1. p-adic Galois representations

* $K/_{\mathbb{Q}_p} < \infty$, $\mathcal{O}_K \supset \underline{m}_K = (\pi_K)$, $k := \mathcal{O}_K/\underline{m}_K$.

$\Rightarrow K_0 := W(k)[\frac{1}{p}]$ ⇐ max. unramified subext of K over $\mathbb{Q}_p$.

(Later, will assume $K = \mathbb{Q}_p$).

- $G_K := \mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \twoheadrightarrow \mathrm{Gal}(L/K)$, ${}^{\forall}L/K$ finite + Galois.

$$\Rightarrow G_K \xrightarrow{\sim} \varprojlim_{L} \mathrm{Gal}(L/K) \underset{\text{closed}}{\subset} \prod_{L} \mathrm{Gal}(L/K)$$

↖ product top.

$\Rightarrow$ Topology of $G_K = \langle gN \mid g \in G_K,\ N \triangleleft G_K,\ [G_K : N] < \infty \rangle$

$\Rightarrow G_K$ is a topological group i.e.

$$G_K \times G_K \to G_K, \quad G_K \to G_K \quad \text{are cont.}$$
$$g,\ h \mapsto g \cdot h \qquad g \mapsto g^{-1}$$

+ compact, + Hausdorff, + "every open set containing 1 contains open normal subgroup".

- $G_K \circlearrowright \overline{\mathbb{Q}}_p \Rightarrow G_K \circlearrowright \overline{\mathbb{Z}}_p,\ \underline{m}_{\overline{\mathbb{Z}}_p} \Rightarrow G_K \circlearrowright \overline{\mathbb{Z}}_p/\underline{m}_{\overline{\mathbb{Z}}_p} \simeq \overline{\mathbb{F}}_p$

$$\Rightarrow 0 \to I_K \to G_K \to G_k \to 0$$

$G_k = \mathrm{Gal}(\overline{\mathbb{F}}_p/k) \simeq \hat{\mathbb{Z}} \simeq \prod_{\ell} \mathbb{Z}_\ell$

I_K ↖ the inertia subgroup.

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$$\begin{tikzcd}
\overline{K} \arrow[d, dash] \arrow[d, bend left=60, phantom, "P_K"'] \arrow[dd, bend right=60, phantom, "I_K"] \\
K^{\text{tame}} := \bigcup_{(n,p)=1} K^{ur}(\pi_K^{\frac{1}{n}}) \arrow[d, dash] \\
\overline{K}^{I_K} := K^{ur} \arrow[d, dash] \\
K
\end{tikzcd}$$

$$\Rightarrow 1 \to P_K \to I_K \xrightarrow{t_\zeta} \prod_{\ell \neq p} \mathbb{Z}_\ell \to 0$$

$$\sigma \longmapsto t_\zeta(\sigma), \qquad \frac{\sigma(\pi_K^{\frac{1}{n}})}{\pi_K^{\frac{1}{n}}} = \zeta_n^{t_\zeta(\sigma)} \ (\mathrm{mod}\ n)$$

$\zeta := (\zeta_n)_{(n,p)=1}$ ⇐ compatible system of primitive roots of unity

Def Let $\mathcal{R}$ be a top. ring.

① A **Galois rep-n** is a cont. homom. $\rho : G_K \to GL_n(\mathcal{R})$.

② A free $\mathcal{R}$-module V of finite rank is a Galois rep-n if V is equipped with cont. $\mathcal{R}$-linear action of G_K.

$$G_K \times V \longrightarrow V$$

Exercise) ① $\Longleftrightarrow$ ② + choice of a basis.

e.g.) ① Cyclotomic character.

$$\begin{array}{ccc} \varepsilon_p : G_K & \longrightarrow & GL_1(\mathbb{Z}_p) = \mathbb{Z}_p^\times \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \sigma & \longmapsto & \varepsilon_p(\sigma), \quad \sigma(\zeta_{p^n}) = \zeta_{p^n}^{\varepsilon_p(\sigma) \pmod{p^n}} \end{array}$$

② $\varepsilon_p \longleftrightarrow \mathbb{Z}_p(1)$

$\varepsilon_p^i \longleftrightarrow \mathbb{Z}_p(i)$

$V(i) := V \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(i)$.

③ If $\mathcal{R}$ is a finite ext-n of $\mathbb{Q}_p$, then V is called a **p-adic Galois rep-n**.

