NCMW - Modular Forms and Galois Representations

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L-values and congruences of modular forms

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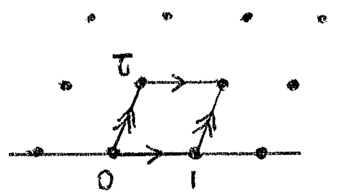
$\mathfrak{h} = \{\tau \in \mathbb{C} : \operatorname{Im}(\tau) > 0\}$ upper half-plane

Given $\tau \in \mathfrak{h}$, lattice $\Lambda_\tau = \langle 1, \tau \rangle_{\mathbb{Z}} = \mathbb{Z} \oplus \mathbb{Z}\tau$



Complex torus $\mathbb{C}/\Lambda_\tau$, a compact connected Riemann surface of genus one.

$H_1(\mathbb{C}/\Lambda_\tau) \simeq \Lambda_\tau$

Space of holomorphic differentials on $\mathbb{C}/\Lambda_\tau$ is $\langle dz \rangle_{\mathbb{C}}$
dimension = genus = 1.

If $\alpha \in \mathbb{C} \setminus \{0\}$,

$$\mathbb{C}/\Lambda_\tau \xrightarrow{\sim} \mathbb{C}/\alpha\Lambda_\tau \qquad \text{(Isomorphism of Riemann surfaces.)}$$
$$[z] \longmapsto [\alpha z]$$

Suppose $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, so $a, b, c, d \in \mathbb{Z}$ and $\det(\gamma) = 1$.

$\Lambda_\tau = \langle 1, \tau \rangle_{\mathbb{Z}} = \langle c\tau + d, a\tau + b \rangle_{\mathbb{Z}}$

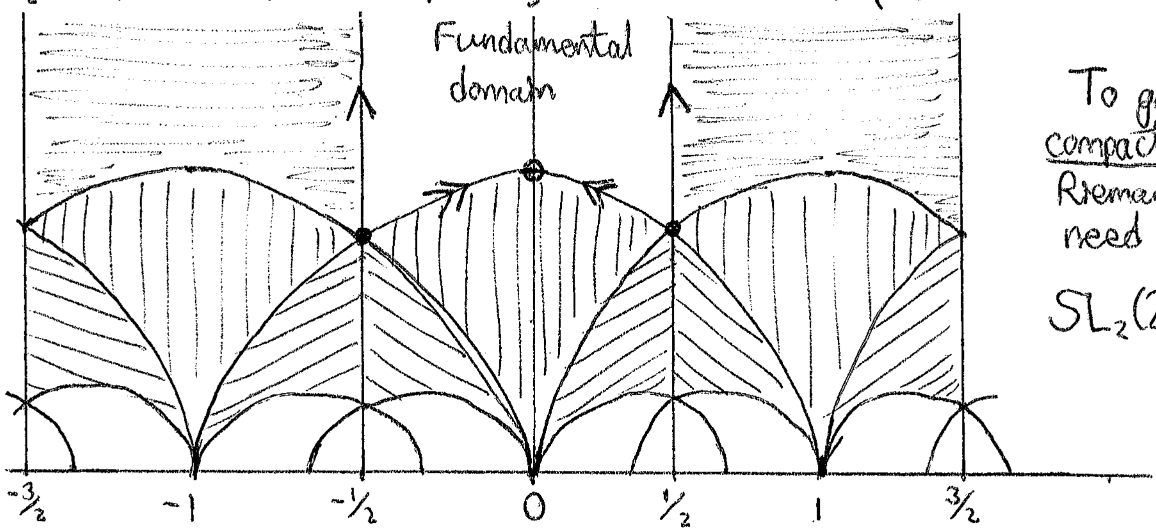
Letting $\alpha = (c\tau + d)^{-1}$, $\alpha \Lambda_\tau = \langle 1, \frac{a\tau+b}{c\tau+d} \rangle$
$\uparrow \in \mathfrak{h}$, since $\det(\gamma) = +1$.

so $\mathbb{C}/\Lambda_\tau \simeq \mathbb{C}/\Lambda_{\frac{a\tau+b}{c\tau+d}} =: \mathbb{C}/\Lambda_{\gamma(\tau)}$

Group action of $SL_2(\mathbb{Z})$ on $\mathfrak{h}$.

Natural bijection of sets

$\{\text{Complex tori up to isomorphism}\} \xrightarrow{\sim} SL_2(\mathbb{Z}) \backslash \mathfrak{h}$



To get a <u>compact</u> connected Riemann surface, need

$SL_2(\mathbb{Z}) \backslash \mathfrak{h}^*$

$\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q}) = \mathfrak{h} \cup \mathbb{Q} \cup \{\infty\}$

Given a positive integer N,

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ : \ c \equiv 0 \pmod N \right\}$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ : \ \begin{matrix} c \equiv 0 \pmod N \\ d \equiv 1 \pmod N \end{matrix} \right\}$$

$$\{\text{Complex tori up to isomorphism}\} \xrightarrow{\sim} SL_2(\mathbb{Z}) \backslash \mathfrak{h}$$

$$\mathbb{C}/\Lambda_\tau \longmapsto [\tau] = SL_2(\mathbb{Z})\tau$$

extends to

$$\{(\text{Complex torus, Point of order } N)\} \xrightarrow{\sim} \Gamma_1(N) \backslash \mathfrak{h}$$

$$(\mathbb{C}/\Lambda_\tau, 1/N) \longmapsto \Gamma_1(N)\tau$$

and

$$\{(\text{Complex torus, Cyclic subgroup of order } N)\} \xrightarrow{\sim} \Gamma_0(N) \backslash \mathfrak{h}$$

$$(\mathbb{C}/\Lambda_\tau, \langle 1/N \rangle) \longmapsto \Gamma_0(N)\tau$$

How does this work?

Given $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

$$(\mathbb{C}/\Lambda_\tau, 1/N) \xrightarrow{\sim} \left(\mathbb{C}/\Lambda_{\frac{a\tau+b}{c\tau+d}}, \frac{1}{N(c\tau+d)}\right)$$

$$\left(z \longmapsto \frac{z}{c\tau+d}\right)$$

$$\simeq \left(\mathbb{C}/\Lambda_{\frac{a\tau+b}{c\tau+d}}, \frac{a}{N} - \frac{c}{N}\frac{a\tau+b}{c\tau+d}\right)$$

$$\simeq \left(\mathbb{C}/\Lambda_{\frac{a\tau+b}{c\tau+d}}, \frac{1}{N}\right) \iff \frac{c}{N} \in \mathbb{Z} \text{ and } a \equiv 1 \pmod N$$

$$\iff \gamma \in \Gamma_1(N)$$

Similarly $(\mathbb{C}/\Lambda_\tau, \langle 1/N \rangle) \simeq \left(\mathbb{C}/\Lambda_{\frac{a\tau+b}{c\tau+d}}, \langle 1/N \rangle\right)$

$$\iff \frac{c}{N} \in \mathbb{Z} \iff \gamma \in \Gamma_0(N).$$

$$Y_0(N) := \Gamma_0(N) \backslash \mathfrak{h}$$

$$X_0(N) := \Gamma_0(N) \backslash \mathfrak{h}^*$$ Compact, connected Riemann surface, genus g

Let $\{\omega_1, \ldots, \omega_g\}$ be a basis for the space of holomorphic differentials on $X_0(N)$.

Define $i : X_0(N) \longrightarrow \mathbb{C}^g$

$$P \longmapsto \left(\int_\infty^P \omega_1, \ldots, \int_\infty^P \omega_g\right)$$

Path from ∞ to P is only defined up to loops $c \in H_1(X_0(N)) \simeq \mathbb{Z}^{2g}$ so we really only have a well-defined map

$$i : X_0(N) \longrightarrow \mathbb{C}^g/\Gamma, \quad \text{where } \Gamma \text{ is a rank } 2g \text{ lattice } \left\{ \left(\int_c \omega_1, \ldots, \int_c \omega_g \right) : c \in H_1(X_0(N)) \right\}$$

$$\mathbb{C}^g/\Gamma = J_0(N)$$

~~generated by~~

More generally, given a divisor $D = \sum_{P \in X_0(N)} n_P (P)$, a formal finite $\mathbb{Z}$-linear combination of points,

$$i(D) := \sum n_P \, i(P)$$

$$i : \mathrm{Div}(X_0(N)) \longrightarrow J_0(N)$$

Restrict to subgroup $\mathrm{Div}^0(X_0(N))$ of degree-zero divisors $(\sum n_P = 0)$

<u>Abel's Theorem</u>

$$\ker i = \{ \mathrm{div}(f) : f \text{ non-zero meromorphic function on } X_0(N) \} =: \mathrm{Pr}(X_0(N))$$

where $\mathrm{div}(f) := \sum_{P \in X_0(N)} \mathrm{ord}_P(f)(P)$

Let $\mathrm{Pic}^0(X_0(N)) := \mathrm{Div}^0(X_0(N)) / \mathrm{Pr}(X_0(N))$.

Then $i : \mathrm{Pic}^0(X_0(N)) \xrightarrow{\sim} J_0(N)$.

Fixing a prime number $p \nmid N$, we have a holomorphic map

$$\pi_1 : X_0(Np) \longrightarrow X_0(N)$$

$$\Gamma_0(Np)\tau \longmapsto \Gamma_0(N)\tau \qquad \text{Well-defined since } \Gamma_0(Np) \leq \Gamma_0(N).$$

Degree $[\Gamma_0(N) : \Gamma_0(Np)] = p+1$.

There is another map, also of degree $p+1$,

$$\pi_2 : X_0(Np) \longrightarrow X_0(N)$$

$$\Gamma_0(Np)\tau \longmapsto \Gamma_0(N)p\tau$$

Well-defined since if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(Np)$ then $p\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{a(p\tau)+pb}{(c/p)(p\tau)+d}$

and $\begin{pmatrix} a & pb \\ c/p & d \end{pmatrix} \in \Gamma_0(N)$.

Hecke correspondence $T_p = (\pi_2)_* \pi_1^* : J_0(N) \longrightarrow J_0(N)$

($(\pi_2)_*$: push-forward of divisor classes; π_1^*: pullback of divisor classes)

Let $\omega = f(\tau)\,d\tau$ be a holomorphic differential on $Y_0(N) = \Gamma_0(N)\backslash\mathfrak{h}$ (or perhaps its pullback to $\mathfrak{h}$).

ω must be invariant under $\Gamma_0(N)$, so given any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$,

$$f(\tau)\,d\tau = f\left(\frac{a\tau+b}{c\tau+d}\right) d\left(\frac{a\tau+b}{c\tau+d}\right)$$

Now $d\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{(c\tau+d)a-(a\tau+b)c}{(c\tau+d)^2}\,d\tau = \frac{d\tau}{(c\tau+d)^2}$ since $ad-bc=1$.

Hence $$\boxed{f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 f(\tau)} \quad *$$

Putting $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $f(\tau+1) = f(\tau)$ so f has a Fourier series

$$f(\tau) = \sum_{n\in\mathbb{Z}} a_n q^n \quad \text{where } q = e^{2\pi i\tau}$$

$dq = 2\pi i\, e^{2\pi i \tau}\,d\tau$

so $\frac{dq}{q} = 2\pi i\,d\tau$

Hence for $f(\tau)\,d\tau = \frac{1}{2\pi i}\sum_{n\in\mathbb{Z}} a_n q^n \frac{dq}{q}$ to be holomorphic at $\tau = i\infty$ $(q=0)$

must be $f(\tau) = \sum_{n\geq 1} a_n q^n$.

In other words, f vanishes at the cusp $i\infty$.

To extend to a holomorphic differential on $X_0(N)$, similarly f vanishes at all the cusps $\Gamma_0(N)\backslash\mathbb{P}^1(\mathbb{Q})$.

Holomorphic f satisfying $*$ (for all $\gamma \in \Gamma_0(N)$) and holomorphic at the cusps ~~and vanishing at all the cusps~~ is called a <u>modular form of weight 2 for $\Gamma_0(N)$</u> ($f \in M_2(\Gamma_0(N))$).

