Topics Covered

Week 1:

Mythily Ramawamy: Sobolev spaces

Distributions:

- Motivation for the weak solution.
- Definition of the test functions and distributions.
- Schwarz space and Fourier transform.
- Fundamental solution.

Sobolev spaces:

- Definition and some properties $W^{1,p}(\Omega)$.
- Regularity of $W^{1,p}(\mathbb{R})$ functions.
- Density results and definition of $W_0^{1,p}(\Omega)$.
- Extension operator.
- Poincaré inequality.

Embeddings, dual spaces and trace

- Definition of H^s spaces.
- Embeddings for $1 \leq p < n,\ p = n,\ p > n$.
- Rellich-Kondrachov compactness theorem.
- Dual spaces of $W_0^{1,p}(\Omega)$
- Trace theorems and Green's formula.

Senthil Rani: Review of basic measure theory

- Basic measure theory.
- Hausdorff measures $\mathcal{H}^n$.
    - Isodiametric inequality.
    - On $\mathbb{R}^n$, $\mathcal{L}^n \equiv \mathcal{H}^n$.
    - Example: Cantor $1/3^{rd}$ set.
- Densities and $\mathcal{H}^n, 0 \leq s \leq n$.
- Lipschitz map and $\mathcal{H}^n$.

Week 2:

S. Kesavan: Geometric Measure Theory

Schwarz Symmetrization:

- Definition and examples.
- Properties of Schwarz symmetrization.
- Hardy-Littlewood inequality.
- Increasing rearrangement.
- Riesz inequality.

Classical Inequalities:

- Classical isoperimetric inequality.
- Minkowski content, Hausdorff measure.
- De-Giorgi primeter.
- Fleming-Rischel theorem.

Measure Theory and Multivariable calculus

- Definition of measure.
- Frechet differentiation, chain rule and mean value theorem.
- Sard's theorem.
- Change of variable formlas for Borel measurable functions.
- Hausdorff measure and isodiametric inequality.
- Rademacher's theorem for Lipschitz continuous functions.
- Linear maps and Jacobians.
- Polar decomposition.
- Co-area formulas and polar coordinates.
- Integration over level sets.
- Isoperimetric inequalities.

T. V. Anoop: Weak Solutions of Elliptic PDE

- Classical solutions of Dirichlet problem.
- Green's representation formula.
- Review of Dirichlet principle, Poincaré sweeping out process and Perron's method.
- Abstract variational problems and Lax-Milgram theorem.
- Weak solution for elliptic boundary value problems with various boundary conditions, like Dirichlet, Neumann and mixed boundaries and the existence of weak solutions.
- Regularity of weak solutions.
- Maximum principle.
- The Dirichlet eigenvalues of Lapalcian.

Sarath Sasi: Monotonicity Methods

- Definition and examples of monotone operators on Hilbert spaces and on ordered Banach spaces.
- The monotone iterative method.

Tutorials

We have grouped the participants in to 8 groups. The tutorial problems were based on the topics covered on the same day. The tutorial sessions were very interactive and the feedbacks from the participants reflects the same.

Some of the tutorial problems:

Sobolev Spaces

1. Find the distributional derivative of the following functions:

   (a) $f_1(x) = |x|$

   (b)

$$f_2(x) = \begin{cases} 1, & x \leq -1, \\ x^2, & x \in (-1, 1]. \\ x, & x \in (1, 2), \\ 1 + x, & x \geq 2, \end{cases}$$

2. (a) Let $\phi \in \mathcal{D}(\mathbb{R})$. Show that there exits $\psi \in \mathcal{D}(\mathbb{R})$ such that $\psi' = \phi$ iff,

$$\int_{\mathbb{R}} \phi(x)\, dx = 0.$$

(b) Deduce that if $T \in \mathcal{D}'(\mathbb{R})$ such that $\frac{dT}{dx} = 0$, then $T = T_c$, *i.e.*, distribution generated by a constant function.

(c) Prove that the constant c is unique.

3. Let $\Omega \subset \mathbb{R}^N$ be a domain. Let $Du = 0$ a.e. in $\mathcal{D}'(\Omega)$. Prove that $u \equiv c$ a.e. in Ω.