④ If $\rho : G_K \to GL_n(\overline{\mathbb{Q}}_p)$ is a Galois rep-n, then

$\exists\, E/\mathbb{Q}_p < \infty$ s.t.

$$\begin{tikzcd} G_K \arrow[rr, "\rho"] \arrow[dr] & \circlearrowleft & GL_n(\overline{\mathbb{Q}}_p) \\ & GL_n(E) \arrow[ur] & \end{tikzcd}$$

* p-adic Galois rep'n

· **Picture** $(E, \mathcal{O}_E, \mathbb{F})$.

Can you describe any single example? which is irred. + 2-dim.

$\{\rho : G_K \to GL_n(\overline{\mathbb{F}_p})\} =: \mathrm{Rep}_E G_K$

$\cup$

$\{\text{Hodge-Tate rep'ns}\} =: \mathrm{Rep}_E^{HT} G_K$ $\cong$ $\{\text{Semi-linear algebra objects}\}$

$\cup$

$\{\text{de Rham rep'ns}\} =: \mathrm{Rep}_E^{dR} G_K$

$\cup$

$\{\text{Semi-stable rep'ns}\}$

$\cup$

$\{\text{crystalline rep'ns}\}$

comes from geometry

$\mathbb{C}_p := \widehat{\overline{\mathbb{Q}_p}}$ $\Leftarrow$ algebraically closed with $\mathbb{C}_p^{G_K} = K$.

· de Rham period ring $\mathbb{B}_{dR}$ $(\cong \mathbb{C}_p((t))$ non-canonically$)$

① $\mathbb{B}_{dR}$ is a field with a ~~subfield~~ subring $\mathbb{B}_{dR}^+$ $(\cong \mathbb{C}_p[\![t]\!])$,

and $\mathbb{B}_{dR}^+$ is a D.V.R. with a uniformizer t and $\mathbb{B}_{dR} = \mathbb{B}_{dR}^+[\frac{1}{t}]$.

② $\mathbb{B}_{dR}^+$ has a cont. action of G_K that extends to $\mathbb{B}_{dR}$

s.t. - $g.t = \varepsilon_p(g) \cdot t \quad \forall g \in G_K$.

- $\mathbb{B}_{dR}^{G_K} = (\mathbb{B}_{dR}^+)^{G_K} = K$.

③ $\forall i \in \mathbb{Z}$, $\mathrm{Fil}^i \mathbb{B}_{dR} := t^i \cdot \mathbb{B}_{dR}^+$ and $\exists$ a G_K-equiv. exact seq.

$$0 \longrightarrow \mathrm{Fil}^{i+1}\mathbb{B}_{dR} \longrightarrow \mathrm{Fil}^i \mathbb{B}_{dR} \longrightarrow \underset{\mathbb{C}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(i)}{\underset{\parallel}{\mathbb{C}_p(i)}} \longrightarrow 0$$

· Let $V_{/E}$ be a p-adic Galois rep-n of $G_{\mathbb{Q}_p}$.

$$\Rightarrow \quad D_{dR}(V) := (\mathbb{B}_{dR} \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}} \quad \in \quad \overset{\overset{E}{\shortparallel}}{\mathbb{Q}_p \otimes_{\mathbb{Q}_p} E} \text{ - module}$$

$$\Rightarrow \quad \alpha_{dR} : \mathbb{B}_{dR} \otimes_{\mathbb{Q}_p} D_{dR}(V) \hookrightarrow \mathbb{B}_{dR} \otimes_{\mathbb{Q}_p} V \qquad \text{not easy, but elementary.}$$

$$x \otimes y \longmapsto x \cdot y$$

is $G_{\mathbb{Q}_p}$-equiv. + $\mathbb{B}_{dR} \otimes_{\mathbb{Q}_p} E$-module map.

Def V is de Rham if $\dim_{E} D_{dR}(V) = \dim_E V$

· Let V be a de Rham rep-n.

