2. Galois stable lattices + their mod p reduction.

Lemma Let $V \in \mathrm{Rep}_E G_K$

$\Rightarrow \exists$ a Galois stable lattice $T \subset V$ i.e.

a free $\mathcal{O}_E$-module $T \subset V$ that are stable under G_K-action s.t. $T \otimes_{\mathcal{O}_E} E \cong V$.

proof) $\because$ G_K is compact. + Exercise.

Lemma Let $V \in \mathrm{Rep}_E G_K$, and T_1, T_2 be Galois stable lattices of V.

$\Rightarrow$ $J.H.(\overline{T_1}) = J.H.(\overline{T_2})$, where $\overline{T_i} := T_i \otimes_{\mathcal{O}_E} \mathbb{F}$.

Lemma Let $V \in \mathrm{Rep}_E G_K$.

If mod p reduction of V is irreducible, then

$\exists$ a unique Galois stable lattice of V up to homothety.

proof) Exercise.

Lemma Let $\overline{V} \in \mathrm{Rep}_{\mathbb{F}} G_K$ be an irred. rep-n.

$\Rightarrow$ P_K acts on $\overline{V}$ trivially.

proof) Exercise. $\because$ P_K = pro-p group $\triangleleft\, G_K$.

* Fontaine – Laffaille Theory

Assume $0 < r < p-1$.

$$\begin{array}{ccc}
\mathrm{Rep}_E^{\mathrm{cris},[0,r]} G_{\mathbb{Q}_p} & \xrightarrow[\sim]{D_{st}^*} & \left\{ \begin{array}{l}\text{admissible filtered } \varphi\text{-modules}/E \\ \text{with jumps} \in [0,r] \end{array} \right\} \\
\uparrow & \circlearrowright & \uparrow \\
\mathrm{Rep}_{\mathcal{O}_E}^{\mathrm{cris},[0,r]} G_{\mathbb{Q}_p} & \xleftarrow[\sim]{T_{cris}^*} & \left\{ \begin{array}{l}\text{Fontaine - Laffaille modules}/\mathcal{O}_E \\ \text{of wt } r \end{array} \right\} \\
\downarrow & \circlearrowright & \downarrow \\
\mathrm{Rep}_{\mathbb{F}} G_{\mathbb{Q}_p} & \underset{\text{f.f.}}{\overset{T_{cris}^*}{\longleftrightarrow}} & \left\{ \begin{array}{l}\text{Fontaine - Laffaille moduls}/\mathbb{F} \\ \text{of wt } r. \end{array} \right\}
\end{array}$$

<u>Def</u> · A <u>Fontaine - Laffaille module</u>$/\mathcal{O}_E$ of weight r is a free $\mathcal{O}_E$-module M of finite rank together with $(\{\mathrm{Fil}^i M\}_{i\in\mathbb{Z}}, \{\phi_i\}_{i\in\mathbb{Z}})$,

where

- a decreasing filtration $\{\mathrm{Fil}^i M\}_{i\in\mathbb{Z}}$ by $\mathcal{O}_E$-submodules
  s.t. · $\mathrm{Fil}^{i+1} M$ is a $\mathbb{Z}_p$-direct summand of $\mathrm{Fil}^i M$ for all i, and
  · $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{r+1} M = 0$.
- $\mathcal{O}_E$-linear maps $\phi_i : \mathrm{Fil}^i M \to M$ s.t.
  $\phi_i|_{\mathrm{Fil}^{i+1}M} = p\cdot\phi_i$ and $\sum_{i=0}^{r} \phi_i(\mathrm{Fil}^i M) = M$

· Morphisms are $\mathcal{O}_E$-linear maps compatible with ϕ_i and filtration.

<u>Thm</u> ∃ an exact, anti-equivalence of categories

$$T^*_{cris} : \left\{ \begin{array}{c} \text{F.-L. modules}/\mathcal{O}_E \\ \text{of wt } r \end{array} \right\} \xrightarrow{\sim} \mathrm{Rep}^{cris,[0,r]}_{\mathcal{O}_E} G_{\mathbb{Q}_p}$$

that fits the diagram.

