Report on Module I, Field Theory

There were four lectures of one and half hour duration and two tutorials of two hours each. The following topics were covered during the week 20^{th} to 25^{th} June in Field Theory (Module I).

1. Examples of fields, review of field of fractions of an integral domain, quotient of a commutative ring by a maximal ideal, quotients of polynomial rings, rational function fields, prime fields, characteristic of a field; field extensions, algebraic and transcendental elements, finite extensions, transitivity of degree, sum and product of algebraic elements are algebraic. Quadratic extensions.

2. Classical problems of Greek geometry, ruler and compass construction; converting the problem to algebra, constructible complex numbers, criterion for a complex number to be constructible in terms of square root tower, impossibility of squaring a circle, duplicating a cube and trisecting an arbitrary angle.

3. Construction of regular n-gons, Gauss' criterion, partial proof and examples. Transitivity of algebraic extensions, splitting field of a polynomial, construction of finite fields.

4. Existence of algebraic closure (Artin's proof), isomorphism of field extensions, uniqueness of finite fields.

The tutorial sheets were distributed in advance. Student participation in the tutorials (and lectures) was good.

Parvati Shastri.

Tutorials: I & II

AFS III, Algebra, Module I

(Field Extensions)

20^{th} June 2016

IISER, Trivendrum

Parvati Shastri
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N. B. Through out the exercises, K denotes a field, unless otherwise stated.

1. Let K_1, K_2 be fields. Prove that any homomorphism $f : K_1 \to K_2$ is injective.

2. Let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the field of p elements, where p is a prime. For all odd primes p, prove that $\mathbb{F}_p^*/\mathbb{F}_p^{*^2}$ is a group of order 2. Can you describe all quadratic extensions of $\mathbb{F}_p$?

3. Let K be a field of characteristic p. Prove that $(x+y)^p = x^p + y^p$.

4. Find the minimal polynomial of $\sqrt{2} - \sqrt{3}$ over $\mathbb{Q}$ and also find all the roots of this polynomial. Do the same over K, where $K = \mathbb{R}, \mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_5$ respectively.

5. Determine the degree $\left[\mathbb{Q}\left(\sqrt{3+2\sqrt{2}}\right) : \mathbb{Q}\right]$.

6. Let p be an odd prime. Prove that the polynomial
$$X^{p-1} + X^{p-2} + \cdots + X + 1$$
is irreducible over $\mathbb{Q}$. Is it irreducible over $\mathbb{R}$? Compute the degree $[\mathbb{Q}(\zeta) : \mathbb{Q}]$, where ζ is a primitive p^{th} root of unity, p being a prime number.

7. Let $\omega = e^{\frac{\pi i}{6}}$. Prove that $[\mathbb{Q}(\omega) : \mathbb{Q}] = 4$.

8. Prove that $\mathbb{Q}(\zeta_5)$ is not a subfield of $\mathbb{Q}(\zeta_7)$, where $\zeta_n = e^{\frac{2\pi i}{n}}$.

9. Determine whether the element $i = \sqrt{-1}$ is in the field K or not, in case (i) $K = \mathbb{Q}(\sqrt{2})$, (ii) $K = \mathbb{Q}(\sqrt[4]{-2})$ and (iii) $K = \mathbb{Q}(\alpha)$, where $\alpha^3 + \alpha + 1 = 0$.

10. Prove or disprove: The fields $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are isomorphic.

11. Let $L = K(\alpha)$ where α is algebraic of odd degree (i.e., minimum polynomial of α is of odd degree). Prove that $L = K(\alpha^2)$.

12. Let R be an integral domain containing a field K. Suppose that the dimension of R as a vector space over K is finite. Prove that R is a field.

13. Let $K = \mathbb{Q}(u)$, where $u^3 - u^2 + u + 2 = 0$. Express $(u^2 + u + 1)(u^2 - 1)$ and $(u - 1)^{-1}$ in the form $au^2 + bu + c$, where $a, b, c \in \mathbb{Q}$.

14. Let $\alpha, \beta \in \mathbb{C}$ be of degree 3 over $\mathbb{Q}$ and $K = \mathbb{Q}(\alpha, \beta)$. Determine the possibilities for $[K : Q]$.

15. Let $\overline{\mathbb{Q}}$ denote the set of all algebraic numbers (i.e., complex numbers which are algebraic over $\mathbb{Q}$). Prove that $\overline{\mathbb{Q}}$ is algebraically closed.

16. Prove that $\overline{\mathbb{Q}}$ is countable. What about the set of transcendental numbers?

17. Let $K \subset L \subset K(t)$, where $K(t)$ is a rational function field in one variable over K. If $K \neq L$, prove that t is algebraic over L. (Hint: Let $\alpha = P(t)/Q(t) \in L \setminus K$, where $P(t)$ and $Q(t)$ are relatively prime. Then t is a root of the polynomial $P(X)\alpha Q(X) \in L[X]$ and its degree is maximum (deg $P(X)$, deg $Q(X)$).)

18. Let L_1, L_2 be finite extensions of K and $[L_i : K] = n_i$, for $i = 1, 2$. Let m be the least common multiple of n_1, n_2. Prove that m divides $[L_1L_2 : K]$. Give an example to show that $[L_1L_2 : K]$ may not be equal to m.