4. Consider the differential operator $L = \frac{d^2}{dx^2} + a\frac{d}{dx} + b$ where a, b are constants. Let f and g satisfy $Lf = 0,\ Lg = 0,\ f(0) = g(0),\ f'(0) - g'(0) = 1$. Consider the function
$$F(x) = \begin{cases} f(x), & x \leq 0, \\ g(x), & x > 0. \end{cases}$$
Show that $-F$ is the fundamental solution for L.

5. Let $1 \leq p < \infty$ and let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Let $u \in L^p(\mathbb{R}^N)$. Assume there exits a constant $C > 0$ such that for every relatively compact open set Ω' in Ω and for every $h \in \mathbb{R}^N$ such that $|h| < d(\Omega', \mathbb{R}^N \setminus \Omega)$, we have
$$\|u(x-h) - u(x)\|_{L^p(\Omega)} \leq C|h|.$$
Show that there exits a constant $C > 0$ such that
$$\Big| \int_\Omega u \frac{\partial \phi}{\partial x_i} dx \Big| \leq C\|\phi\|_{L^q(\Omega)}, \quad \forall \phi \in \mathcal{D}(\Omega),$$
where q is the conjugate exponent of p. Hence show that $u \in W^{1,p}(\Omega)$.

6. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. If $u \in W^{1,p}(\Omega)$, then show that $u^+, u^- \in W^{1,p}(\Omega)$. Moreover,
$$Du^+ = \begin{cases} Du, & u > 0, \\ 0, & u \leq 0. \end{cases}$$
and
$$Du^- = \begin{cases} 0, & u \geq 0, \\ -Du, & u < 0. \end{cases}$$

7. Give counter examples:
(i) Let $N > p$. Then $W^{1,p}(\Omega)$ is not compactly embedded into $L^{p^*}(\Omega)$.
(ii) $W^{1,N}(\Omega)$ is not compactly embedded into $L^\infty(\Omega)$.

8. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Prove that there exits a constant $C > 0$, such that
$$\|u\|^2_{L^2(\Omega)} \leq C\left\{ \|\nabla u\|^2_{L^2(\Omega)} + \left(\int_\Omega u\right)^2 \right\},$$
for all $u \in H^1(\Omega)$.

Review of Basic Measure Theory

1. Give an example such that the inequality in Fatou's lemma is strict.
2. Give an example for which the Monotone convergence theorem fails when a hypothesis is dropped.
3. Give an example of a measure which is Borel but not Borel regular.
4. Give an example of a measure which is not outer regular.
5. Give an example of a measure which is not Radon.
6. Let $f : \mathbb{R} \to \mathbb{R}^2$ such that $f(x) = (x, 0)$. For a non-measurable set $N \subset \mathbb{R}$, show that
   - $f(N)$ is measurable
   - f is Borel measurable
   - $f(N)$ is not Borel
   - $\chi_{f(N)}$ is measurable but not Borel measurable
7. Prove that $\int_X f(x)d\mu(x) = \int_0^\infty \mu(\{x : f(x) \geq t\})dt$.
8. Compute the Hausdorff dimension of Cantor $1/5^{th}$ set.
9. Prove that the Hausdorff dimension of an image under $f : [0, \pi] \to \mathbb{R}$ given by $f(\theta) = (\cos\theta, \sin\theta)$ (in general, can we say under rectifiable curves in $\mathbb{R}^n$?) is 1.
10. Compute the $(n-1)$-Hausdorff measure of the sphere.

Geometric Measure Theory

1. Let $\Omega \subset \mathbb{R}^3$ be a bounded open set, $f \in L^2(\Omega)$. Consider the functional $J : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{6}\int_\Omega u^6 dx - \int_\Omega fu.$$

   (a) Is J well-defined?

   (b) Is J continuous?

   (c) Is J differentiable? If so, find $J'(u)$.
2. Let $F : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y$ be a differentiable function at $x \in X$. Then, show that $F'(x)$ is unique.
3. Let X be a normed linear space and $0 \in \Omega \subseteq X$. Let $T : \Omega \to X$ be such that the image of any bounded set is relatively compact and T is differentiable at 0. Show that $T'(0) : X \to X$ is compact.
4. Let $f : M_n(\mathbb{R}) \to \mathbb{R}$ be defined as $f(A) = \det(A)$. Compute $f'(A)H$.
5. Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be a non-singular linear transformation. Let $E \subset \mathbb{R}^N$ and μ^* is an outer measure on $\mathbb{R}^N$. Show that

   (a) $\mu^*\big(A(E)\big) = |\det(A)|\mu^*(E)$.

