - 02

1. Let $f(\theta) = \sum_{n=-N}^{N} a_n e^{in\theta}$ .Prove that $\widehat{f}(n) = a_n$

2. Verify the following properties of convolution operation $*$

a) $f * g = g * f$

b) $f * (g + h) = f * g + f * h$

c) $cf * g = c(f * g)$

d) $f * (g * h) = (f * g) * h$

e) $\widehat{f * g}(n) = \widehat{f}\widehat{g}$

3. Verify the following properties of $f_n(x) = e^{inx} \quad \forall \ x \in \mathbb{R}$

a) f_n is a periodic function of period 2π.

b) $\frac{1}{2\pi}\int_{-\pi}^{\pi} f_n(x)dx = 1$ if $n = 0$ and it is 0 otherwise.

c) $\frac{1}{2\pi}\int_{-\pi}^{\pi} f_n(x)\overline{f_m}(x)dx = \delta_{mn}$

Here $\delta_{mn} = 1$ if $m = n$ and it is 0 otherwise.

d) Prove that $\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = 1$ if $|m| = |n|$ and 0 otherwise.

e) Prove that $\frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = 1$ if $|m| = |n|$ and 0 otherwise.

4. Suppose f is 2π periodic and integrable on any finite interval .

Prove that if $a, b \in \mathbb{R}$ then show that

a) $\int_a^b f(x)dx = \int_{a+2\pi}^{b+2\pi} f(x)dx = \int_{a-2\pi}^{b-2\pi} f(x)dx.$

b) Also prove that $\int_{-\pi}^{\pi} f(x+a)dx = \int_{-\pi}^{\pi} f(x)dx = \int_{a-\pi}^{a+\pi} f(x)dx.$

5. Consider the 2π periodic odd function defined on $[0, \pi]$ defined by

$f(x) = x(\pi - x)$

a) Draw the graph of f.

b) Find $\widehat{f}(n)$ and show that

$f(x) = \frac{8}{\pi}\sum_{k\geq 1, odd} \frac{\sin kx}{k^3}.$

6. Sawtooth Function is defined as

$f(x) = i(\pi - x) \quad if \quad 0 \leq x \leq \pi; \quad f(x) = -i(\pi + x) \quad if \quad -\pi \leq x < 0.$

Prove that $\widehat{f}(n) = \frac{1}{n}$ for all $n \neq 0$ and $\widehat{f}(0) = 0$.

7. Let f be a 2π periodic Reimann integarble function defined over $\mathbb{R}$.

Prove that the following.

Show that $f(x) \sim \widehat{f}(0) + \sum_{n=1}^{\infty}(\widehat{f}(n) + \widehat{f}(-n))\cos nx + \sum_{n=1}^{\infty} i(\widehat{f}(n) - \widehat{f}(-n))\sin nx,.$

8. Let f be a 2π periodic Reimann integarble function defined over $\mathbb{R}$.

Prove that the following.

a) If f is even then $\widehat{f}(n) = \widehat{f}(-n)$.

b) If f is odd then $\widehat{f}(n) = -\widehat{f}(-n)$.

c) If $f(\pi + x) = f(x)$ for all $x \in \mathbb{R}$ then show that $\widehat{f}(n) = 0$ for all odd n.

d) If f is real valued then show that $\overline{\widehat{f}(n)} = \widehat{f}(-n)$.

IST on Fourier analysis :Problem session - 03

1. Let us consider $\sum_{n=0}^{\infty} a_n$. Let $(s_n)_{n=1}^{\infty}$ be sequence of its partial sums.

   Let $\sigma_N = \frac{s_1+s_2+...+s_{N-1}}{N}$ .We say $\sum_{n=0}^{\infty} a_n$ is Cesaro summable to L if $\sigma_N \longrightarrow L$ as $N \longrightarrow \infty$.

   If $\sum_{n=0}^{\infty} a_n$ converges L then show that $\sum_{n=0}^{\infty} a_n$ is Cesaro summable to L.
   Give a counterexample where converse is not true.

2. Let us consider $\sum_{n=0}^{\infty} a_n$. Let $A(r) = \sum_{n=0}^{\infty} a_n r^n$ converges $0 \le r < 1$.

   We say that $\sum_{n=0}^{\infty} a_n$ is Abel summable to s if $\lim_{r \to 1^-} A(r) = s$. If $\sum_{n=0}^{\infty} a_n$ is Cesaro summable to s then it is Abel summable to s.
   Give counter-example where converse is not true.