$\Rightarrow$ $D_{dR}(V)$ has a decreasing filtration.

$$\mathrm{Fil}^i D_{dR}(V) := (\mathrm{Fil}^i \mathbb{B}_{dR} \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}} \quad \in E\text{-v.s.}$$

s.t. - $\displaystyle\bigcup_{i \in \mathbb{Z}} \mathrm{Fil}^i D_{dR}(V) = D_{dR}(V)$

- $\displaystyle\bigcap_{i \in \mathbb{Z}} \mathrm{Fil}^i D_{dR}(V) = 0.$

$$\Rightarrow \cdot \quad \mathrm{Fil}^i D_{dR}(V) = \begin{cases} D_{dR}(V) & \text{if} \quad i \ll 0 \\ 0 & \text{if} \quad i \gg 0. \end{cases}$$

$$\cdot \quad \dim_E D_{dR}(V) = \sum_{i \in \mathbb{Z}} \dim_E \frac{\mathrm{Fil}^i D_{dR}(V)}{\mathrm{Fil}^{i+1} D_{dR}(V)}.$$

Def Let V be a de Rham.

$\Rightarrow$ Hodge-Tate weights of V,

$$HT(V) := \left\{ -i \in \mathbb{Z} \;\middle|\; \frac{\mathrm{Fil}^i D_{dR}(V)}{\mathrm{Fil}^{i+1} D_{dR}(V)} \neq 0, \text{ counted with multiplicity } \dim_E \frac{\mathrm{Fil}^i D_{dR}(V)}{\mathrm{Fil}^{i+1} D_{dR}(V)} \right\}.$$

e.g.) $\mathbb{Q}_p(1) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)$.

$$\Rightarrow \cdot\ \mathrm{Fil}^{-1}(\mathbb{Q}_p(1)) = \left(\mathrm{Fil}^{-1}\mathbb{B}_{dR} \otimes \mathbb{Q}_p(1)\right)^{G_{\mathbb{Q}_p}} = \mathbb{Q}_p$$

$$\cdot\ \mathrm{Fil}^{0}(\mathbb{Q}_p(1)) = \left(\mathrm{Fil}^{0}\mathbb{B}_{dR} \otimes \mathbb{Q}_p(1)\right)^{G_{\mathbb{Q}_p}} = 0.$$

$$\Rightarrow HT(\mathbb{Q}_p(1)) = \{1\}.$$

- Semi-stable period ring $\mathbb{B}_{st}$

① $\mathbb{B}_{st}$ is a non-canonical subring of $\mathbb{B}_{dR}$ containing t, and the action of G_K on $\mathbb{B}_{dR}$ gives an action on $\mathbb{B}_{st}$ with $\mathbb{B}_{st}^{G_K} = K_0$.

② $\exists$ an injective map $\phi : \mathbb{B}_{st} \longrightarrow \mathbb{B}_{st}$ that

- is Frob. semilinear on $\mathbb{Q}_p^{ur}$
- commutes with the action of G_K.
- $\phi(t) = pt$.

③ $\exists$ $\mathbb{Q}_p^{ur}$-linear map $N : \mathbb{B}_{st} \longrightarrow \mathbb{B}_{st}$ that

- commutes with the action of G_K
- satisfies $N\phi = p\phi N$.

④ $\mathbb{B}_{st,K} := K \otimes_{K_0} \mathbb{B}_{st} \hookrightarrow \mathbb{B}_{dR}$ and $\mathbb{B}_{st,K}$ has a decreasing filtration $\mathrm{Fil}^i \mathbb{B}_{st,K} := \mathbb{B}_{st,K} \cap \mathrm{Fil}^i \mathbb{B}_{dR}$.

- Let $V \in \mathrm{Rep}_E G_{\mathbb{Q}_p}$

  $\Rightarrow D_{st}(V) := (\mathbb{B}_{st} \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}} \in$ E-v.s.

  and $\dim_E D_{st}(V) \leq \dim_E V$.

Def V is *semi-stable* if $\dim_E D_{st}(V) = \dim_E(V)$.

$\Rightarrow D_{st}(V)$ has a decreasing filtration of E-v.s.

$$\mathrm{Fil}^i D_{st}(V) := (\mathrm{Fil}^i \mathbb{B}_{st} \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$$

- *Lemma* A semi-stable rep-n V is de Rham,

  in which case, $\mathrm{Fil}^i D_{st}(V) \cong \mathrm{Fil}^i D_{dR}(V) \quad \forall i \in \mathbb{Z}$.

proof). · $\dim_E V = \dim_E D_{st}(V) \leq$ ~~dim~~ $\dim_E D_{dR}(V) \leq \dim_E V$.