If f <u>vanishes</u> at the cusps it is called a <u>cusp form</u> ($f \in S_2(\Gamma_0(N))$).

$$S_2(\Gamma_0(N)) \xrightarrow{\sim} \{\text{Hol. diffs. on } X_0(N)\}$$
$$f \longmapsto f(\tau)\,d\tau$$

T_p naturally acts on $S_2(\Gamma_0(N))$:

$$T_p(f)(\tau) = p\,f(p\tau) + \frac{1}{p}\sum_{b=0}^{p-1} f\left(\frac{\tau+b}{p}\right)$$

If $f(\tau) = \sum_{n=1}^{\infty} a_n q^n$ then $(T_p f)(\tau) = \sum_{n=1}^{\infty} b_n q^n$

where $b_n = \begin{cases} a_{np} & \text{if } p \nmid n; \\ a_{np} + p\,a_{n/p} & \text{if } p \mid n. \end{cases}$ $**$

In fact there are Hecke operators T_n for all $n \geq 1$, all commuting with each other. The T_p for p prime generate all the others.

Petersson inner product on $S_2(\Gamma_0(N))$

$$\langle f, g\rangle = \int_{\Gamma_0(N)\backslash \mathfrak{h}} f(\tau)\overline{g(\tau)}\, dx\,dy \qquad \text{where } \tau = x+iy.$$

If $\gcd(n,N)=1$ then T_n is self-adjoint with respect to $\langle , \rangle$:
$\langle T_n f, g\rangle = \langle f, T_n g\rangle$

Since the T_n commute with each other, $S_2(\Gamma_0(N))$ has a basis of simultaneous eigenforms for all the T_n with $\gcd(n,N)=1$.

Let $S_2(\Gamma_0(N))^{old}$ be the subspace generated by elements of the form $f(\tau) = g(d\tau)$ where $g \in S_2(\Gamma_0(M))$ with $M \mid N$ $(M<N)$ and $d \mid (N/M)$.

Let $S_2(\Gamma_0(N))^{new}$ be the orthogonal complement of $S_2(\Gamma_0(N))^{old}$ with respect to $\langle , \rangle$.

<u>Fact</u> $S_2(\Gamma_0(N))^{new}$ has a basis of simultaneous eigenforms for <u>all</u> T_n.
Such a "newform" $f = \sum_{n=1}^{\infty} a_n q^n$ can be scaled so that $a_1 = 1$ (normalised)
and then formulas such as ** show that $\boxed{T_n f = a_n f}$

i.e. the Fourier coefficients are the Hecke eigenvalues.

The T_n act naturally on the $\mathbb{Z}$-module $H_1(X_0(N))$.
It follows that the eigenvalues a_n are algebraic integers.
For a given f they generate a finite extension $K_f : \mathbb{Q}$.

Special case: $K_f = \mathbb{Q}$, the a_n are rational integers.

<u>Example</u> $\dim S_2(\Gamma_0(11)) = 1$. For $X_0(11)$, $g=1$. $i : X_0(11) \xrightarrow{\sim} J_0(11)$.

$S_2(\Gamma_0(11))$ spanned by a normalised newform $f = q\prod_{n=1}^{\infty}\left[(1-q^n)^2(1-q^{11n})^2\right]$

$$= q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 - 2q^{10} + q^{11} - 2q^{12} + 4q^{13} + 4q^{14} - q^{15} \cdots$$

What is the meaning of these coefficients?

Up to now, we have looked at $\mathbb{C}/\Lambda_\tau$ and $X_0(N) = \Gamma_0(N)\backslash \mathfrak{h}^*$ purely as Riemann surfaces.
Henceforth, we need to look at them as algebraic curves.

Given $\Lambda_\tau = \langle 1, \tau\rangle_{\mathbb{Z}}$ define the Weierstrass $\wp$-function

$$\wp_{\Lambda_\tau}(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda_\tau \setminus \{0\}} \left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right)$$

This is doubly periodic: $\wp_{\Lambda_\tau}(z) = \wp_{\Lambda_\tau}(z+\omega) \quad \forall \omega \in \Lambda_\tau$

It has a double pole at each lattice point and is holomorphic elsewhere on $\mathbb{C}$.

It satisfies the identity

$$\wp'(z)^2 = 4\wp(z)^3 - g_2(\Lambda_\tau)\wp(z) - g_3(\Lambda_\tau),$$

where $g_2(\Lambda_\tau) = 60\,G_4(\tau)$

$g_3(\Lambda_\tau) = 140\,G_6(\tau)$

and for even $k>2$ the Eisenstein series $G_k(\tau) = \sum_{\substack{\omega\in\Lambda_\tau \\ \omega\neq 0}} \frac{1}{\omega^k} = \sum_{\substack{(m,n) \\ \in \mathbb{Z}^2\setminus\{(0,0)\}}} \frac{1}{(m\tau+n)^k}$

Since, for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, $\Lambda_{\frac{a\tau+b}{c\tau+d}} = (c\tau+d)^{-1}\Lambda_\tau$,

$G_k\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k G_k(\tau)$.

Also, $\lim_{\tau\to i\infty} \sum_{n\in\mathbb{Z}} \frac{1}{n^k} = 2\zeta(k)$ is finite, so $G_k \in M_k(SL_2(\mathbb{Z}))$.

Let E_τ be the algebraic curve $y^2 = 4x^3 - g_2(\Lambda_\tau)x - g_3(\Lambda_\tau)$

$\left(Y^2 Z = 4X^3 - g_2 XZ^2 - g_3 Z^3 \text{ as a projective curve}\right)$

Then we have an isomorphism of Riemann surfaces

$$\mathbb{C}/\Lambda_\tau \xrightarrow{\sim} E_\tau(\mathbb{C})$$

$$z+\Lambda_\tau \longmapsto \begin{cases} (x,y) = (\wp_{\Lambda_\tau}(z), \wp'_{\Lambda_\tau}(z)) & z\notin\Lambda_\tau \\ O = \text{“}\infty\text{”} & z\in\Lambda_\tau \end{cases}$$

i.e. $z+\Lambda_\tau \longmapsto [X,Y,Z] = \begin{cases} [\wp_{\Lambda_\tau}(z), \wp'_{\Lambda_\tau}(z), 1] & z\notin\Lambda_\tau \\ [0, 1, 0] & z\in\Lambda_\tau \end{cases}$

Inverse map $E_\tau(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}/\Lambda_\tau$

$$P = (x,y) \longmapsto \int_O^P \frac{dx}{y} \quad \text{modulo } \Lambda_\tau$$

Note that $\frac{dx}{y} = \frac{d\wp(z)}{\wp'(z)} = \frac{\wp'(z)\,dz}{\wp'(z)} = dz$, so the integral of $\frac{dx}{y}$ around a loop on $E_\tau(\mathbb{C})$ is $\int_0^\omega dz = \omega$ for some $\omega\in\Lambda_\tau$.

Since $\mathbb{C}/\Lambda_\tau$ is an abelian group under addition,

$E_\tau(\mathbb{C})$ has the structure of an abelian group, neutral element O (at infinity).

$E_\tau(\mathbb{C}) \simeq \mathrm{Pic}^0(E_\tau(\mathbb{C}))$ Group isomorphism

$P \longmapsto [(P)-(O)]$.

Geometrically, the three intersection points of $E_\tau(\mathbb{C})$ with a (projective) line (counted with multiplicity) add up to O.

If $(x_1,y_1)+(x_2,y_2) = (x_3,y_3)$ then

$x_3, y_3 \in \mathbb{Q}(x_1,y_1,x_2,y_2,g_2,g_3)$

$(*)$ $\left[\begin{array}{l} y^2+xy = x^3 - \frac{36}{j-1728}x - \frac{1}{j-1728} \\ \text{has } j\text{-invariant } j. \end{array} \right.$

E_τ is an algebraic group (elliptic curve) defined over the field $\mathbb{Q}(g_2,g_3)$.

$g_2^3,\ g_3^2 \in M_{12}(SL_2(\mathbb{Z}))$, in fact

$\Delta := g_2^3 - 27g_3^2 \in S_{12}(SL_2(\mathbb{Z}))$.

$\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n \qquad q=e^{2\pi i\tau}$

($\tau(n)$: Ramanujan tau function.)

$j := \dfrac{1728\, g_2^3}{\Delta} = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \dots$

$j\left(\frac{a\tau+b}{c\tau+d}\right) = j(\tau) \quad \forall \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{Z}),\ \tau\in\mathfrak{h}$. (A modular <u>function</u>.)

$j : SL_2(\mathbb{Z})\backslash\mathfrak{h}^* \xrightarrow{\sim} \mathbb{C}\cup\{\infty\} = \mathbb{P}^1(\mathbb{C})$, isomorphism of Riemann surfaces.

So now we see $X_0(1)$ as the set of $\mathbb{C}$-points of an algebraic curve $\mathbb{P}^1$ defined over $\mathbb{Q}$.

From now on we write $X_0(1)(\mathbb{C})$ instead of $X_0(1)$ for $SL_2(\mathbb{Z})\backslash\mathfrak{h}^*$, and write $X_0(1)$ for the algebraic curve

Similarly we write $X_0(N)(\mathbb{C})$ for $\Gamma_0(N)\backslash\mathfrak{h}^*$

$X_1(N)(\mathbb{C})$ for $\Gamma_1(N)\backslash\mathfrak{h}^*$

and $X_0(N)$, $X_1(N)$ are algebraic curves defined over $\mathbb{Q}$.

(This follows from $(*)$) [This is technically not quite the true story, but is intended to give a rough idea: 3 variables x,y,j, two equations: that of E, and from (x,y) being an N-torsion pt.]

<u>Example</u> $X_0(11)$ is the elliptic curve $E: y^2+y = x^3-x^2-10x-20$

The group $E(\mathbb{Q}) \simeq \mathbb{Z}/5\mathbb{Z}$, generated by the point $P=(5,5)$.

(In general Mordell's Thm $\Rightarrow$ $E(\mathbb{Q})$ finitely generated.)

$[5]P = O$ ($[5]$: multiplication by 5 map.)

In fact, on $X_0(11)$, P is the cusp $0 \longmapsto [(0)-(\infty)]$ in $\mathrm{Pic}^0(X_0(11)) = J_0(11)$.

Given elliptic curves E_1, E_2 and an isogeny $\psi : E_1 \to E_2$
(a non-constant morphism of curves, taking O_1 to O_2,
necessarily a group homomorphism),

$\psi : \mathrm{Pic}^0(E_1) \longrightarrow \mathrm{Pic}^0(E_2)$ is ψ_*, push-forward of divisor classes.

Pull-back $\psi^* : \mathrm{Pic}^0(E_2) \to \mathrm{Pic}^0(E_1)$
comes from a <u>dual</u> isogeny $\hat{\psi} : E_2 \to E_1$.

$\hat{\psi} \circ \psi : E_1 \to E_1$ (i.e. $\psi^* \psi_* : \mathrm{Pic}^0(E_1) \to \mathrm{Pic}^0(E_1)$)

is $[\deg \psi]$, multiplication by the degree.

For each prime $p \neq 11$, $E = X_0(11)$ has <u>good reduction</u> at p :

$\tilde{E}/\mathbb{F}_p$ is nonsingular, an elliptic curve in characteristic p.
$\tilde{E} : y^2 + y = x^3 - x^2 - 10x - 20$

May count the points in the finite set $\tilde{E}(\mathbb{F}_p)$.

e.g. $\tilde{E}(\mathbb{F}_2) = \{O, (0,0), (0,1), (1,0), (1,1)\}$ $\quad \# \tilde{E}(\mathbb{F}_2) = 5$

$\tilde{E}(\mathbb{F}_3) = \{O, (1,0), (1,2), (2,0), (2,2)\}$ $\quad \# \tilde{E}(\mathbb{F}_3) = 5$

In general, Frobenius map $\varphi_p : \tilde{E}(\overline{\mathbb{F}}_p) \to \tilde{E}(\overline{\mathbb{F}}_p)$

$(x, y) \mapsto (x^p, y^p)$

This is a purely inseparable isogeny of degree p.
$\ker \varphi_p = \varphi_p^{-1}\{O\} = \{O\}$ has size 1, not $\deg \varphi_p = p$.

$$\tilde{E}(\mathbb{F}_p) = \{ Q \in \tilde{E}(\overline{\mathbb{F}}_p) : \varphi_p(Q) = Q \}$$

$$= \ker(\underbrace{1 - \varphi_p}_{\text{separable isogeny}})$$

$$[\# \tilde{E}(\mathbb{F}_p)] = [\deg(1 - \varphi_p)] = \widehat{(1 - \varphi_p)} \circ (1 - \varphi_p)$$

$$= 1 - (\hat{\varphi}_p + \varphi_p) + \hat{\varphi}_p \circ \varphi_p$$

$$= 1 + p - (\varphi_p + \hat{\varphi}_p)$$

<u>Eichler-Shimura Congruence Relation</u>

Viewing E ~~and~~ as $X_0(11) \simeq J_0(11)$,

$\tilde{T}_p = \varphi_p + \hat{\varphi}_p$ in $\mathrm{End}(\tilde{J}_0(11))$

Hence $\boxed{\# \tilde{E}(\mathbb{F}_p) = 1 + p - a_p}$ $\quad (\Rightarrow |a_p| < 2\sqrt{p})$

p	a_p	$\#\tilde{E}(\mathbb{F}_p) = 1+p-a_p$
2	-2	5
3	-1	5
5	1	5
7	-2	10
[11	1	11]
13	4	10
17	-2	20

Notice always $5 \mid \#\tilde{E}(\mathbb{F}_p)$ (for $p \neq 11$)

This is because the reduction of $P = (5,5)$ gives an element of order 5.