(b) E is Lebesgue measurable iff $A(E)$ is Lebesgue measurable.

6. Let $\triangle$ be the triangle with vertices $(x_i, y_i),\ i = 1, 2, 3$. Then, show that

$$Area(\triangle) = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

7. Let A be a $n \times n$ symmetric positive definite matrix. Then compute

$$\int_{\mathbb{R}^n} e^{-\sum\limits_{i,j=1}^{n} a_{ij}x_ix_j} \, dm_n.$$

8. Let ω_N denotes the measure of the unit ball in $\mathbb{R}^N$. Then, show that

$$\lim_{N\to\infty} \omega_N = 0.$$

Weak Solutions of Elliptic PDEs

1. Construct the Green's function for Laplacian in certain domains, like square, ball, half-plane etc. in $\mathbb{R}^2$.

2. (a) Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Show that the bilinear form defined on $H^1(\Omega)$ by

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx + \left(\int_\Omega u dx\right)\left(\int_\Omega v dx\right)$$

is $H^1(\Omega)$-elliptic.

(b) Deduce that if $f \in L^2(\Omega)$, there exists a unique $u \in H^1(\Omega)$ such that, for every $v \in H^1(\Omega)$, we have

$$a(u, v) = \int_\Omega f v dx$$

and that

$$\int_\Omega u dx = \frac{1}{|\Omega|}\int_\Omega f dx.$$

(c) If $\int_\Omega f dx = 0$, what is the BVP solved by u?

3. (a) Consider the BVP:

$$\begin{aligned} -u'' + u &= f \text{ in } (0, 1), \\ u(0) &= u(1), \\ u'(0) &= u'(1). \end{aligned}$$

If $f \in L^2(0, 1)$, give the weak formulation of this problem in a suitable subspace of $H^1(0, 1)$ and prove the existence and the uniqueness of the weak solution.

(b) If $f \in H^1(0, 1)$, show that the weak solution is in fact a classical solution.

4. (a) (The Obstacle problem) Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Let $\chi \in H^2(\Omega)$ with $\chi \leq 0$ on $\partial\Omega$. Define

$$K = \left\{ v \in H_0^1(\Omega) : v \geq \chi \text{ a.e. in } \Omega \right\}$$

Let $f \in L^2(\Omega)$. Show that there exists a unique $u \in K$ such that

$$J(u) = \inf_{v \in K} J(v)$$

where

$$J(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 dx - \int_\Omega f v dx.$$

(b) If u is smooth enough, show that it satisfies the conditions:

$$\begin{aligned} u &\geq \chi \text{ in } \Omega, \\ -\Delta u &= f \text{ on the set } \{x \in \Omega : u(x) > \chi(x)\}, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

5. State and prove the mean value property for the harmonic, sub-harmonic and super-harmonic functions.

6. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $\lambda_1(\Omega)$ and $\mu_1(\Omega)$ be the first Dirichlet and first Neumann eigenvalues of $-\Delta$ in Ω. Show that $\lambda_1(\Omega)$ and $\mu_1(\Omega)$ are simple and principal eigenvalues.

Monotonicity Methods

1. Let $T : H \to H$ be an operator which is strongly monotone and Lipschitz. Prove that $T(u) = h$ has a unique solution.

2. Find the existence of solutions of the following problems:

   (i)

$$\begin{aligned} -x''(t) + g(x(t)) &= f(t),\ t \in (0,1), \\ x(0) &= x(1), \end{aligned}$$

   such that $g : \mathbb{R} \to \mathbb{R}$ is continuous.

   (ii) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and

$$\begin{aligned} -\Delta u(x) &= g(x, u(x)) \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

3. Find a subsolution and a super solution of the following problem:

$$-\Delta u = \lambda f(u), \text{ in } \Omega, \tag{1}$$
$$u = 0 \text{ on } \partial\Omega, \tag{2}$$

   such that

   (i) $f : \mathbb{R} \to \mathbb{R}$ is continuous,

   (ii) $f(0) > 0$ and f is increasing,

   (iii) $\lim\limits_{s \to \infty} \frac{f(s)}{s} = 0$.