· $\mathrm{Fil}^i D_{st}(V) \hookrightarrow \mathrm{Fil}^i D_{dR}(V)$

$$\Rightarrow \bigoplus \frac{\mathrm{Fil}^i D_{st}(V)}{\mathrm{Fil}^{i+1} D_{st}(V)} \hookrightarrow \bigoplus \frac{\mathrm{Fil}^i D_{dR}(V)}{\mathrm{Fil}^{i+1} D_{dR}(V)}$$

$$\Rightarrow \dim_E \downarrow = \dim_E D_{st}(V) = \dim_E(V) = \dim_E D_{dR}(V) = \dim_E \downarrow$$

$\square$

- $\varphi := \varphi \otimes 1$, $N := N \otimes 1$ $\circlearrowleft D_{st}(V) \Rightarrow N\varphi = p\varphi N$. on $D_{st}(V)$

$\therefore$ V is semi-stable

$\Rightarrow$ · V is de Rham (define $HT(V)$)

· $D_{st}(V)$ is an E-v.s. with

  - a decreasing filtration $\mathrm{Fil}^i D_{st}(V) \cong \mathrm{Fil}^i D_{dR}(V)$
  - $\varphi, N \circlearrowleft D_{st}(V)$ with $N\varphi = p\varphi N$.

- **Def** A filtered (ϕ, N)-module of rank d is

an d-dimensional v.s. D together with $(\phi, N, \{\mathrm{Fil}^i D\}_{i\in\mathbb{Z}})$

where

- the Frobenius map $\phi : D \to D$ is an E-linear automorphism
- the monodromy operator $N : D \to D$ is an E-linear endomorphism (nilpotent)

  s.t. $N\phi = p\phi N$
- the Hodge filtration $\{\mathrm{Fil}^i D\}_{i\in\mathbb{Z}}$ is a decreasing filtration

  on D. with

  the sub E-v.s. $\mathrm{Fil}^i D = \begin{cases} D & \text{if } i \ll 0 \\ 0 & \text{if } i \gg 0. \end{cases}$

- Morphism between filtered (ϕ, N)-modules is

  an E-linear map preserving ϕ, N. $\{\mathrm{Fil}^i D\}_{i\in\mathbb{Z}}$.

Prop D_{st} gives a fully faithful, exact, covariant functor

from $\mathrm{Rep}_E^{st} G_{\mathbb{Q}_p}$ to $\{\text{filtered } (\phi, N)\text{-modules}/E\}$.

- What is the essential image?

Def Let D be a filtered (ϕ, N)-module

$\Rightarrow$ · $t_H(D) := \sum_{i\in\mathbb{Z}} i \cdot \dim_E \dfrac{\mathrm{Fil}^i D}{\mathrm{Fil}^{i+1} D}$ ← Hodge number

· $t_N(D) := v_p(\det \phi)$ ← Newton number

$$v_p : \overline{\mathbb{Q}_p}^{\times} \to \mathbb{Q}_p$$
$$p \longmapsto 1$$

- Let D' be a subspace of D that is stable under ϕ, N.

$\Rightarrow$ D' is a filtered (ϕ, N)-module with

- $\phi' := \phi|_D$
- $N' := N|_D$
- $\mathrm{Fil}^i D' := D' \cap \mathrm{Fil}^i D$.

Def A filtered (ϕ, N)-module D is **(weakly) admissible** if

- $t_H(D) = t_N(D)$
- $t_H(D') \leq t_N(D')$ $\forall (\phi, N)$-stable sub. v.s. $D' \subset D$.

Thm (Colmez – Fontaine)

$$\mathrm{Rep}_E^{st} G_{\mathbb{Q}_p} \xrightarrow[\sim]{D_{st}} \{\text{weakly admissible filtered } (\varphi, N)\text{-modules}\} =: \mathrm{FM}_E^{wa}(\phi, N)$$

Def V is **crystalline** if V is semi-stable

$+$ N on $D_{st}(V)$ is 0.

Facts

① V is de Rham (resp. semi-stable, resp. crystalline)

iff $V^\vee := \mathrm{Hom}(V, E)$ is.

② $-i \in HT(V) \iff i \in HT(V^\vee)$.

③ $HT(V(i)) = i + HT(V)$ $\Rightarrow$ Harmless to assume

$HT(V) \subset \mathbb{Z}_{\geq 0}$

and $0 = \min HT(V)$.

· $D^*_{st}(V) := D_{st}(V^\vee)$.

$\Rightarrow$ $\mathrm{Rep}^{st}_E G_{\mathbb{Q}_p} \xrightarrow[\sim]{D^*_{st}} \mathrm{FilM}^{w.a}_E(\phi, N)$

with a quasi-inverse $V^*_{st}(D) := \mathrm{Hom}_{\phi, N, \mathrm{Fil}}(D, \mathbb{B}_{st})$.