In other words $\boxed{a_p \equiv 1+p \pmod 5 \quad \forall \text{ primes } p \neq 11.}$

The Dirichlet series

$$L_f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \frac{1}{(1-11^{-s})} \prod_{p \neq 11} \frac{1}{1-a_p p^{-s} + p^{1-2s}}$$

is also $L_E(s)$.

As $L_E(s)$, it converges for $\mathrm{Re}(s) > 3/2$ (since $|a_p| < 2p^{1/2}$)

But as $L_f(s)$,

$$\int_0^{\infty} f(iy) y^s \frac{dy}{y} = (2\pi)^{-s} \Gamma(s) L_f(s)$$

Using this, may show that (i) $L_f(s)$ is analytic on $\mathbb{C}$

(ii) $\xi(s) = \pm \xi(2-s)$

where $\xi(s) = 11^{s/2} (2\pi)^{-s} \Gamma(s) L_f(s)$.

Central point $s=1$.

__Birch and Swinnerton-Dyer Conjecture__

$$\mathrm{Rank}\, E(\mathbb{Q}) = \operatorname*{ord}_{s=1} L_E(s).$$

Known by theorem of Kolyvagin, Gross, Zagier when RHS $\leq$ 1.

In this example $L_E(1) \neq 0$ and $E(\mathbb{Q})$ is finite.

Refined conjecture

$$L_E(1) = \frac{c_{11} \cdot \# Ш}{(\# E(\mathbb{Q}))^2} \int_{E(\mathbb{R})} \omega$$

$\omega = \dfrac{dx}{2y+1}$ where $y^2 + y = x^3 - x^2 - 10x - 20$ Global minimal Weierstrass equation

$\Delta = -11^5$.

$c_{11} = 5$, $\# Ш = 1$, $\# E(\mathbb{Q}) = 5$.

$$\int_{E(\mathbb{R})} \omega \approx 1.26920930428$$

$$L_E(1) \approx 0.253841860856$$

As a holomorphic differential on $X_0(11)(\mathbb{C}) = \Gamma_0(11)\backslash\mathfrak{h}^*$, ω must be some multiple of $f(\tau)d\tau$

Recall that $f(\tau)\,d\tau = \frac{1}{2\pi i}\left(\sum_{n=1}^{\infty} a_n q^n\right)\frac{dq}{q}$ $\quad q = e^{2\pi i \tau}$

In fact, since ω is defined over $\mathbb{Q}$, must have $\omega = c\, 2\pi i f(\tau)\,d\tau$ for some $c \in \mathbb{Q}^\times$ (Manin constant)

Conjecturally $c = 1$ (certainly true in this example).

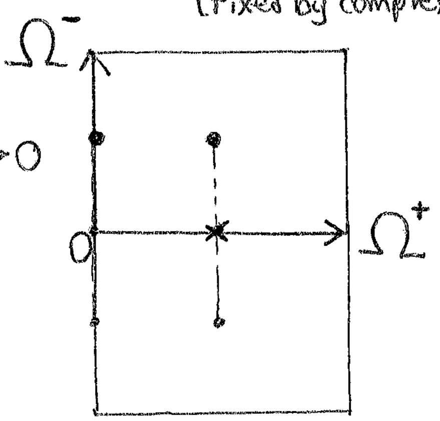
We have the Riemann surface $X_0(11)(\mathbb{C})$ as $\Gamma_0(11)\backslash\mathfrak{h}^*$ but as $E(\mathbb{C})$ it is $\mathbb{C}/\Lambda$ for some lattice Λ.

What does Λ look like?

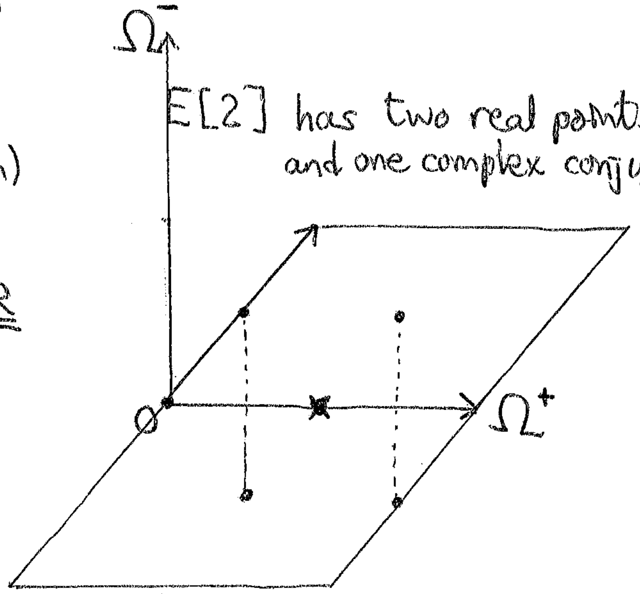
$\underbrace{E[2]}_{\text{2-torsion points}} \subseteq E(\mathbb{R})$ (fixed by complex conjugation)

$E[2]$ has two real points and one complex conjugate pair

$\Delta > 0$

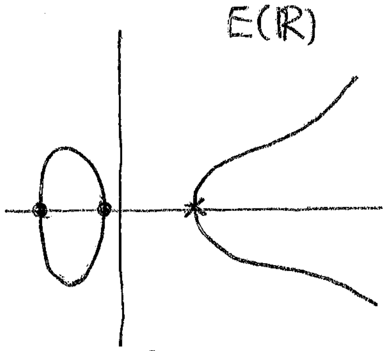


OR



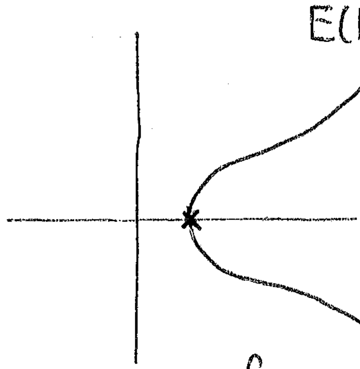
$\Delta < 0$

E(ℝ)



$\int_{E(\mathbb{R})} \omega = 2\Omega^+$

E(ℝ)



$\int_{E(\mathbb{R})} \omega = \Omega^+$

[$E = X_0(11)$ is this case.]

$$\int_{E(\mathbb{C})} \omega\wedge\overline{\omega} = \int_{\mathbb{C}/\Lambda} dz\wedge d\bar{z} = 2i\int_{\mathbb{C}/\Lambda} dx\,dy = \Omega^+\Omega^-$$

But $\omega = 2\pi i f(\tau)\,d\tau \Rightarrow \int_{E(\mathbb{C})} \omega\wedge\overline{\omega} = \int_{X_0(11)(\mathbb{C})} (2\pi)^2 |f(\tau)|^2\, d\tau\wedge d\bar{\tau} = 2i(2\pi)^2\langle f,f\rangle$

Hence $\boxed{\Omega^+\Omega^- = 2i\,(2\pi)^2\langle f,f\rangle}$ (Would be $i(2\pi)^2\langle f,f\rangle$ in other case.)

Another example

$\dim(S_2(\Gamma_0(37))) = 2$, spanned by normalised newforms

$$f = q - 2q^2 - 3q^3 + 2q^4 - 2q^5 + 6q^6 - q^7 + 6q^9 + 4q^{10} - 5q^{11} - 6q^{12} - 2q^{13} \dots$$

$$g = q + q^3 - 2q^4 - q^7 - 2q^9 + 3q^{11} - 2q^{12} - 4q^{13} \dots$$

$X_0(37)$ has genus 2 so is not an elliptic curve.

Neither is $J_0(37)$, which is a 2-dimensional abelian variety $/\mathbb{Q}$.

All the Hecke correspondences $T_n \in \mathrm{End}(J_0(14))$ are defined over $\mathbb{Q}$. Let $\mathbb{T}$ be the ring they generate, over $\mathbb{Z}$.

We have a homomorphism $\theta_f : \mathbb{T} \to \mathbb{Z}$, $T_n \mapsto a_n(f)$

Let $I_f = \ker \theta_f$

Then $E_f := J_0(37) / I_f J_0(37)$ is an elliptic curve $/\mathbb{Q}$, on which T_n acts as $[a_n]$. (Eichler-Shimura construction.)

For primes $p \nmid 14$, E_f has good reduction at p, and
$$\#\tilde{E}_f(\mathbb{F}_p) = 1 + p - a_p(f).$$

$$X_0(37) \xrightarrow{i} J_0(37) \xrightarrow{\psi_f} E_f$$

$\psi_f : J_0(37) \to E_f$ is $(\pi_f)_* : \mathrm{Pic}^0(X_0(37)) \to \mathrm{Pic}^0(E_f)$

Also have $\widehat{\psi_f} : E_f \to J_0(37)$ which is $(\pi_f)^* : \mathrm{Pic}^0(E) \to \mathrm{Pic}^0(X_0(37))$.

$\psi_f \circ \widehat{\psi_f} : E_f \to E_f$ is $[\deg \pi_f]$

↑ "modular degree" m_{E_f}

$E_f : y^2 + y = x^3 - x$ $\qquad E_f(\mathbb{Q}) \simeq \mathbb{Z}$, generated by $P = (0,0)$

Similarly have $E_g : y^2 + y = x^3 + x^2 - 23x - 50$

$E_g(\mathbb{Q}) \simeq \mathbb{Z}/3\mathbb{Z}$, generated by $P = (8,18)$

In fact
$$0 \to E_g \xrightarrow{\widehat{\psi_g}} J_0(37) \xrightarrow{\psi_f} E_f \to 0$$
$$0 \to E_f \xrightarrow{\widehat{\psi_f}} J_0(37) \xrightarrow{\psi_g} E_g \to 0$$
and $\deg \pi_f = \deg \pi_g = 2$.

Inside $J_0(37)$, $E_f \cap E_g$ (i.e. $\hat{\psi}_f(E_f) \cap \hat{\psi}_g(E_g)$) is $E_f[2] = E_g[2]$, since $E_f[2] = E_f[\psi_f \circ \hat{\psi}_f] \simeq \hat{\psi}_f(E_f) \cap \ker \psi_f = \hat{\psi}_f(E_f) \cap \hat{\psi}_g(E_g)$.

Since T_n acts as $a_n(f)$ on E_f
$a_n(g)$ on E_g

$E_f[2] = E_g[2]$ inside $J_0(37)$ $\Rightarrow$ $\boxed{a_n(f) \equiv a_n(g) \pmod 2 \quad \forall n \geq 1}$

In general we have $\pi_f : X_0(N) \longrightarrow E_f$
(where $f \in S_2(\Gamma_0(N))$ is a normalised newform with rational integer coefficents).