3. The poisson Kernel is given by $P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$.

   Show that $P_r(\theta) = \dfrac{1-r^2}{1-2r\cos\theta + r^2}$ if $0 \le < 1$.
   Also show that poisson Kernel is good Kernel.

4. Let $f = \chi_{[a,b]}$ be characteristic function of $[a,b]$. Show that Fourier series of f is given by $f(x) \sim \dfrac{b-a}{2\pi} + \sum_{n \neq 0} \dfrac{e^{-ina} - e^{-inb}}{2\pi i n} e^{inx}$.
   a) If $a \neq -\pi$ or $b \neq \pi$ and $a \neq b$ then show that Fourier series does not converge absolutely for any x.
   Also prove that ,Fourier series converges at any point x.
   b) What happens if $a = -\pi, \;\; b = \pi$ Note that Fourier series converges means its sequence of symmetric partial sums converges.

5. Suppose f is 2π periodic function which belongs to class C^k .Show that $\widehat{f}(n) = O(\frac{1}{|n|^k})$ as $|n| \to \infty$.

6. Let f be 2π periodic function defined by $f(\theta) = |\theta|$
   Show that $\widehat{f}(0) = \frac{\pi}{2}$ and $\widehat{f}(n) = \frac{-1+(-1)^n}{\pi n^2} \quad if \quad n \neq 0$.
   Also write Fourier series of f in terms of sin and cos .
   Putting $\theta = 0$ in Fourier series we get $\sum_{n \ge 1, odd} \dfrac{1}{n^2} = \dfrac{\pi^2}{8}$.

7. Do similar analysis of the 2π periodic function $f(\theta) = \frac{(\pi-\theta)^2}{4}$, $0 \le \theta \le 2\pi$.
   Hence conclude that $\sum_{n \ge 1} \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$.

8. Use Ex 6 and Ex 7 and Parseval's identity to conclude that
$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$ and $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

IST on Fourier analysis :Problem session - 04

1. Consider the sequence $(a_n)_{n=-\infty}^{\infty}$ defined as $a_n = \frac{1}{n} \quad if \quad n \geq 1$, otherwise.
   Show that there is no Riemann integrable function f such that $\widehat{f}(n) = a_n$ for all n.

2. Give an example of a funcion whose Fourier series converges for all $x \in \mathbb{R}$,but does not converge absolutely.

3. Let f is 2π periodic continuous function .Prove that Fourier series of f converges to f.

4. Construct the sequence of Reimann integrable functions $(f_k)_{k=1}^{\infty}$ on $[0, 2\pi]$ such that
   $\lim_{k\to\infty} \int_0^{2\pi} |f_k(\theta)|^2 d\theta = 0$,but $\lim_{k\to\infty} f_k(\theta)$ fail to exist for any θ.

5. Let us consider vector space $R[0, 2\pi] = all\ Riemann\ integrble\ \ functions\ \ on\ \ [0, 2\pi]$.We Define Equivalence relation on this space as below $f \sim g$ iff $f = g$ almost everywhere on $[0, 2\pi]$.
   Show that $<,>$ defined by $< f, g >= \frac{1}{2\pi} \int_0^{2\pi} f(x)\overline{g(x)}dx$ is inner product on space of these equivalence classes.

6. Let Dirichlet's Kernel is given by $D_N(\theta) = \sum_{k=-N}^{N} e^{ik\theta}$. Show that $D_N(\theta) = \frac{\sin(\frac{N+1}{2}\theta)}{\sin(\frac{\theta}{2})}$.
   Show that D_N is not a good Kernel.

7. Show that $\int_{-\pi}^{\pi} D_N(\theta)d\theta = 2\pi$ .Hence By Using Riemann Lebesgue Lemma prove that $\int_0^{\infty} \frac{sin\theta}{\theta} d\theta = \frac{\pi}{2}$.

8. Prove that the Fourier series of continuously differentiable function f on the circle is absolutely convergent.

9. Let f is 2π periodic Reimann integrable unction on $[-\pi, \pi]$ .If f satisfiy
   $|f(x+h) - f(x)| \leq C|h|^\alpha$ for some $0 < \alpha \leq 1$, some $C > 0$ and all x, h .
   Show that $\widehat{f}(n) = O(\frac{1}{|n|^\alpha})$.