19. Find the splitting field of $X^3 - 5$. Does $i = \sqrt{-1}$ belong to this field?

20. Home Work: Revise your school geometry of constructions with ruler and compass.

(i) How to draw a line perpendicular to a given line through a given point, draw a line parallel to a given line and passing through a given point etc..

(ii) Using only ruler and compass,

Can you bisect a given angle? Can you trisect a given angle?

Tutorials: III& IV

AFS III, Algebra, Module I

(Field Extensions)

25^{th} June 2016

IIISER, Trivendrum

Parvati Shastri
Mumbai University

N. B. Through out the exercises, K denotes a field, unless otherwise stated.

1. Determine whether or not a 9-gon can be constructed by ruler and compass.

2. Show that $\arccos \frac{11}{16}$ is constructible by ruler and compass, but it can not be trisected.

3. Show that an angle of n degrees, $n \in N$, is constructible if and only if $3|n$.

4. Let $L|K$ be a finite extension. Let for any $\alpha \in L$, $\mu_\alpha : L \to L$ be the multiplication by α map; i.e., $\mu_\alpha(x) = \alpha x$, $\forall x \in L$. Prove that μ_α is a K-linear map. If $[L : K] = n$, prove that the correspondence $\alpha \to \mu_\alpha$ gives an isomorphism of L onto a subring of $M_n(K)$. Compute the image of this isomorphism for $L = \mathbb{C}$ over $K = \mathbb{R}$.

5. Let $L|K$ be a finite extension of degree n and suppose that $L = K(\alpha)$. Prove that the characteristic polynomial of μ_α is the minimal polynomial of α.

7. Show that the degree of a splitting field of a polynomial of degree n is at most $n!$.

8. Prove that a finite subgroup of the multiplicative group of any field is cyclic.

9. Give an example of a field extension $L|K$, such that $[L : K]$ is finite, K has only finitely many roots of unity but L has infinitely many roots of unity.

10. Prove that the minimal polynomial of $\alpha = \sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$ is monic with integer coefficients and the reduction of this polynomial modulo p, is reducible (over $\mathbb{F}_p$,) for all p. Can you replace $2, 3$ above by arbitrary $m, n, \in \mathbb{N}$? Justify.

11. Prove or disprove: $\mathbb{Q}(\pi) \cong \mathbb{Q}(e)$.

12. Determine the splitting field of $X^{p^r} - 1$ over $\mathbb{F}_p$.

13. Let $K = \mathbb{F}_p(Y)$ be the rational function field in one variable over $\mathbb{F}_p$. Determine the splitting field of $X^p - Y$.

14. Prove or disprove: $\mathbb{Z}/p^r\mathbb{Z}$ is isomorphic to the field of p^r elements.

15. Prove that the set of all constructible numbers is dense in $\mathbb{C}$. Is it countable? Justify.

16. If $2^r + 1$ is a prime number, prove that $r = 2^t$, for some $t \in \mathbb{N} \cup \{0\}$. (Such primes are called Fermat primes.)

17. Prove that a field of p^r elements contains a field of p^s elements if and only if $s|r$.

18.

(a) Let F be a field of characteristic p. Show that if $X^p - X - a$ is reducible in $F[X]$, then it splits into distinct factors in $F[X]$. (Hint: If θ is a root of this polynomial, then $\theta + 1$ is also a root.)

(b) For every prime p, show that $X^p - X - 1$ is irreducible in $\mathbb{Q}[X]$.

19. (This exercise may be of some interest in later part of the course.) Prove that the alternating group A_4 has no subgroup of order 6.

AFS III Report

I taught in the second and third week of AFS III program. I was able to cover the entire syllabus as prescribed. The attendance and participation of the students was excellent. They asked many questions. During tutorial sessions, I made the students come up to the board and solve the problems. Many students volunteered and it seemed quite clear that they understood the material. This was one of the very satisfying AFS programs I have been involved in.

The topics covered by me included Splitting fields and algebraic closures; existence and isomorphisms, Characterization of perfect fields of positive characteristic. Separable and inseparable extensions. Finite subgroup of the multiplicative group of a field is cyclic. Primitive element theorem. Finite separable extensions have a primitive element. Normal extensions and their examples. Characterization of normal extensions in terms of embeddings and splitting field. Galois extensions. Galois groups of finite extensions of finite fields and quadratic extensions. Galois groups of biquadratic extensions. Galois groups of a separable cubic polynomials. Fundamental Theorem of Galois theory (FTGT). Artin's Theorem about fixed field of a finite group of automorphisms.Behavior of Galois group under isomorphisms. Normal subgroups of the Galois groups and their fixed fields. Fundamental theorem of algebra via FTGT. Roots of unity in a field. Galois group of n-th roots of unity. Inverse Galois problem for finite abelian groups.

Overall it was a very satisfying program, and the quality of students was better than usual, which made the program all the more fun and meaningful.