If $E_f : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$ global minimal Weierstrass equation

$$\omega = \frac{dx}{2y + a_1 x + a_3}$$

$\pi_f^*(\omega) = c \, 2\pi i \, f(\tau) \, d\tau$ for $c \in \mathbb{Q}^\times$ (conjecturally $c = 1$)

$$\int_{X_0(N)} \pi_f^*(\omega \wedge \bar{\omega}) = \deg \pi_f \int_{E(\mathbb{C})} \omega \wedge \bar{\omega}$$

$$\Rightarrow \boxed{\frac{4i}{\#E(\mathbb{R})[2]} (2\pi)^2 \langle f, f \rangle = (\deg \pi_f) \, \Omega^+ \Omega^-}$$

Yet another example

$1058 = 2 \cdot 23^2$ $\qquad \dim(S_2(\Gamma_0(1058))) = 115$

Among the normalised newforms are

$$f = q - q^2 - 2q^3 + q^4 - 3q^5 + 2q^6 - 2q^7 - q^8 + q^9 + 3q^{10} - 6q^{11} - 2q^{12} - q^{13} + 2q^{14} \ldots$$

$$g = q - q^2 + 3q^3 + q^4 + 2q^5 - 3q^6 - 2q^7 - q^8 + 6q^9 - 2q^{10} + 4q^{11} + 3q^{12} + 4q^{13} + 2q^{14} \ldots$$

LMFDB
1058.a2 $E_f : y^2 + xy + y = x^3 + 2$ $\qquad \deg \pi_f = 2^4 \cdot 5$

1058.e1 $E_g : y^2 + xy = x^3 - x^2 - 332311x - 73733731$ $\qquad \deg \pi_g = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 23$

In fact $E_f[5] = E_g[5]$ inside $J_0(1058)$, $*$

and $\boxed{a_n(f) \equiv a_n(g) \pmod 5 \quad \forall n \geq 1}$

$E_f(\mathbb{Q}) \simeq \mathbb{Z}^2$, generated by $(-1, 1), (0, 1)$

$E_g(\mathbb{Q}) = \{O\}$

$L_f(1) = 0$ $\qquad \dfrac{L_g(1)}{\Omega^+} = 25$

(in agreement with Birch & Swinnerton-Dyer)

Actually, $* \Rightarrow \left(L_f(1) = 0 \Rightarrow \frac{L_g(1)}{\Omega^+} \equiv 0 \pmod 5 \right)$

BSD says $\frac{L_g(1)}{\Omega^+} = \frac{c_2 c_{23} \# Ш}{(\# E(\mathbb{Q})_{tors})^2}$ but $c_2 = c_{23} = 1$ and $\# E(\mathbb{Q})_{tors} = 1$

so should have $\# Ш = 25$

(Visibility) Construction of Cremona and Mazur directly links rational points of infinite order on E_f with elements in $Ш[5]$ for E_g, via the mod **5** congruence.

Let $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We have ~~an exact~~ a short exact sequence of $G_{\mathbb{Q}}$-modules:

$$0 \longrightarrow E_f(\overline{\mathbb{Q}})[5] \longrightarrow E_f(\overline{\mathbb{Q}}) \xrightarrow{[5]} E_f(\overline{\mathbb{Q}}) \to 0$$

This gives rise to a long exact sequence in Galois cohomology:

$$0 \to H^0(G_{\mathbb{Q}}, E_f[5]) \to H^0(G_{\mathbb{Q}}, E_f(\overline{\mathbb{Q}})) \xrightarrow{[5]} H^0(G_{\mathbb{Q}}, E_f(\overline{\mathbb{Q}})) \xrightarrow{\delta}$$
$$H^1(G_{\mathbb{Q}}, E_f[5]) \to H^1(G_{\mathbb{Q}}, E_f(\overline{\mathbb{Q}})) \to H^1(G_{\mathbb{Q}}, E_f(\overline{\mathbb{Q}})) \to \dots$$

Part of this is

$$E_f(\mathbb{Q}) \xrightarrow{[5]} E_f(\mathbb{Q}) \xrightarrow{\delta} H^1(G_{\mathbb{Q}}, E_f[5])$$

$$H^1(G_{\mathbb{Q}}, E_f[5]) = \frac{Z^1(G_{\mathbb{Q}}, E_f[5])}{B^1(G_{\mathbb{Q}}, E_f[5])} \quad \left(\frac{\text{cocycles}}{\text{coboundaries}} \right)$$

where $Z^1(G_{\mathbb{Q}}, E_f[5]) = \{ c : G_{\mathbb{Q}} \to E_f[5] \mid c(\sigma\tau) = c(\sigma) + \sigma c(\tau) \ \forall \sigma, \tau \in G_{\mathbb{Q}} \}$

$B^1(G_{\mathbb{Q}}, E_f[5]) = \{ c : G_{\mathbb{Q}} \to E_f[5] \mid c(\sigma) = \sigma(m) - m \ \forall \sigma \in G_{\mathbb{Q}},$ for some $m \in E_f[5] \}$

$B^1 \subseteq Z^1$ because $\sigma\tau(m) - m = [\sigma(m) - m] + \sigma[\tau(m) - m]$

Given $P \in E_f(\mathbb{Q})$, $\exists Q \in E_f(\overline{\mathbb{Q}})$ such that $[5]Q = P$ (Q determined up to addition of element of $E_f[5]$).

$\delta(P)$ is represented by the cocycle

$$\sigma \longmapsto \sigma(Q) - Q$$

Remarks (i) This is a cocycle, for the same reason that $B^1 \subseteq Z^1$.

(ii) It is valued in $E_f[5]$, since $[5](\sigma(Q) - Q) = \sigma([5]Q) - [5]Q = \sigma(P) - P = 0$.

(iii) A different choice of Q only changes it by a coboundary.

For any prime p, fixing $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ so $G_{\mathbb{Q}_p} \hookrightarrow G_{\mathbb{Q}}$

$$\begin{array}{ccc} E_f(\mathbb{Q}) & \xrightarrow{\delta} & H^1(G_{\mathbb{Q}}, E_f[5]) \\ \downarrow & \circlearrowright & \downarrow \text{res.} \\ E_f(\mathbb{Q}_p) & \xrightarrow{\delta_p} & H^1(G_{\mathbb{Q}_p}, E_f[5]) \end{array}$$

so $\mathrm{res}(\delta(E_f(\mathbb{Q}))) \subseteq \delta_p(E_f(\mathbb{Q}_p))$ (similarly for $p=\infty$, $\mathbb{Q}_p = \mathbb{R}$)

i.e. $\delta(E_f(\mathbb{Q})) \subseteq \mathrm{Sel}_5(E_f)$.

Since $E_f(\mathbb{Q}) \simeq \mathbb{Z}^2$, $E_f(\mathbb{Q})/5E_f(\mathbb{Q}) \hookrightarrow H^1(G_{\mathbb{Q}}, E_f[5])$ gives a $(\mathbb{Z}/5\mathbb{Z})^2$ inside $H^1(G_{\mathbb{Q}}, E_f[5])$, in fact in $\mathrm{Sel}_5(E_f)$.

Recall $*$, that $E_f[5] = E_g[5]$ inside $J_0(1058)$, so certainly $E_f[5] \simeq E_g[5]$ as $G_{\mathbb{Q}}$-modules

and the $(\mathbb{Z}/5\mathbb{Z})^2$ inside $H^1(G_{\mathbb{Q}}, E_f[5])$, i.e. $\delta(E_f(\mathbb{Q}))$, is equally inside $H^1(G_{\mathbb{Q}}, E_g[5])$.

One may show it is even inside $\mathrm{Sel}_5(E_g)$.

But $E_g(\mathbb{Q}) = \{0\}$, so it can't be in the image of $E_g(\mathbb{Q})$.
It must map on to a $(\mathbb{Z}/5\mathbb{Z})^2$ in $H^1(G_{\mathbb{Q}}, E_g(\overline{\mathbb{Q}}))$, everywhere locally trivial

i.e. to $(\mathbb{Z}/5\mathbb{Z})^2$ inside $Ш_{E_g}$.

<u>Another look at $E = X_0(11)$</u>, $f = q\prod_{n=1}^{\infty}[(1-q^n)^2(1-q^{11n})^2]$

<u>and the congruence $a_p \equiv 1+p \pmod 5$</u> $\forall$ prime $p \neq 11$,

which came from the rational point $P=(5,5)$ of order 5.
(5 divides $\#\tilde{E}(\mathbb{F}_p) = 1+p-a_p$.)

$E[5]$ is a 2-dimensional $\mathbb{F}_5$-vector space, with a linear action of $G_{\mathbb{Q}}$.
$P \in E[5]$ is fixed by $G_{\mathbb{Q}}$
The same is true of any multiple of P.
So $G_{\mathbb{Q}}$ acts trivially on the line spanned by P inside $E[5]$.
As a representation of $G_{\mathbb{Q}}$, $E[5]$ is reducible, with a trivial 1-dimensional submodule.

Completing P to a basis for $E[5]$ we get a homomorphism

$$\rho: G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_5)$$

$$\sigma \longmapsto \begin{bmatrix} 1 & * \\ 0 & \chi(\sigma) \end{bmatrix}$$

What is the character χ (the other composition factor)?

Weil pairing $E[5] \times E[5] \longrightarrow \mu_5$ (group of 5th roots of unity)

is bilinear, non-degenerate, skew symmetric and $G_{\mathbb{Q}}$-equivariant $(\sigma Q, \sigma R) = \sigma(Q,R)$.

It follows that $\chi = \det\rho$ is the cyclotomic character by which $G_{\mathbb{Q}}$ acts on μ_5.

If $\mu_5 = \langle \zeta_5 \rangle$ then $G_{\mathbb{Q}}$ acts on μ_5 via the quotient

$$Gal(\mathbb{Q}(\zeta_5)/\mathbb{Q}) \simeq (\mathbb{Z}/5\mathbb{Z})^\times$$

$$(\zeta_5 \mapsto \zeta_5^t) \longleftarrow [t]$$

On the 1-dimensional $\mathbb{F}_5$-vector space μ_5, this is scalar multiplication by $[t] \in \mathbb{F}_5$.

For $p \neq 5$, μ_5 survives reduction mod p,

and $G_{\mathbb{Q}_p}$ acts on μ_5 via $Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p)$

If $Frob_p \in G_{\mathbb{Q}}$ is any element reducing to $(x \mapsto x^p)$ in $Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p)$, then $Frob_p(\zeta_5) = \zeta_5^p$ so $\chi(Frob_p) = p$.

$$\rho(Frob_p) = \begin{bmatrix} 1 & * \\ 0 & p \end{bmatrix} \quad (p \neq 5)$$

and $tr\,\rho(Frob_p) = 1 + p$ (in $\mathbb{F}_5$)

But Eichler-Shimura $\Rightarrow tr\,\rho(Frob_p) = a_p$ (for $p \neq 11, 5$)

We recover $a_p \equiv 1 + p \pmod 5$.

<u>A similar example for higher weight</u>

$$\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n. \qquad S_{12}(SL_2(\mathbb{Z})) = \langle\Delta\rangle_{\mathbb{C}}.$$

Ramanujan's congruence

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691} \quad \forall n \geq 1.$$

In particular, for prime p,

$$\tau(p) \equiv 1 + p^{11} \pmod{691}.$$

This is a congruence of Hecke eigenvalues between Δ and the Eisenstein series G_{12}.

A theorem of Deligne implies the following.

For any prime ℓ, there exists a continuous representation $\rho_\ell : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{Q}_\ell)$ such that

(i) ρ_ℓ is unramified at any prime $p \neq \ell$, meaning $G_{\mathbb{Q}_p} \subset G_{\mathbb{Q}}$ acts through its quotient $G_{\mathbb{Q}_p}/I_p \simeq G_{\mathbb{F}_p}$

generated by Frob_p — I_p: inertia subgroup

(ii) $\rho_\ell(\mathrm{Frob}_p^{-1})$ has characteristic polynomial

$$X^2 - \tau(p)X + p^{11}$$

In particular $\mathrm{tr}\,\rho_\ell(\mathrm{Frob}_p^{-1}) = \tau(p)$.
$\det \rho_\ell(\mathrm{Frob}_p^{-1}) = p^{11}$.

<u>NB</u> The Euler factor at p in $L_\Delta(s)$ is

$$[\det(I - \rho_\ell(\mathrm{Frob}_p^{-1})p^{-s})]^{-1}.$$

Let $T_\ell\mu := \varprojlim_n \mu_{\ell^n}$, isomorphic to $\mathbb{Z}_\ell$ as a group.

$G_{\mathbb{Q}}$ acts on $T_\ell\mu$ via the ℓ-adic cyclotomic character

$$\varepsilon_\ell : G_{\mathbb{Q}} \longrightarrow \mathbb{Z}_\ell^\times \subset \mathbb{Q}_\ell^\times$$
$$\mathrm{Frob}_p \longmapsto p$$

and on μ_ℓ via the reduction $\bar{\varepsilon}_\ell : G_{\mathbb{Q}} \rightarrow \mathbb{F}_\ell^\times$

Given a $\mathbb{Z}_\ell[G_{\mathbb{Q}}]$-module M (or $\mathbb{Q}_\ell[G_{\mathbb{Q}}]$-module, or $\mathbb{F}_\ell[G_{\mathbb{Q}}]$-module), define the Tate twist $M(n)$ (for $n \in \mathbb{Z}$) to be the same as M, but with the $G_{\mathbb{Q}}$-action multiplied by the n^{th} power of ε_ℓ, so the action of Frob_p is multiplied by p^n.
i.e. $M(n) = M \otimes (T_\ell\mu)^{\otimes n}$

$$\det \rho_\ell(\mathrm{Frob}_p^{-1}) = p^{11} \Rightarrow \det \rho_\ell = \varepsilon_\ell^{-11}.$$

Choosing a $G_{\mathbb{Q}}$-invariant $\mathbb{Z}_\ell$-lattice, $\rho_\ell : G_{\mathbb{Q}} \rightarrow GL_2(\mathbb{Z}_\ell)$
Reduction $\bar{\rho}_\ell : G_{\mathbb{Q}} \rightarrow GL_2(\mathbb{F}_\ell)$.