IST on Fourier analysis :Problem session - 05

1. If f is bounded monotonic function on $[-\pi, \pi]$ then show that $\widehat{f}(n) = O(\frac{1}{|n|})$.

2. Show that Fourier coefficeints of a continuous function can tend to 0 arbitrary slowly by proving that for any sequence of nonnegative real numbers $(\epsilon_n)_{n=1}^{\infty}$ converging to 0 ,there exists a continuous function f such that $|\widehat{f}(n)| \geq \epsilon_n$ for infinitely many values of n.

3. Prove that the sequence $(\gamma_n)_{n=1}^{\infty}$,where γ_n is fractional part of $(\frac{1+\sqrt{5}}{2})^n$ is not equidistributed in $[0, 1]$

4. Let $\gamma \in \mathbb{R} - \mathbb{Q}$ then prove that the sequence $\{< n\gamma >\}_{n=1}^{\infty}$ is equidistributed in $[0, 1)$ .Here $< x >$ denotes the fractional part of x.

5. Show that the sequence $0, \frac{1}{2}, 0\frac{1}{3}, \frac{2}{3}, 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 0, \frac{1}{5}, \frac{2}{5}...$ is equidistributed in $[0, 1)$.

6. Suppose f is a periodic function on $\mathbb{R}$ of period 1,and $(a_n)_{n=1}^{\infty}$ is sequence which is eqidistributed in $[0, 1)$
   Prove that:If f continuous and satisfies $\int_0^1 f(x)dx = 0$ then $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x + a_n) = 0$ uniformly in x.

7. Suppose f is a periodic function on $\mathbb{R}$ of period 1,and $(a_n)_{n=1}^{\infty}$ is sequence which is eqidistributed in $[0, 1)$
   Prove that:If f is Reimann integrable on $[0, 1]$ and satisfies $\int_0^1 f(x)dx = 0$ then
   $\lim_{N\to\infty} \int_0^1 |\frac{1}{N}\sum_{n=1}^{N} f(x + a_n)|^2 dx = 0$.

8. Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers equidistributed in $[0, 1)$ then show that for all $k \in \mathbb{Z} - \{0\}$, $\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k a_n} \longrightarrow 0$ as $N \longrightarrow \infty$

IST on Fourier analysis :Problem session - 06

1. Let $f \in L^1(\mathbb{R})$ and $\widehat{f}(s) = \int_{\mathbb{R}} f(x)e^{-2\pi i x s}dx$.
   Prove that
   (a)$\widehat{\tau_\alpha f}(s) = e^{2\pi i s\alpha}\widehat{f}(s)$, where $\tau_\alpha f(x) = f(x + \alpha)$.
   (b) $\widehat{e^{2\pi\eta} f}(s) = \widehat{f}(s - \eta)$.
   (c)$\widehat{D_a f}(s) = \widehat{f}(as)$,where $D_a(f)(x) = \frac{1}{a}f(\frac{x}{a})$

2. Let $f, g \in L^1(\mathbb{R})$ .Prove that $(f * g)(x) = \int_{\mathbb{R}} f(y)g(x - y)dy = \int_{\mathbb{R}} f(x - y)g(y)dy$

3. Let $f(x) = e^{-\pi x^2}$ on $\mathbb{R}$ then show that $\widehat{f}(s) = e^{-\pi s^2}$.

4. Show that if $f \in L^p(\mathbb{R})$, $\quad g \in L^{p'}(\mathbb{R})$ then show that $f * g$ is continuous.

5. Let $f, g \in C_c(\mathbb{R})$ .Show that $\tau_x(f * g) = \tau_x(f) * g = f * \tau_x(g)$.

6. Let $T : L^p(\mathbb{R}) \longrightarrow L^p(\mathbb{R})$ be bounded linear translation invariant operator.Show that $T(f * g) = T(f) * g = f * T(g)$.

7. Let $a, b\imath l_1(\mathbb{Z})$ .Define $(a * b)(n) = \sum_{m\in\mathbb{Z}} a(m)b(n - m)$. Show that $a * b \in l_1(\mathbb{Z})$ .Adress the problem of existence of identity with respect to $*$.