<u>Def</u> . A <u>Fontaine-Laffaille module</u>$/\mathbb{F}$ of weight r is a $\mathbb{F}$-v.s. M of finite dimension together with

$$\left(\{ \mathrm{Fil}^i M \}_{i \in \mathbb{Z}}, \{ \phi_i \}_{i \in \mathbb{Z}} \right)$$

where

- a decreasing filtration $\{ \mathrm{Fil}^i M \}_{i \in \mathbb{Z}}$ by $\mathbb{F}$-subspaces s.t.
  - $\mathrm{Fil}^{i+1} M$ is a $\mathbb{F}_p$-direct summand of $\mathrm{Fil}^i M$ for all $i \in \mathbb{Z}$, and
  - $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{r+1} M = 0$.
- $\mathbb{F}$-linear map $\phi_i : \mathrm{Fil}^i M \to M$ s.t. $\mathrm{Fil}^{i+1} M \subset \mathrm{Ker}\, \phi_i$ and $\sum_{i=0}^{r} \phi_i (\mathrm{Fil}^i M) = M$.

. Morphisms are $\mathbb{F}$-linear maps compatible with ϕ_i and filtration.

<u>Thm</u> ∃ an exact, fully faithful contravariant functor

$$T^*_{cris} : \left\{ \begin{array}{c} \text{FL modules}/\mathbb{F} \\ \text{of weight } r \end{array} \right\} \longrightarrow \mathrm{Rep}_{\mathbb{F}} G_{\mathbb{Q}_p}.$$

e.g.) Consider the example (1) - ©. + assume $0 < r < p-1$.

- $M := \mathcal{O}_E(e_1, e_2)$

$$\mathrm{Fil}^i M := M \cap \mathrm{Fil}^i D = \begin{cases} M & i \le 0 \\ \mathcal{O}_E e_1 & 0 < i \le r \\ 0 & r < i \end{cases}$$

$$\phi_i := \frac{1}{p^i}\phi : \mathrm{Fil}^i M \to M.$$

$$\Rightarrow \cdot\ \phi_0 = \phi : \mathrm{Fil}^0 M \to M$$
$$e_1 \longmapsto \lambda e_2$$
$$e_2 \longmapsto -e_1 + \eta e_2$$

$$\cdot\ \phi_1 : \mathcal{O}_E e_1 \to M, \quad e_1 \longmapsto \frac{\lambda}{p} e_2$$

$$\vdots$$

$$\cdot\ \phi_r : \mathcal{O}_E e_1 \to M, \quad e_1 \longmapsto \frac{\lambda}{p^r} e_2$$

- $\overline{M} = \mathbb{F}(e_1, e_2)$

$$\mathrm{Fil}^i \overline{M} = \begin{cases} \overline{M} & i \le 0 \\ \mathbb{F} e_1 & 0 < i \le r \\ 0 & r < i \end{cases}$$

$$\phi_i : \mathrm{Fil}^i \overline{M} \to \overline{M} \Rightarrow \phi_0 : e_1 \mapsto 0, \quad e_2 \mapsto -e_1$$

$$\phi_1 : e_1 \longmapsto 0$$

$$\vdots$$

$$\phi_r : e_1 \longmapsto \overline{\left(\frac{\lambda}{p^r}\right)} e_2$$

Exercise : $\cdot\ T^*_{cris}(\overline{M})$ is irreducible

In fact; $\bar{\rho}|_{I_{\mathbb{Q}_p}} \sim \omega_2^r \oplus \omega_2^{pr}$ where $\bar{\rho} := T^*_{cris}(M)$.

* Strongly divisible modules

Assume $0 < r < p-1$.

$$\mathrm{Rep}_E^{st,[0,r]} G_{\mathbb{Q}_p} \xrightarrow[\sim]{D_{st}^*} \left\{ \begin{array}{l} \text{adm. filtered } (\phi, N)\text{-mod}/E \\ \text{with jumps} \in [0,r] \end{array} \right\} \underset{f.f.}{\hookleftarrow} \left\{ \begin{array}{l} \text{certain} \\ S_E\text{-modules} \end{array} \right\}.$$

$$\uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow$$

$$\mathrm{Rep}_{\mathcal{O}_E}^{st,[0,r]} G_{\mathbb{Q}_p} \xleftarrow[\sim]{T_{st}^* \ (\text{Breuil} + \text{Liu})} \left\{ \begin{array}{l} \text{Strongly divisible} \\ \text{modules}/S_{\mathcal{O}_E} \text{ of wt. } r \end{array} \right\}$$

$$\downarrow \qquad\qquad \circlearrowleft \qquad\qquad \downarrow$$

$$\mathrm{Rep}_{\mathbb{F}} G_{\mathbb{Q}_p} \xleftarrow[f.f.]{T_{st}^*} \left\{ \begin{array}{l} \text{Breuil modules}/S_{\mathbb{F}} \\ \text{of wt } r. \end{array} \right\}$$

Warning!! The bottom T_{st}^* is only faithful, in general.

Known fully faithful when $e \cdot r < p-1$, where $e := [K:K_0]$.