Viji Z Thomas

AFS-III-2015
Report from Anant Shastri

Anant R. Shastri
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November 17, 2016

I arrived at the venue on 19th June. The campus is being built up in a valley with beautiful natural surroundings. Our program was to be held in a new building which was partly complete. All around there was hectic building activity going on which caused some concern. The director arrived for the inaugural much in advance and assured us that he will see personally to minize the disturbance. And he kept his words. The program went well smoothly, thanks to the pains taken by the local organizer.

In the first week I lectured on Basic Algebraic Topology according to the plan and as indicated in the syllabus. Ofc ourse I needed to bring down the level only a little bit. Students enjoyed the extra session of playing with the Mobius band and making platonic solids. By the end of the first week I had already established good repert with the students. They were freely mixing with us to the extent possible.

In the Third week, I lectured on Complex analysis, from whereever, Porf. S. Bhattacharya had left. Exccept for some extra discussion about winding number, the rest of the talk was as per the plan, modulo the proof of Runge's theorem.

In the last week, I lectured on Homology theory. After building up the bare necessary algebra of chain complexes (Snake lemma), directly discussed construction of singular chain complex, stated all its fundamental properties, verified them except the excision axiom. After that I discussed its applications even completing the proof of Brouwer's invariance of domain.

The reception was reasonably well as expected. Students were serious and did try to solve exercises.

Report on AFS III organized at IISER Trivandrum

I gave a short course on Complex Analysis in this program and covered the following topics :

1) Analytic functions. Power series and radius of convergence. Exponential function and branches of logarithm. Cauchy-Riemann equations.

2) Integration of a function with respect to a path. Index of a point with respect to a closed curve.

3) Cauchy's theorem and Cauchy's integral formula. Power series representation of an analytic function. Zero's of an analytic function.

4) Integral formula for derivatives. Liouville's theorem. Fundamental theorem of Algebra.

5) Removable singularities, poles, and essential singularities. Laurent series expansions.

6) Number of zero's of an analytic function inside a closed curve. Rouche's theorem. Open mapping theorem. Maximum modulus principle.

The overall performance of the students was highly satisfactory. The attendance rate was high and many of them actively participated in the lectures. They also showed a lot of enthusiasm during the tutorial sessions and collectively solved almost all the problems given to them.

Siddhartha Bhattacharya

AFS III Report

In my four lectures, I have done the following:

1. Geometric Simplicial Complexes 2. Abstract Simplicial Complexes 3. Geometric Realization of Simplicial Complexes 4. Barycentric Subdivision 5. Simplicial Approximation Theorem

During the lectures, most people, except few, followed well and also everybody tried problems in tutorial hours. I think that few students enjoyed this topics. Some of them felt difficulties in solving problems.

Ramesh Kasilingam

REPORT ON COVERING SPACES

Lecture 1 : -
Introduced covering spaces, discussed some examples, showed that an even group action on a topological space gives rise to a covering space. Stated the lifting problems and proved path lifting property for covering spaces.
Lecture 2 : -
Explained the homotopy lifting property leading to the definition of fibration, proved that covering projections are fibration. Started exploring the relations of the fundamental groups with the lifting problem : gave a criteria when loops can be lifted to loops. Defined normal coverings.
Lecture 3 :-
Solved the lifting problem for covering projections.Defined equivalence of coverings and gave a criteria for two coverings to be equivalent. Introduced the deck transformation groups, started the discussion to explicitly describe the deck transformation group for normal coverings.
Lecture 4 :-
Completed the description of deck transformation groups for normal covering. Defined universal covering, showed simply connected covers are universal, stated the criteria for existence of simply connected covers, showed that universal cover is unique upto equivalence of coverings. Proved that subgroups of $\pi_1(X)$ upto conjugation are in bijective correspondence with equivalence classes of connected coverings for a connected, locally path connected, semi locally simply connected spaces.

I covered the following topics---- Tony

1. Cyclic extensions of degree p over fields with characteristics p. Solvable groups. Simplicity of An.
2. Galois group of composite extensions.
3. Radical extensions. Solvability by radicals and solvable Galois groups. A quintic polynomial which is not solvable by radicals.
4. Cardano's method for roots of cubic equations.
5. Galois group as a group of permutations. Irreducibility and transitivity.Galois groups of quartics.
6. The norm and the trace function. Multiplicative form of Hilbert's Theorem 90. Cyclic extensions of degree n: Additive version of Hilbert's 90. Cyclic extensions of prime degree: Artin-Schreier Theorem

Comments on the AFS-III conducted at IISER, TVM

I was a lecturer for Complex Analysis in the fourth week of the programme. The topics assigned to me were Harmonic Functions, Dirichlet Problem, Green?s Functions and an outline of the Riemann Mapping Theorem. I think the students were responsive. Since these topics are not usually covered in the MSc syllabus, these topics should continue to be included in the AFS programmes. Of course, one week is a little short to do these topics in the AFS style. Efforts should be done to get more students to the programme. I feel that to get thirty students for the programme, one should give admission to forty or fifty students. In the unlikely event of more than thirty turning up, the organiser of the programme should have the freedom to do some changes in the budget and accommodate the extra students. The accommodation given to me was very nice and I enjoyed the lectures.

Joseph Mathew