8. Is there exist identity with respect to convolution in $L^1(\mathbb{R})$? Give justification.

IST on Fourier analysis :Problem session - 07

1. Give an example of function which is of moderate decrease but not in Schwartz space.
2. Give an example of function which is decrese rapidly at infinity but not in Schwartz space.
3. Let f is continuous and of Moderate decrease then prove that $\widehat{f}$ is continuous and $\widehat{f}(s) \to 0$ as $s \to \infty$.
4. Let f is continuous and of Moderate decrease then prove that $\widehat{f}$ is continuous and if $\widehat{f}(s) = 0$ for all s then prove that f is identically 0.
5. Prove that convolution of two moderate decrease functions is of moderate decrease.
6. Give example of complactly supported functions in Schwartz space $S(\mathbb{R})$.
7. Show that Fourier transform of a function of Moderate decrease is need not be f Moderate decrease.

IST on Fourier analysis :Problem session - 08

1. Let $1 < p < 2$ and $f \in L^p(\mathbb{R})$ .Let $A = \{x \mid \ |f(x)| > 1\}$.
Show that $f_1 = f\chi_A \in L^1(\mathbb{R})$ and $f_2 = f\chi_{\mathbb{R}-A} \in L^2(\mathbb{R})$ and $f = f_1 + f_2$.
2. Find maximal function of $\chi_{[a,b]}$.
3. Let f be a function on circle T .For ech $N \geq 1$ the discrete Fourier corfficients of f are defined as $a_N(n) = \dfrac{1}{N}\displaystyle\sum_{k=0}^{N-1} f(e^{2\pi ik/N})e^{-2\pi ik/N}$ or all $n \in \mathbb{Z}$. Also let $a(n) = \displaystyle\int_0^1 f(e^{2\pi ix})e^{-2\pi ix}dx$.Show that $a_N(n) = a_N(n+N)$ .Also show that $a_N(n) \longrightarrow a(n)$ as $N \longrightarrow \infty$
4. Let $\lambda > 0$. Show that $|\{x \in B \ : \ M(f)(x) > 2\lambda\}| \leq \dfrac{1}{\lambda}\displaystyle\int_{\{x\in B \ : \ |f(x)|>\lambda\}} |f(x)|$
Here B is finite ball in $\mathbb{R}^n$ and $M(f)$ is maximal function of f.

IST on Fourier analysis :Problem session - 08

1. Let $g \in C^\infty(\mathbb{R})$ and $u(x,t) = \displaystyle\sum_{m=0}^{\infty} g^{(m)}(t)\frac{x^{2n}}{(2n)!}$.
Verify that $u_t = u_{xx}$.
2. Let $a > 0$. Let $g(t) = e^{-t^{-a}}$ if $t > 0$ and 0 otherwise. Show that $g \in C^\infty(\mathbb{R})$.
Prove that there exist $\theta \in (0,1)$ such that $|g^{(k)}(t)| \leq \dfrac{k!}{\theta t}e^{-\frac{1}{2t^a}}$.
Also show that if $t > 0$ then $\displaystyle\sum_{n=0}^{\infty} g^{(n)}(t)\frac{x^{2n}}{(2n)!}$ converges for all $x \in \mathbb{R}$.
3. Let P_D is poisson Kernel for unit disk D .It is given by $P_D(r,\theta) = Real\ \ part\ \ of\ \ (\frac{1-z}{1+z})$ .Let $\mathbb{H}$ denote upper half plane . We know that $\phi(z) = \frac{z-i}{z+i} : \mathbb{H} \to D$ is Caley map. Use it to find Poisson Kernel $P_{\mathbb{H}}$ on $\mathbb{H}$.

4. Solve the Laplace's equation $u_{xx} + u_{yy} = 0$ for semiinfinite strip $0 < x < 1, 0 < y < \infty$. subject to boundary conditions
   $u(0, y) = 0 \quad if \;\; 0 \leq y < \infty$, $u(1, y) = 0 \quad if \;\; 0 \leq y < \infty$, $u(x, 0) = f(x) \quad if \;\; 0 \leq x \leq 1$.

5. Let r, θ are polar coordinates and x, y are rectangular coordinates .
   Show that $u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$ for any twice differentiable function.