(Caruso).

- Let $\cdot\, E(u) := u - p. \in \mathbb{Z}_p[u]$

$$\cdot\, S := \widehat{\mathbb{Z}_p\left[\tfrac{u^i}{i!} \,\middle|\, i \in \mathbb{N}\right]} \qquad (\ \widehat{\ } \ \text{p-adic completion})$$

$$= \left\{ \sum_{i=0}^{\infty} w_i \cdot \frac{E(u)^i}{i!} \;\middle|\; w_i \in \mathbb{Z}_p,\ w_i \mapsto 0 \right\}$$

$\Rightarrow S$ has additional structure:

- $\phi : S \to S$ is a $\mathbb{Z}_p$-linear map with $\phi(u) = u^p$.
- $N : S \to S$ is a $\mathbb{Z}_p$-linear ~~map~~ derivation s.t. $N(u) = -u$.
- a decreasing filtration $\{\mathrm{Fil}^i S\}_{i \in \mathbb{Z}_{\geq 0}}$, where

$$\mathrm{Fil}^i S := \widehat{\sum_{j \geq i} \frac{E(u)^j}{j!} S}$$

Ex). $\cdot\, N\phi = p\phi N$ on S

$\cdot\, \phi(\mathrm{Fil}^i S) \subset p^i S$ for $0 \leq i \leq p-1$.

Def $S_{\mathcal{O}_E} := S \otimes_{\mathbb{Z}_p} \mathcal{O}_E$

$S_E := S_{\mathcal{O}_E} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = S \otimes_{\mathbb{Z}_p} E.$

Extend the definitions of ϕ, N, $\{\mathrm{Fil}^i S\}_{i \in \mathbb{Z}_{\geq 0}}$

to $S_{\mathcal{O}_E}$ and S_E by $\mathcal{O}_E$-linearity and E-linearity.

Def A filtered (ϕ, N)-modules $/ S_E$ is a free S_E-module $\mathcal{D}$ of finite rank together with

- a $\phi \otimes 1$-semilinear morphism $\phi : \mathcal{D} \to \mathcal{D}$
  s.t. $\det \phi$ is invertible in $S_{\mathcal{O}_p}$
  w.r.t. a $S_{\mathcal{O}_p}$-basis.
- a decreasing filtration on $\mathcal{D}$ by S_E-modules $\{\mathrm{Fil}^i \mathcal{D}\}_{i \in \mathbb{Z}}$
  with
  - $\mathrm{Fil}^i \mathcal{D} = \mathcal{D}$ if $i \leq 0$
  - $\mathrm{Fil}^i S_E \cdot \mathrm{Fil}^j \mathcal{D} \subseteq \mathrm{Fil}^{i+j} \mathcal{D}$
- an E-linear map $N : \mathcal{D} \to \mathcal{D}$ s.t.
  - $N(sx) = N(s) \cdot x + s \cdot N(x)$ $\forall s \in S_E, \forall x \in \mathcal{D}$
  - $N \phi = p \phi N$
  - $N(\mathrm{Fil}^i \mathcal{D}) \subset \mathrm{Fil}^{i-1} \mathcal{D}$ $\forall i \in \mathbb{Z}$.

• Let D be an adm. filtered (ϕ, N)-module$/_E$ with $\mathrm{Fil}^0 D = D$.

$\Rightarrow$ - $\mathcal{D} := S \otimes_{\mathbb{Z}_p} D \quad \Leftarrow S_E$-module

- $\phi := \phi \otimes \phi : \mathcal{D} \to \mathcal{D}$

- $N := N \otimes 1 + 1 \otimes N : \mathcal{D} \to \mathcal{D}$.

- $\mathrm{Fil}^0 \mathcal{D} = \mathcal{D}$ and, by induction,

$$\mathrm{Fil}^{i+1} \mathcal{D} := \left\{ x \in \mathcal{D} \;\middle|\; N(x) \in \mathrm{Fil}^i \mathcal{D} \text{ and } f_p(x) \in \mathrm{Fil}^{i+1} D \right\}$$

where $f_p : \mathcal{D} \to D$ is defined by $s(u) \otimes x \longmapsto s(p) \cdot x$.

<u>Thm</u> (Breuil) $D \longmapsto \mathcal{D} := S \otimes_{\mathbb{Z}_p} D$ gives a fully faithful ~~functor~~ functor from the category of adm. filtered (ϕ, N)-mod$/_E$ with $\mathrm{Fil}^0 D = D$ to ——〃—— of filtered (ϕ, N)-modules$/_{S_E}$.