* Examples

(1) $D = E(e_1, e_2)$, $\underline{e} := (e_1, e_2)$

· $$\mathrm{Fil}^i D = \begin{cases} D & \text{if } i \leq 0 \\ Ee_1 & \text{if } 0 < i \leq r \\ 0 & \text{if } r < i \end{cases}$$

· $\mathrm{Mat}_{\underline{e}}(N) = 0$.

· ⓐ $\mathrm{Mat}_{\underline{e}}(\phi) = \begin{pmatrix} \lambda & \\ & \eta \end{pmatrix}$ $v_p(\lambda) = r$, $v_p(\eta) = 0$.

ⓑ $\mathrm{Mat}_{\underline{e}}(\phi) = \begin{pmatrix} \lambda & 0 \\ p^r & \eta \end{pmatrix}$ $v_p(\lambda) = r$. $v_p(\eta) = 0$.

ⓒ $\mathrm{Mat}_{\underline{e}}(\phi) = \begin{pmatrix} 0 & -1 \\ \lambda & \eta \end{pmatrix}$ $v_p(\lambda) = r$ $v_p(\eta) > 0$.

Lemma ① ~~$V^*_{st}(D)$~~ D is admissible and $V^*_{st}(D)$ is crystalline with $HT(V) = \{0, r\}$.
rep^n of $G_{\mathbb{Q}_p}$

② $$V^*_{st}(D) = \begin{cases} \text{decomposable} & (\text{case ⓐ}) \\ \text{indecomposable + reducible} & (\text{case ⓑ}) \\ \text{irreducible} & (\text{case ⓒ}) \end{cases}$$

<u>proof</u>) ⓐ (ϕ, N)-stable subspaces $= \{ 0, E(e_1), Ee_2, D \}$

- $t_H(Ee_1) = r \leq t_N(Ee_1) = v_p(\lambda) = r$
- $t_H(Ee_2) = 0 \leq t_N(Ee_2) = v_p(\eta) = 0$
- $t_H(D) = r = t_N(D) = v_p(\lambda\eta) = r$

$\therefore$ D in case ⓐ is admissible

- $HT(V_{st}^*(D))$ = jumps at the filtration $= \{0, r\}$.

$N=0 \Rightarrow V_{st}^*(D)$ is crystalline

$V_{st}^*(D) = V_{st}^*(Ee_1) \oplus V_{st}^*(Ee_2)$. $\boxtimes$

ⓑ, ⓒ Exercise.

(2) $D = E(e_1, e_2)$, $\underline{e} := (e_1, e_2)$.

- $$\mathrm{Fil}^i D = \begin{cases} D & \text{if } i \leq 0 \\ E(e_1 + \mathcal{L} e_2) & \text{if } 0 < i \leq r \\ 0 & \text{if } r < i \end{cases}$$

- $$\mathrm{Mat}_{\underline{e}}(\phi) = \begin{pmatrix} p\lambda & \\ & \lambda \end{pmatrix}$$

- $$\mathrm{Mat}_{\underline{e}}(N) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

- $v_p(\lambda) = \frac{r-1}{2}$, $\mathcal{L} \in E$.

Lemma ① D is admissible

② $V_{st}^*(D)$ is a semi-stable, non-crystalline rep-n of $G_{\mathbb{Q}_p}$

with $HT(V_{st}^*(D)) = \{0, r\}$

③ $V_{st}^*(D)$ is reducible iff $r=1$.

proof) ① $\{0, Ee_1, D\} \in (\phi, N)$-stable

- $t_H(Ee_1) = 0 \leq t_N(Ee_1) = v_p(x) = \frac{r-1}{2}$
- $t_H(D) = r = t_N(D) = r$

② $N \neq 0 \Rightarrow$ semi-stable, non-crystalline

jumps at $\{0, r\} \Rightarrow HT(V_{st}^*(D)) = \{0, r\}$

③ Obvious. $\boxtimes$

Prop ① The examples above exhaust all the semi-stable rep-ns of $G_{\mathbb{Q}_p}$ with HT wts $= \{0, r\}$ for $r > 0$.

② No overlap between these examples.

proof exercise

Def V is potentially semistable (resp. potentially crystalline)

if $\exists\, L/K$ finite Galois s.t.

$V|_{G_L}$ is semi-stable (resp. crystalline).