Taking now $\ell = 691$, $\tau(p) \equiv 1 + p^{11} \pmod{691}$

$$\Rightarrow \mathrm{tr}\,\bar{\rho}_{691}(\mathrm{Frob}_p^{-1}) = 1 + p^{11} = \mathrm{id}(\mathrm{Frob}_p^{-1}) + \bar{\varepsilon}_{691}^{-11}(\mathrm{Frob}_p^{-1}).$$

Hence $\bar{\rho}_{691}$ is reducible, with composition factors id, $\bar{\varepsilon}_{691}^{-11}$
i.e. $\mathbb{F}_{691}$, $\mathbb{F}_{691}(-11)$.

The choice of $G_{\mathbb{Q}}$-invariant $\mathbb{Z}_{691}$-lattice determines whether we have

$$0 \to \mathbb{F}_{691} \to W \to \mathbb{F}_{691}(-11) \to 0$$

or $$0 \to \mathbb{F}_{691}(-11) \xrightarrow{i} W \xrightarrow{\pi} \mathbb{F}_{691} \to 0,$$

where W is the 2-dimensional $\mathbb{F}_{691}$-vector space on which $G_{\mathbb{Q}}$ acts via $\bar{\rho}_{691}$. Let's choose the latter.

By a theorem of Ribet, $\rho_{691} : G_{\mathbb{Q}} \to GL_2(\mathbb{Q}_{691})$ is irreducible.

It follows (by an argument also due to Ribet) that we may further refine the choice of $\mathbb{Z}_{691}$-lattice so that $W \not\cong \mathbb{F}_{691}(-11) \oplus \mathbb{F}_{691}$,

i.e. W is a non-split extension of $\mathbb{F}_{691}$ by $\mathbb{F}_{691}(-11)$.

We can use this to define a cocycle $c \in Z^1(G_{\mathbb{Q}}, \mathbb{F}_{691}(-11))$ in the following way. Choose $w \in W$ such that $\pi(w) = 1$.

For any $\sigma \in G_{\mathbb{Q}}$, $\pi(\sigma(w) - w) = \pi(\sigma(w)) - \pi(w) = \sigma(\pi(w)) - \pi(w)$
$= \sigma(1) - 1 = 0$.

Hence $\sigma(w) - w \in \ker \pi = \operatorname{im} i$ so $\sigma(w) - w \in \mathbb{F}_{691}(-11)$
(meaning $i(\mathbb{F}_{691}(-11))$).

c is defined by $c(\sigma) = \sigma(w) - w$

Changing c by a coboundary exactly corresponds to adjusting w by an element of $i(\mathbb{F}_{691}(-11))$, so we have a well-defined cohomology class

$[c] \in H^1(G_{\mathbb{Q}}, \mathbb{F}_{691}(-11))$ $\qquad$ W non-split $\Rightarrow [c] \neq 0$.

Moreover, it follows from the fact that ρ_{691} is $\begin{cases} \text{unramified at all } p \neq 691 \\ \text{crystalline at } p = 691 \end{cases}$

that $[c]$, or rather its (non-zero) image in

$H^1(G_{\mathbb{Q}}, \mathbb{Q}_{691}/\mathbb{Z}_{691}(-11))$, satisfies the local restriction conditions

to belong to the Bloch-Kato Selmer group $H^1_f(G_{\mathbb{Q}}, \mathbb{Q}_{691}/\mathbb{Z}_{691}(-11))$.

By a theorem of Bloch and Kato (analogue of BSD)

$$\operatorname{ord}_\ell\left(\frac{\zeta(n)}{(2\pi i)^n}\right) = \frac{\# H^1_f(G_{\mathbb{Q}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1-n))}{\# H^0(G_{\mathbb{Q}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1-n)) \# H^0(G_{\mathbb{Q}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))}$$

for positive even n and $\ell > n$.

So letting $n = 12$, $\ell = 691$, the existence of $[c]$ fits with the known fact that $691 \mid \frac{\zeta(12)}{\pi^{12}}$.

Consider once again $E = X_0(11) : y^2 + y = x^3 - x^2 - 10x - 20$

with its rational 5-torsion point $P = (5,5)$.

$$0 \longrightarrow \mathbb{F}_5 \longrightarrow E[5] \longrightarrow \mathbb{F}_5(1) \longrightarrow 0$$

This is not a direct sum, whereas for the 5-isogenous curve $E' = X_1(11) : y^2 + y = x^3 - x^2$

with rational 5-torsion point $(0,0)$, we would have $E'[5] \simeq \mathbb{F}_5 \oplus \mathbb{F}_5(1)$.

The 5-adic Tate modules $T_5[E] = \varprojlim_n E[5^n]$ and $T_5[E']$

are different choices of $G_{\mathbb{Q}}$-invariant $\mathbb{Z}_5$-lattice in a common 2-dimensional $\mathbb{Q}_5$-vector space on which $G_{\mathbb{Q}}$ acts, $V_5 = T_5[E] \otimes_{\mathbb{Z}_5} \mathbb{Q}_5$

Anyway, $0 \to \mathbb{F}_5 \to E[5] \longrightarrow \mathbb{F}_5(1) \to 0$ is non-split, as is its twist

$$0 \to \mathbb{F}_5(-1) \to E[5](-1) \longrightarrow \mathbb{F}_5 \to 0.$$

Arguing as above, we get non-zero $[c] \in H^1(G_{\mathbb{Q}}, \mathbb{F}_5(-1))$.

With $n = 2$, $1 - n = -1$, shouldn't this mean that $5 \mid \frac{\zeta(2)}{(2\pi i)^2}$, whereas in fact $\zeta(2) = \pi^2/6$?

In this case $\rho_5 : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}(V_5) \simeq GL_2(\mathbb{Q}_5)$

is unramified at p not for all $p \neq 5$, only for $p \neq 5, 11$, since 11 is a prime of bad reduction.

The image of $[c]$ in $H^1(G_{\mathbb{Q}}, \mathbb{Q}_5/\mathbb{Z}_5(-1))$ satisfies all the Bloch-Kato local conditions <u>except at $p = 11$.</u>

It lies in $H^1_{\{11\}}(G_{\mathbb{Q}}, \mathbb{Q}_5/\mathbb{Z}_5(-1))$. On the LHS of the Bloch-Kato formula, we need to replace $\zeta(2)$ by $\zeta_{\{11\}}(2)$, where $\zeta_{\{11\}}(s) = \prod_{p \neq 11} \frac{1}{(1 - p^{-s})} = (1 - 11^{-s})\zeta(s)$

Since $1 - 11^{-2} = -\frac{1}{11^2}(11^2 - 1)$ and $11 \equiv 1 \pmod 5$,

we see that indeed $5 \mid \zeta_{\{11\}}(2)/\pi^2$

Consider once again $f, g \in S_2(\Gamma_0(1058))$ $\qquad$ $(1058 = 2 \cdot 23^2)$

$E_f : y^2 + xy + y = x^3 + 2$ $\qquad$, $\deg \pi_f = 2^4 \cdot 5$, $E_f(\mathbb{Q}) \simeq \mathbb{Z}^2$

$E_g : y^2 + xy = x^3 - x^2 - 332311x - 73733731$, $\deg \pi_g = 2^3 3 \cdot 5 \cdot 7 \cdot 23$, $E_g(\mathbb{Q}) = \{O\}$.

$a_n(f) \equiv a_n(g) \pmod 5$, $E_f[5] = E_g[5]$ inside $J_0(1058)$.

$\rho_{f,5}, \rho_{g,5} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{Z}_5)$

$\bar{\rho}_{f,5}, \bar{\rho}_{g,5} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_5)$

Since $E_f[5] = E_g[5]$, may choose bases so that $\bar{\rho}_{f,5} = \bar{\rho}_{g,5}$.

Given $\sigma \in G_{\mathbb{Q}}$, $\rho_{g,5}(\sigma) \equiv (I + 5c(\sigma))\rho_{f,5}(\sigma) \pmod{5^2}$

for some $c(\sigma) \in M_2(\mathbb{F}_5)$.

$\det \rho_{g,5}(\sigma) = \det \rho_{f,5}(\sigma) \; (= \varepsilon_5(\sigma)) \Rightarrow tr(c(\sigma)) = 0$.

Given $\sigma, \tau \in G_{\mathbb{Q}}$, $\rho_{g,5}(\sigma\tau) = \rho_{g,5}(\sigma)\rho_{g,5}(\tau)$

$\equiv (I + 5c(\sigma))\rho_{f,5}(\sigma)(I + 5c(\tau))\rho_{f,5}(\tau) \pmod{5^2}$

$\equiv (I + 5[c(\sigma) + \rho_{f,5}(\sigma)c(\tau)\rho_{f,5}(\sigma)^{-1}])\rho_{f,5}(\sigma)\rho_{f,5}(\tau)$

$\equiv (I + 5[c(\sigma) + \rho_{f,5}(\sigma)c(\tau)\rho_{f,5}(\sigma)^{-1}])\rho_{f,5}(\sigma\tau) \pmod{5^2}$

But $\rho_{g,5}(\sigma\tau) \equiv (I + 5c(\sigma\tau))\rho_{f,5}(\sigma\tau) \pmod{5^2}$.

Hence $c(\sigma\tau) = c(\sigma) + \rho_{f,5}(\sigma)c(\tau)\rho_{f,5}(\sigma)^{-1}$

This is $c(\sigma\tau) = c(\sigma) + \sigma(c(\tau))$, where the action of $G_{\mathbb{Q}}$ on $M_2(\mathbb{F}_5)^{tr=0}$ is via the adjoint action

$ad^0\bar{\rho}_{f,5} : G_{\mathbb{Q}} \longrightarrow Aut_{\mathbb{F}_5}(M_2(\mathbb{F}_5)^{tr=0})$

We write $c \in Z^1(G_{\mathbb{Q}}, ad^0\bar{\rho}_{f,5})$.

$a_3(f) = -2 \not\equiv 3 = a_3(g) \pmod{25} \Rightarrow c \neq 0$, in fact $[c] \neq 0$ in $H^1(G_{\mathbb{Q}}, ad^0\bar{\rho}_{f,5})$.

Moreover (with slight abuse of notation)

$[c] \in H^1_{\{2,23\}}(G_{\mathbb{Q}}, ad^0\bar{\rho}_{f,5})$

There is an adjoint L-function $L(\mathrm{ad}^0(f), s)$.
For a good prime $p \neq 2, 23$, the Euler factor at p is (choosing any $\ell \neq p$)

$$\det(I - \mathrm{ad}^0 \rho_{f,\ell}(\mathrm{Frob}_p^{-1})\, p^{-s})^{-1}$$

$$= \left[(1-p^{-s})\left(1-\frac{\alpha_p}{\beta_p} p^{-s}\right)\left(1-\frac{\beta_p}{\alpha_p} p^{-s}\right)\right]^{-1}$$

where $[(1-\alpha_p p^{-s})(1-\beta_p p^{-s})]^{-1}$ is the Euler factor at p in $L_f(s) = L_{E_f}(s)$.

[Since $\alpha_p \beta_p = p$,

$$(1-p^{-s})\left(1-\frac{\alpha_p}{\beta_p} p^{-s}\right)\left(1-\frac{\beta_p}{\alpha_p} p^{-s}\right) = (1-\alpha_p\beta_p p^{-(1+s)})(1-\alpha_p^2 p^{-(1+s)})(1-\beta_p^2 p^{-(1+s)})$$

and $L(\mathrm{ad}^0(f), s) = L(\mathrm{Sym}^2 E_f, 1+s)$.]

$L(\mathrm{ad}^0(f), s)$ has an analytic continuation to $\mathbb{C}$ and satisfies a functional equation for $s \leftrightarrow 1-s$.

Non-zero $[c] \in H^1_{\{2,23\}}(G_{\mathbb{Q}}, \mathrm{ad}^0 \bar{\rho}_{f,5})$ implies, by Bloch-Kato conjecture (theorem of Diamond, Flach & Guo in this case) that we should see a factor 5 in $\frac{L(\mathrm{ad}^0(f),1)}{\Omega}$ for some carefully chosen Deligne period Ω. (or maybe $\frac{L_{\{2,23\}}(\mathrm{ad}^0(f),1)}{\Omega}$)

In fact, $\Omega = -2(2\pi i)\,\Omega^+_{E_f}\,\Omega^-_{E_f}$

By a theorem of Rankin and Shimura,

$$L(\mathrm{ad}^0(f), 1) = \frac{4^2 \pi^3 \langle f, f\rangle}{1058\, \phi(1058)}$$

Hence

$$\frac{L(\mathrm{ad}^0(f),1)}{\Omega} = \frac{4\pi^2}{1058\,\phi(1058)} \frac{\langle f, f\rangle}{-i\,\Omega^+\Omega^-} = \frac{\frac{1}{2}\deg \pi_f}{1058\,\phi(1058)}$$

so $5 \mid \deg \pi_f \Rightarrow 5 \mid \frac{L(\mathrm{ad}^0(f),1)}{\Omega}$, as expected.

Next we shall look at something similar, but for higher weight.

Given a character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, define a subspace $S_k(\Gamma_0(N), \chi) \subseteq S_k(\Gamma_1(N))$ by the condition

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \chi(d)(c\tau+d)^k f(\tau) \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

Special case $N = D$, the discriminant of a real quadratic field $F = \mathbb{Q}(\sqrt{D})$, $\chi = \chi_D = \left(\frac{D}{\cdot}\right)$ the associated quadratic character.

Let $f \in S_k(\Gamma_0(D), \chi_D)$ be a Hecke eigenform.

Remarks (i) $D > 0 \Rightarrow \chi_D(-1) = 1$, i.e. χ_D is an even character.
Letting $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma_0(D)$, $f(\tau) = \chi(-1)(-1)^k f(\tau) \Rightarrow k$ is even

(ii) χ_D is a primitive character $\Rightarrow f$ is a newform, let's say normalised.

(iii) The T_n for $(n,N)=1$ are not quite self-adjoint, but still commute with their adjoints. For prime $q \nmid D$,

$$\langle T_q f, f \rangle = \chi_D(q) \langle f, T_q f \rangle$$

$$\Rightarrow a_q(f) \langle f, f \rangle = \chi_D(q) \overline{a_q(f)} \langle f, f \rangle$$

$$\Rightarrow a_q(f) \text{ is } \begin{cases} \text{real} & \text{if } \chi_D(q) = 1 \quad (\text{i.e. } q \text{ splits in } F) \\ \text{purely imaginary} & \text{if } \chi_D(q) = -1 \quad (\text{i.e. } q \text{ is inert in } F). \end{cases}$$

Examples $S_8(\Gamma_0(5), \chi_5)$ is spanned by a complex conjugate pair of Hecke eigenforms

$$f = q + 2\sqrt{-29}\,q^2 + 6\sqrt{-29}\,q^3 + 12q^4 + (75 - 50\sqrt{-29})q^5 - 348q^6 - 78\sqrt{-29}\,q^7 + \dots$$

$$f^c = q - 2\sqrt{-29}\,q^2 - 6\sqrt{-29}\,q^3 + 12q^4 + (75 + 50\sqrt{-29})q^5 - 348q^6 + 78\sqrt{-29}\,q^7 + \dots$$

$S_6(\Gamma_0(5), \chi_5)$ is spanned by a complex conjugate pair of Hecke eigenforms

$$g = q - 2\sqrt{-11}\,q^2 + 6\sqrt{-11}\,q^3 - 12q^4 + (-45 - 10\sqrt{-11})q^5 + 132q^6 + 18\sqrt{-11}\,q^7 + \dots$$

$$g^c = q + 2\sqrt{-11}\,q^2 - 6\sqrt{-11}\,q^3 - 12q^4 + (-45 + 10\sqrt{-11})q^5 + 132q^6 - 18\sqrt{-11}\,q^7 + \dots$$

$S_4(\Gamma_0(8), \chi_8)$ is spanned by

$$h = q + (-1 - \sqrt{-7})q^2 + 2\sqrt{-7}\,q^3 + (-6 + 2\sqrt{-7})q^4 - 4\sqrt{-7}\,q^5 + (14 - 2\sqrt{-7})q^6 - 8q^7 + \dots \text{ and } h^c$$

$S_{14}(\Gamma_0(8), \chi_8)$ is spanned by

$$r = q + (-56 - 8\sqrt{-79})q^2 + 258\sqrt{-79}\,q^3 + (-1920 + 896\sqrt{-79})q^4 - 2540\sqrt{-79}\,q^5 + (163056 - 14448\sqrt{-79})q^6 - 175832q^7 + \dots,$$
and r^c

Observe the following congruences:

$$a_n(f) \equiv a_n(f^c) \pmod{\sqrt{-29}} \quad \forall n \geq 1.$$

$$a_n(g) \equiv a_n(g^c) \pmod{\sqrt{-11}} \quad \forall n \geq 1.$$

$$a_n(h) \equiv a_n(h^c) \pmod{\sqrt{-7}} \quad \forall n \geq 1.$$

$$a_n(r) \equiv a_n(r^c) \pmod{\sqrt{-79}} \quad \forall n \geq 1.$$

It follows that $\sqrt{-29} \mid m(f)$, a "cohomological congruence ideal", analogous to the modular degree.

$\sqrt{-11} \mid m(g)$, $\sqrt{-7} \mid m(h)$, $\sqrt{-79} \mid m(r)$.

By a theorem of Hida

$$\langle f, f \rangle = i^{k-1} \Omega^+ \Omega^- \, m(f) \quad \text{(up to divisors of } D(k!))$$

where Ω^+, Ω^- are certain "canonical periods".

The natural Deligne period for $L(ad^\circ(f), 1)$ is

$$\Omega = 2(2\pi i)^{k+1} \Omega^+ \Omega^-,$$

and by the theorem of Rankin and Shimura

$$L(ad^\circ(f), 1) = \frac{4^k \pi^{k+1} \langle f, f \rangle}{(k-1)! \, D^2}$$

Hence $\dfrac{L(ad^\circ(f), 1)}{\Omega}$ is like $\dfrac{m(f)}{D^2}$

$\dfrac{L_{\{5\}}(ad^\circ(f), 1)}{\Omega}$ is like $\dfrac{(1-5^{-1})\, m(f)}{5^2}$

To account for the $q \mid m(f)$ where $q = \sqrt{-29}$, Bloch-Kato conjecture (theorem of Diamond, Flach & Guo) says there is a non-zero element in $H^1_{\{5\}}(G_{\mathbb{Q}}, ad^\circ \bar{\rho}_{f,q})$

We can construct this using $\bar{\rho}_{f,q} \simeq \bar{\rho}_{f^c,q}$ just like we did for $E_f[5] = E_g[5]$ (different f, g) earlier.

The right-of-centre "critical" values for $L(\mathrm{ad}^0(f), s)$ are at $s = 1, 3, 5, \ldots, k-1$. The Deligne period is $\Omega_s := (2\pi i)^{2(s-1)}\Omega$.

For $s > 1$, a theorem of Katsurada allows the critical values to be computed.

It says something a bit like

$$G_s(z) = \frac{L_{\{S\}}(\mathrm{ad}^0(f), s)}{\pi^{k+2s-1}\langle f, f\rangle} f(z) + \frac{L_{\{S\}}(\mathrm{ad}^0(f^c), s)}{\pi^{k+2s-1}\langle f^c, f^c\rangle} f^c(z)$$

in our case where $S_k(\Gamma_0(D), \chi_D) = \langle f, f^c\rangle_{\mathbb{C}}$.

$G_s(z)$ is a certain cusp form obtained from a genus-2 Siegel Eisenstein series by applying differential operators and restriction from $\mathfrak{h}_2$ to $\mathfrak{h}_1 \times \mathfrak{h}_1$.

Looking at the coefficients of some q^{n_1}, q^{n_2} gives linear equations

$$\begin{cases} a_{n_1}(G_s) = a_{n_1}(f)\, x(f) + a_{n_1}(f^c)\, x(f^c) \\ a_{n_2}(G_s) = a_{n_2}(f)\, x(f) + a_{n_2}(f^c)\, x(f^c) \end{cases}$$

where $x(f) = \frac{L_{\{S\}}(\mathrm{ad}^0(f), s)}{\pi^{k+2s-1}\langle f, f\rangle}$, $x(f^c) = \frac{L_{\{S\}}(\mathrm{ad}^0(f^c), s)}{\pi^{k+2s-1}\langle f^c, f^c\rangle}$

Since $a_n(f) \equiv a_n(f^c) \pmod{\mathfrak{q}}$, the coefficient matrix

$$\begin{bmatrix} a_{n_1}(f) & a_{n_1}(f^c) \\ a_{n_2}(f) & a_{n_2}(f^c) \end{bmatrix}$$ is singular $(\mathrm{mod}\ \mathfrak{q})$, so $\mathfrak{q} \mid \det\begin{bmatrix} a_{n_1}(f) & a_{n_1}(f^c) \\ a_{n_2}(f) & a_{n_2}(f^c) \end{bmatrix}$

and one would expect Cramer's rule to put $\mathfrak{q}$ in the denominators of the solutions $x(f)$, $x(f^c)$, thus avoiding $\mathfrak{q} \mid \frac{L_{\{S\}}(\mathrm{ad}^0(f), s)}{\Omega_s}$ despite $\mathfrak{q} \mid \frac{\langle f, f\rangle}{\Omega}$.

This is what would typically happen where there is a congruence between two newforms (same weight, level, character), but in this special situation of a congruence between complex conjugates, $x(f)$ and $x(f^c)$ are actually the same.

[Recall that for prime $q \nmid D$, $a_q(f) = \chi_D(q)\overline{a_q(f)} = \chi_D(q)\, a_q(f^c)$.
Since $a_q(f) = \mathrm{tr}\, \rho_{f,\lambda}(\mathrm{Frob}_q^{-1})$ etc., it follows that

$$\rho_{f,\lambda} \simeq \rho_{f^c,\lambda} \otimes \chi_D, \text{ hence } \mathrm{ad}^0(\rho_{f,\lambda}) \simeq \mathrm{ad}^0(\rho_{f^c,\lambda})$$]

So, taking $n_1=1$, we actually have a single equation in one unknown:

$$2x(f) = a_1(G_s)$$

The RHS is complicated, but one can show it is integral at q.
So in fact $q \mid \frac{L_{\{5\}}(ad^0(f),s)}{\Omega_s}$ not just for $s=1$, but

for all $s=1,3,5,\dots,k-1$.

In summary (neglecting powers of π)

each $L_{\{5\}}(ad^0(f),s)$ is a q-integral multiple of $\langle f,f\rangle$

(Rankin-Shimura for $s=1$, Katsurada for $s>1$).

But $q \mid \frac{\langle f,f\rangle}{\Omega_s}$ as a consequence of $a_n(f) \equiv a_n(f^c) \pmod{q}$

so $q \mid \frac{L_{\{5\}}(ad^0(f),s)}{\Omega_s}$ for $s=1,3,\dots,k-1$.

For $s=1$, $1-s=0$, we already constructed a non-zero element in $H^1_{\{5\}}(G_{\mathbb{Q}}, ad^0(\overline{\rho}_{f,q}))$.

Bloch-Kato conjecture $\Rightarrow$ for the $s>1$ we should also have non-zero elements in $H^1_{\{5\}}(G_{\mathbb{Q}}, ad^0(\overline{\rho}_{f,q})(1-s))$.

How can we construct these?

We will need some more congruences, involving Hilbert modular forms.

Newform $f \in S_k(\Gamma_0(D), \chi_D)$ $F = \mathbb{Q}(\sqrt{D})$ real quadratic field.

$a_n(f) \equiv a_n(f^c) \pmod{q}$ $\forall n \geq 1$

For $s = 3, 5, \ldots, k-1$ we want a non-zero element of
$H^1_{\{p|D\}}(G_{\mathbb{Q}}, ad^0(\bar{\rho}_{f,q})(1-s))$, in accord with the Bloch-Kato conjecture,

since $q \mid m(f) \Rightarrow q \mid \frac{L_{\{p|D\}}(ad^0(f), s)}{\Omega_s}$

We need a closer look at $\rho_{f,q}$ and $\bar{\rho}_{f,q}$.

$\rho_{f,q} : G_{\mathbb{Q}} \longrightarrow GL_2(K_q)$ (e.g. $K = \mathbb{Q}(\sqrt{-29})$, $q = (\sqrt{-29})$)

is irreducible, and is unramified at all primes $p \nmid qD$
(i.e. $\rho_{f,q}|_{I_p}$ is trivial for such p).

At $p \mid D$ it is known that $\rho_{f,q}|_{I_p} \simeq id. \oplus \chi_D$
where $\chi_D : (\mathbb{Z}/D\mathbb{Z})^\times \to \{\pm 1\}$
is thought of as a character of $G_{\mathbb{Q}}$, using $(\mathbb{Z}/D\mathbb{Z})^\times \simeq Gal(\mathbb{Q}(\mu_D)/\mathbb{Q})$

In particular, if $\mathfrak{P} | p$ in F then $\rho_{f,q}|_{I_{\mathfrak{P}}}$ is trivial,
i.e. $\rho_{f,q}|_{G_F}$ is unramified at all primes $\mathfrak{P} | D$.
It can be ramified at most at primes dividing q.
The same applies to $\bar{\rho}_{f,q}|_{G_F}$. *

Recall that $\rho_{f,q} \simeq \rho_{f^c,q} \otimes \chi_D$

Substituting $\bar{\rho}_{f^c,q} \simeq \bar{\rho}_{f,q}$, we get

$$\bar{\rho}_{f,q} \simeq \bar{\rho}_{f,q} \otimes \chi_D \qquad **$$

One can check in our examples that
(i) $\bar{\rho}_{f,q}$ is absolutely irreducible
(ii) f is <u>ordinary</u> at q: $q \nmid a_q(f)$

Using also * and **, it follows that $(q) = \mathfrak{Q}\mathfrak{Q}^\sigma$ splits in F
(say $Gal(F/\mathbb{Q}) = \langle\sigma\rangle$)

and that $\bar{\rho}_{f,q} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_q) \subset GL_2(\bar{\mathbb{F}}_q)$

is induced from a character $\psi : G_F \to \bar{\mathbb{F}}_q^{\times}$
ramified only at Q (among finite primes)

Equally it is induced from $\psi^{\sigma} : G_F \to \bar{\mathbb{F}}_q^{\times}$
ramified only at Q^{σ} (among finite primes),

where $\psi^{\sigma}(g) := \psi(\sigma g \sigma^{-1}) \quad \forall g \in G_F$.

$\bar{\rho}_{f,q}|_{G_F} \simeq \psi \oplus \psi^{\sigma}$.

Using $\det \bar{\rho}_{f,q} \simeq \chi_D \bar{\varepsilon}_q^{\,1-k} \simeq \mathbb{F}_q(1-k) \otimes \chi_D$,

in fact the finite parts of the conductors of ψ and ψ^{σ} are precisely Q and Q^{σ}, respectively, and we can say precisely what these characters must be.

For simplicity, assume from now on that O_F has narrow class number one — every ideal is generated by a totally positive element. This is certainly true for $F = \mathbb{Q}(\sqrt{5})$ and $F = \mathbb{Q}(\sqrt{2})$.

As an abstract field, $F = \mathbb{Q}(\alpha)$ with $\alpha^2 = D$.
Two real embeddings $\theta_1, \theta_2 : F \hookrightarrow \mathbb{R}$
$\theta_1 : \alpha \mapsto \sqrt{D}$, $\quad \theta_2 : \alpha \mapsto -\sqrt{D}$.
Associated valuations $|x|_{\infty_1} = |\theta_1(x)|$, $\quad |x|_{\infty_2} = |\theta_2(x)|$.

Unit group $O_F^{\times}$ is generated by -1 and a fundamental unit u.

$\psi : G_F \to \bar{\mathbb{F}}_q^{\times}$ factors through $\mathrm{Gal}(H_{Q\infty_1}/F)$, where

$H_{Q\infty_1}$ is the ray class field of conductor $Q\infty_1$.

$\mathrm{Gal}(H_{Q\infty_1}/F) \simeq C_{Q\infty_1}$, a generalised ideal class group.

$$C_{Q\infty_1} \simeq I_{Q\infty_1} / P_{Q\infty_1}, \text{ where}$$

$I_{Q\infty_1}$ is the group of fractional ideals of F coprime to Q

$P_{Q\infty_1}$ is the subgroup of principal ideals (x) with
$x \equiv 1 \pmod{Q}$, ~~$\theta_1(x), \theta_2(x) > 0$ (totally positive)~~
$\theta_1(x) > 0$.

We define $\psi : C_{Q\infty_1} \longrightarrow \mathbb{F}_q^\times$ (hence : $G_F \longrightarrow \mathbb{F}_q^\times$)

in the following way, to make $\psi\psi^\sigma = \varepsilon_q^{1-k} \circ \mathrm{Norm}_{F/\mathbb{Q}}$.

Given $\mathfrak{a} \in I_{Q\infty_1}$, $\exists\ a \in F^\times$, $\mathrm{ord}_Q(a) = 0$ such that $\mathfrak{a} = (a)$ and $\theta_1(a), \theta_2(a) > 0$ (a is totally positive).

This is because $\mathcal{O}_F$ has narrow class number 1, i.e. $C_{\infty_1\infty_2}$ is trivial.

Define $\psi : I_{Q\infty_1} \longrightarrow \mathbb{F}_q^\times$ by

$$\psi(\mathfrak{a}) = \bar{a}^{1-k}$$, where $\bar{a}$ is the image of a in $\mathcal{O}_F/Q \simeq \mathbb{F}_q$.

For this to be well-defined independent of the choice of a, we must have $(u^2)^{1-k} \equiv 1 \pmod{Q}$, ⊛

since $u^2 a$ is also a totally positive generator of $\mathfrak{a}$.

Note that u is not totally positive. It is a consequence of $\mathcal{O}_F$ having narrow class number 1 that $\theta_1(u)$ and $\theta_2(u)$ are opposite in sign.

By ⊛, $u^{k-1} \equiv \pm 1 \pmod{Q}$.

Replacing u by $-u$ if necessary (k is even) we may assume $u^{k-1} \equiv 1 \pmod{Q}$,

and replacing u by u^σ if necessary (and Q by Q^σ) we may assume $\theta_1(u) > 0$, $\theta_2(u) < 0$.

Now if $(x) \in P_{Q\infty_1}$ with $x \equiv 1 \pmod{Q}$, $\theta_1(x) > 0$

there are two possibilities.

(i) If $\theta_2(x) > 0$ then $\psi((x)) = \bar{x}^{1-k} = 1$ since $x \equiv 1 \pmod{Q}$.

(ii) If $\theta_2(x) < 0$ then ux is a totally positive generator of (x), so $\psi((x)) = \overline{ux}^{1-k} = 1$ since $u^{k-1} \equiv x \equiv 1 \pmod{Q}$.

We have shown that $\psi : I_{Q\infty_1} \longrightarrow \mathbb{F}_q^\times$ factors through $I_{Q\infty_1}/P_{Q\infty_1} \simeq C_{Q\infty_1} \simeq \mathrm{Gal}(H_{Q\infty_1}/F)$, as desired.

Example: $F = \mathbb{Q}(\alpha)$, $\alpha^2 = 5$ $\quad u = \frac{1+\alpha}{2}$, $k = 8$, $q = 29$.

$\frac{1+\sqrt{5}}{2} > 0$, $\frac{1-\sqrt{5}}{2} < 0$

$x^2 - 5 \equiv (x+11)(x-11) \pmod{29}$

$Q = (29, \alpha+11)$, $Q^\sigma = (29, \alpha - 11)$

$\alpha \equiv -11 \pmod{Q}$

$u \equiv -5 \pmod{Q}$

$u^7 \equiv 1 \pmod{Q}$.

Suppose that $(p) = \mathfrak{p}\mathfrak{p}^\sigma$ splits in F, and let $\mathfrak{p} = (a)$ with a totally positive.

We have $\mathrm{Frob}_p = \mathrm{Frob}_{\mathfrak{p}} \in G_F \subseteq G_{\mathbb{Q}}$.

Recall that $\bar{\rho}_{f,q}|_{G_F} \simeq \psi \oplus \psi^\sigma$.

Evaluating at $\mathrm{Frob}_{\mathfrak{p}}^{-1}$ and taking the trace, we should find

$$a_p(f) \equiv a^{k-1} + (a^\sigma)^{k-1} \pmod{\mathcal{Q}}.$$

p	a	$a \pmod{\mathcal{Q}}$	$a^\sigma \pmod{\mathcal{Q}}$	$a^7 \pmod{\mathcal{Q}}$	$(a^\sigma)^7 \pmod{\mathcal{Q}}$	$a_p(f)$
11	$4+\alpha$	-7	15	-1	17	$-6828 \equiv 16$
19	$12+5\alpha$	15	9	17	-1	$6860 \equiv 16$
31	$6+\alpha$	-5	17	1	12	$82112 \equiv 13$
29	$7+2\alpha$	14	0	12	0	$25590 \equiv 12$

Another example: $F = \mathbb{Q}(\alpha)$, $\alpha^2 = 5$, $u = \frac{1+\alpha}{2}$, $k = 6$, $q = 11$

$x^2 - 5 \equiv (x+4)(x-4) \pmod{11}$

$\mathcal{Q} = (11, \alpha+4)$, $\mathcal{Q}^\sigma = (11, \alpha-4)$

$\alpha \equiv -4 \pmod{\mathcal{Q}}$
$u \equiv 4 \pmod{\mathcal{Q}}$
$u^5 \equiv 1 \pmod{\mathcal{Q}}$

p	a	$a \pmod{\mathcal{Q}}$	$a^\sigma \pmod{\mathcal{Q}}$	$a^5 \pmod{\mathcal{Q}}$	$(a^\sigma)^5 \pmod{\mathcal{Q}}$	$a_p(g)$
11	$4+\alpha$	0	8	0	-1	$252 \equiv -1$
19	$12+5\alpha$	3	-1	1	-1	$-220 \equiv 0$
29	$7+2\alpha$	-1	4	-1	1	$6930 \equiv 0$
31	$6+\alpha$	2	-1	-1	-1	$6752 \equiv -2$

This is consistent with $a_p(g) \equiv a^{k-1} + (a^\sigma)^{k-1} \pmod{\mathcal{Q}}$

and $\bar{\rho}_{g,q}|_{G_F} \simeq \psi \oplus \psi^\sigma$

$$\bar{\rho}_{g,q} \simeq \mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}} \psi \simeq \mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}} \psi^\sigma.$$

There exists a Hilbert modular form $\hat{f} \in S_{[k,k]}(SL_2(\mathcal{O}_F))$

the <u>base change</u> of f.

$\hat{f} : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathbb{C}$ holomorphic,

$$\hat{f}\left(\left(\frac{a_1\tau_1 + b_1}{c_1\tau_1 + d_1}, \frac{a_2\tau_2 + b_2}{c_2\tau_2 + d_2}\right)\right) = (c_1\tau_1 + d_1)^k (c_2\tau_2 + d_2)^k \hat{f}(\tau_1, \tau_2)$$

$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}_F)$, where $a_1 = \theta_1(a)$, $a_2 = \theta_2(a)$, etc.

$\hat{f}$ vanishes at the cusps.

$\hat{f}$ is an eigenform for Hecke operators T_β (β prime ideals of $\mathcal{O}_F$)

$$T_\beta(\hat{f}) = a_\beta(\hat{f})\,\hat{f}$$

There is an irreducible, continuous $\rho_{\hat{f},q} : G_F \longrightarrow GL_2(K_q)$

unramified at $\beta \nmid qD$

$$\mathrm{tr}\left(\rho_{\hat{f},q}(\mathrm{Frob}_\beta^{-1})\right) = a_\beta(\hat{f})$$

To say $\hat{f}$ is the base-change of f is simply to say that

$$\rho_{\hat{f},q} = \rho_{f,q}\big|_{G_F}.$$

Put another way,
$$a_\beta(\hat{f}) = \begin{cases} a_p(f) & (p) = \beta\beta^\sigma \text{ split;} \\ a_p(f)^2 - 2p^{k-1} & (p) = \beta \text{ inert.} \end{cases}$$

Note that $\overline{\rho}_{\hat{f},q} \simeq \psi \oplus \psi^\sigma$

For $s = 3, 5, \ldots, k-1$, what we actually need is a Hecke eigenform $F_s \in S_{[k+s-1,\,k+1-s]}(SL_2(\mathcal{O}_F))$

[of non-parallel weight, $F_s\left(\left(\frac{a_1\tau_1+b_1}{c_1\tau_1+d_1}, \frac{a_2\tau_2+b_2}{c_2\tau_2+d_2}\right)\right) = (c_1\tau_1+d_1)^{k+s-1}(c_2\tau_2+d_2)^{k+1-s} F_s(\tau_1,\tau_2)$
$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}_F)$, $(\tau_1,\tau_2) \in \mathfrak{h}\times\mathfrak{h}$]

such that $\overline{\rho}^{\,ss}_{F_s,q} \simeq (\psi \otimes \overline{\varepsilon}_q^{\,1-s}) \oplus \psi^\sigma$

For $s=1$, $\hat{f}$ plays this rôle, but it appears that F_s exists in general.

For $(p) = \beta\beta^\sigma$ split, $\beta = (a)$, a totally positive, we would have $a_\beta(F_s) \equiv (a^\sigma)^{k-1} + a^{k-1}p^{s-1} \pmod{Q}$

Example: $F = \mathbb{Q}(\sqrt{5})$, $k=8$, $q=29$

p	a	$(a^\sigma)^7 \pmod{Q}$	$a^7 \pmod{Q}$	$(a^\sigma)^7 + a^7 p^{s-1} \pmod{Q}$ for $s=3$	$s=5$	$s=7$
11	$4+\alpha$	17	-1	12	21	8
19	$12+5\alpha$	-1	17	17	1	25
29	$7+2\alpha$	0	12	0	0	0
31	$6+\alpha$	12	1	16	-1	18

$\wp$	$F_3 \in S_{[10,6]}(SL_2(\mathcal{O}_F))$ $a_\wp(F_3)$	$F_5 \in S_{[12,4]}(SL_2(\mathcal{O}_F))$ $a_\wp(F_5)$	$F_7 \in S_{[14,2]}(SL_2(\mathcal{O}_F))$ $a_\wp(F_7)$
11	$1092 - 22680\alpha \equiv 12$	$-397788 + 156240\alpha \equiv 21$	$4303332 - 1580040\alpha \equiv 8$
19	$-66820 + 214200\alpha \equiv 17$	$10814300 \mp 1386000\alpha \equiv 1$	$166703900 + 52353000\alpha \equiv 25$
29	$1085070 + 982800\alpha \equiv 0$	$-18469050 - 14364000\alpha \equiv 0$	$-3830443650 + 128142000\alpha \equiv 0$
31	$-315328 + 2227680\alpha \equiv 16$	$46411232 + 20784960\alpha \equiv -1$	$-100191808 - 947691360\alpha \equiv 18$

Assuming that F_5 exists, here is what we can do with it.

We have $\rho_{F_5,q} : G_F \longrightarrow GL_2(K_q)$ (irreducible)

Also $\rho^{\sigma}_{F_5,q}$ defined by $\rho^{\sigma}_{F_5,q}(g) = \rho_{F_5,q}(\sigma g \sigma^{-1})$ $\forall g \in G_F$.

Let $\rho := \mathrm{Ind}_{G_F}^{G_\mathbb{Q}} \rho_{F_5,q} : G_\mathbb{Q} \longrightarrow GL_4(K_q)$.

Since $\rho^{\sigma}_{F_5,q} \not\simeq \rho_{F_5,q}$, ρ is irreducible.

But $\bar{\rho}$ is reducible. We can find its composition factors as follows:

$$\bar{\rho}^{ss} = \mathrm{Ind}_{G_F}^{G_\mathbb{Q}} \bar{\rho}^{ss}_{F_5,q} \simeq \mathrm{Ind}_{G_F}^{G_\mathbb{Q}} [(\psi \otimes \bar{\varepsilon}_q^{1-s}) \oplus \psi^{\sigma}]$$
$$\simeq \mathrm{Ind}_{G_F}^{G_\mathbb{Q}} (\psi \otimes \bar{\varepsilon}_q^{1-s}) \oplus \mathrm{Ind}_{G_F}^{G_\mathbb{Q}} (\psi^{\sigma})$$
$$\simeq \mathrm{Ind}_{G_F}^{G_\mathbb{Q}} (\psi) \otimes \bar{\varepsilon}_q^{1-s} \oplus \mathrm{Ind}_{G_F}^{G_\mathbb{Q}} (\psi^{\sigma})$$
$$\simeq \bar{\rho}_{f,q} \otimes \bar{\varepsilon}_q^{1-s} \oplus \bar{\rho}_{f,q}$$

Since ρ is irreducible, a Ribet-style argument shows that we can choose the $G_\mathbb{Q}$-invariant $\mathcal{O}_q$-lattice for ρ in such a way that we have a <u>non-split</u> extension of $\mathbb{F}_q[G_\mathbb{Q}]$-modules

$$0 \longrightarrow \bar{\rho}_{f,q} \otimes \bar{\varepsilon}_q^{1-s} \longrightarrow \bar{\rho} \longrightarrow \bar{\rho}_{f,q} \longrightarrow 0$$

i.e. $$0 \longrightarrow \bar{\rho}_{f,q}(1-s) \xrightarrow{i} \bar{\rho} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \bar{\rho}_{f,q} \longrightarrow 0$$

i and π are maps of $\mathbb{F}_q[G_\mathbb{Q}]$-modules

Choose a section λ, an $\mathbb{F}_q$-linear map such that $\pi \circ \lambda = id$.
Since the extension is non-split, λ is not an $\mathbb{F}_q[G_\mathbb{Q}]$-module map.

Now define $c : G_{\mathbb{Q}} \longrightarrow \mathrm{Hom}_{\mathbb{F}_q}(\tilde{\rho}_{f,q}, \bar{\rho}_{f,q}(1-s))$

by $c(g)(w) := g\lambda(g^{-1}w) - \lambda(w) \qquad \forall g \in G_{\mathbb{Q}}$.

Strictly speaking it is $i^{-1}(g\lambda(g^{-1}w) - \lambda(w))$

This makes sense, since

$$\pi(g\lambda(g^{-1}w) - \lambda(w)) = g(\pi\circ\lambda)(g^{-1}w) - (\pi\circ\lambda)w$$

(since π is an $\mathbb{F}_q[G_{\mathbb{Q}}]$-module map)

$$= g(g^{-1}w) - w$$

(since $\pi\circ\lambda = id.$)

$$= w - w = 0$$

and $\ker(\pi) = \mathrm{im}(i)$.

Given $g, h \in G_{\mathbb{Q}}$,

$$c(gh)(w) = gh\lambda((gh)^{-1}w) - \lambda(w)$$
$$= gh\lambda(h^{-1}g^{-1}w) - g\lambda(g^{-1}w) + g\lambda(g^{-1}w) - \lambda(w)$$
$$= g\,c(h)(g^{-1}w) + c(g)(w)$$

Hence c is a cocycle with values in $(\mathrm{ad}\,\bar{\rho}_{f,q})(1-s)$

A different choice of section λ only changes c by a coboundary.

So we have $[c] \in H^1(G_{\mathbb{Q}}, (\mathrm{ad}\,\bar{\rho}_{f,q})(1-s))$

Non-split extension $\Rightarrow [c] \neq 0$.

Special form of $\rho = \mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}} \rho_{F_s,q} \Rightarrow [c] \in H^1(G_{\mathbb{Q}}, (\mathrm{ad}^0\bar{\rho}_{f,q})(1-s))$

ρ unramified at $p \nmid D$, crystalline at q

$\Rightarrow [c] \in H^1_{\{p|D\}}(G_{\mathbb{Q}}, \mathrm{ad}^0(\bar{\rho}_{f,q})(1-s))$.

This is what we wanted.

Further example: $F = \mathbb{Q}(\sqrt{5}) = \mathbb{Q}(\alpha)$, $\alpha^2 = 5$, $k = 6$, $q = 11$

$Q = (11, \alpha + 4)$

$\alpha \equiv -4 \pmod{Q}$

p	a	$(a^\sigma)^5 \pmod{Q}$	$a^5 \pmod{Q}$	$(a^\sigma)^5 + a^5 p^{s-1} \pmod{Q}$ for $s=3$	$s=5$
11	$4+\alpha$	-1	0	-1	-1
19	$12+5\alpha$	-1	1	8	3
29	$7+2\alpha$	1	-1	7	9
31	$6+\alpha$	-1	-1	6	9

$G_3 \in S_{[8,4]}(SL_2(\mathcal{O}_F))$ Problem: $S_{[10,2]}(SL_2(\mathcal{O}_F)) = \{0\}$

p	$a_p(G_3)$	~~$a_p(G_5)$~~
11	$2172+1800\alpha \equiv -1$	
19	$-21340-9000\alpha \equiv 8$	
29	$86790+54000\alpha \equiv 7$	
31	$106112+79200\alpha \equiv 6$	

Explanation: When G_s exists, we use it to construct a non-zero
$[c] \in H^1_{\{5\}}(G_{\mathbb{Q}}, \mathrm{ad}^0(\bar{\rho}_{g,q})(1-s))$
accounting for $q \mid \dfrac{L_{\{5\}}(\mathrm{ad}^0(g), s)}{\Omega_s}$.

But one can show that actually $[c] \in H^1_f(G_{\mathbb{Q}}, \mathrm{ad}^0(\rho_{g,q})(1-s))$
so we should have $q \mid \dfrac{L(\mathrm{ad}^0(g), s)}{\Omega_s} = \dfrac{1}{1-5^{-s}} \dfrac{L_{\{5\}}(\mathrm{ad}^0(g), s)}{\Omega_s}$

$1-5^{-s} = \frac{1}{5^s}(5^s - 1)$ When $s=5$ this is divisible by 11, since $5^5 \equiv 1 \pmod{11}$,
so it can cancel the q we had in $\dfrac{L_{\{5\}}(\mathrm{ad}^0(g), 5)}{\Omega_5}$, making

$q \nmid \dfrac{L(\mathrm{ad}^0(g), 5)}{\Omega_5}$, so G_5 had better not exist.

Remarks (i) When G_s exists, there is an associated vector-valued Siegel modular Hecke eigenform of genus 2, $\mathcal{F}_s$, of weight $\det^{s+1} \otimes \mathrm{Sym}^{k-(s+1)}$, paramodular level $K(D^2)$. (Generalisation of Johnson-Leung Roberts lift.) Its associated 4-dimensional representations of $G_{\mathbb{Q}}$ are $\mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}}$ of 2-dim reps. attached to G_s.

When G_s doesn't exist, a suitable $\mathcal{F}_s$ that is not lifted from a Hilbert modular form may still exist, and provide the Galois representation required to construct $[c] \in H^1_{\{p \nmid D\}}(G_{\mathbb{Q}}, \mathrm{ad}^0(\bar{\rho}_{g,q})(1-s))$

If $\lambda_p(\mathcal{F}_s)$ is the Hecke eigenvalue of "$T(p)$" on $\mathcal{F}_s$ (lifted from G_s) then

$$\lambda_p(\mathcal{F}_s) = a_{\mathfrak{p}}(G_s) + a_{\mathfrak{p}^\sigma}(G_s)$$
$$\equiv \left[(a^\sigma)^{k-1} + a^{k-1}p^{s-1}\right] + \left[a^{k-1} + (a^\sigma)^{k-1}p^{s-1}\right] \pmod{\mathfrak{Q}}$$
$$\equiv (a^{k-1} + (a^\sigma)^{k-1})(1 + p^{s-1})$$
$$\equiv a_p(g)(1 + p^{s-1}) \pmod{\mathfrak{Q}}$$

This is a congruence of Hecke eigenvalues between $\mathcal{F}_s$ and a Klingen-Eisenstein series (same weight and level) attached to g.

(ii) Another place where non-parallel weight Hilbert modular forms appear.

The projectivisation of the quintic $P(x,y) = P(z,w)$ where $P(x,y) = x^5 + y^5 - (5xy-5)(x^2+y^2-x-y)$, has 120 singular points, which may be blown up to give a non-singular Calabi-Yau 3-fold, the Consani-Scholten quintic X.

The representation of $G_{\mathbb{Q}}$ on $H^3_{\text{ét}}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$ is 4-dimensional.

In 2012, Dieulefait, Pacetti and Schütt proved that this is induced from the representation of $G_{\mathbb{Q}(\sqrt{5})}$ attached to a Hilbert modular eigenform of weight $[4,2]$, level 30.

(iii) Hida informs me that numerical examples of congruences of this type, for non-parallel weight Hilbert modular forms, were discovered over 30 years ago by Naganuma (unpublished).

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