<u>Lemma</u> Consider the example (2), and assume $r = 2$

$\Rightarrow$ • $\mathrm{Fil}^0 \mathcal{D} = \mathcal{D}$

• $\mathrm{Fil}^1 \mathcal{D} = S_E (e_1 + \alpha e_2) + \mathrm{Fil}^1 S_E \cdot \mathcal{D}$

• $\mathrm{Fil}^2 \mathcal{D} = S_E \left(e_1 + \alpha e_2 + \frac{u-p}{p} e_2 \right) + \mathrm{Fil}^2 S_E \cdot \mathcal{D}$

• $\mathrm{Fil}^i \mathcal{D} = \mathrm{Fil}^{i-2} S_E \left(e_1 + \alpha e_2 + \frac{u-p}{p} e_2 \right) + \mathrm{Fil}^i S_E \cdot \mathcal{D} \quad \forall_{i \geq 2}$.

<u>proof</u>) Exercise.

Tip!! $N\left(e_1 + \alpha e_2 + \frac{u-p}{p} e_2 \right) = e_2 + 0 - \frac{u}{p} e_2 = -\frac{u-p}{p} e_2 \in \mathrm{Fil}^1 \mathcal{D}$.

Def) A <u>strongly divisible module</u> $/ S_{\mathcal{O}_E}$ of weight r is

a free $S_{\mathcal{O}_E}$-module $\underline{\mathcal{M}}$ of finite rank

with
- an $S_{\mathcal{O}_E}$-submodule $\mathrm{Fil}^r \underline{\mathcal{M}}$
- additive maps $\phi, N : \underline{\mathcal{M}} \longrightarrow \underline{\mathcal{M}}$

s.t.
- $\mathrm{Fil}^r S_{\mathcal{O}_E} \cdot \underline{\mathcal{M}} \subseteq \mathrm{Fil}^r \underline{\mathcal{M}}$
- $\mathrm{Fil}^r \underline{\mathcal{M}} \cap I \cdot \underline{\mathcal{M}} = I \cdot \mathrm{Fil}^r \underline{\mathcal{M}} \quad {}^\forall I \underset{\text{ideal}}{\subseteq} \mathcal{O}_E$
- $\phi(sx) = \phi(s) \cdot \phi(x) \quad {}^\forall s \in S_{\mathcal{O}_E},\ {}^\forall x \in \underline{\mathcal{M}}.$
- $\phi(\mathrm{Fil}^r \underline{\mathcal{M}}) \subset p^r \underline{\mathcal{M}}$ and generate it over $S_{\mathcal{O}_E}$.
- $N(sx) = N(s) \cdot x + s \cdot N(x) \quad {}^\forall s \in S_{\mathcal{O}_E} \quad {}^\forall x \in \underline{\mathcal{M}}$
- $N\phi = p\phi N$
- $E(u) \cdot N(\mathrm{Fil}^r \underline{\mathcal{M}}) \subset \mathrm{Fil}^r \underline{\mathcal{M}}.$

<u>**Thm**</u> let $0 \leq r < p-1$

$$\mathrm{Rep}^{st,[0,r]}_{\mathcal{O}_E} G_{\mathbb{Q}_p} \xleftarrow[\sim]{T_{st}^*} \{\text{Strongly div. modules}/S_{\mathcal{O}_E} \text{ of weight } r\}$$

that fits the diagram.

- Breuil proved $G_{\mathbb{Q}_p}$-case + conjectured G_K-case

Liu proved G_K-case.

- Let $\underline{\mathcal{M}}$ be a strongly div. module$/S_{\mathcal{O}_E}$ of wt r.

$\Rightarrow$ · $\mathcal{D} := \underline{\mathcal{M}}[\frac{1}{p}] = \underline{\mathcal{M}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, $\quad \mathrm{Fil}^r \mathcal{D} := \mathrm{Fil}^r \underline{\mathcal{M}}[\frac{1}{p}]$.

· $\phi, N \circlearrowright \mathcal{D}$.

· $$\mathrm{Fil}^i \mathcal{D} := \begin{cases} \mathcal{D} & \text{if } i \le 0 \\ \{x \in \mathcal{D} \mid E(u)^{r-i} x \in \mathrm{Fil}^r \mathcal{D}\} & \text{if } 0 < i \le r \\ \sum_{j=0}^{i-1} (\mathrm{Fil}^{i-j} S_{\mathbb{Q}_p})(\mathrm{Fil}^j \mathcal{D}) & \text{if } i > r, \text{ inductively} \end{cases}$$

$\Rightarrow$ $\mathcal{D}$ is a filtered (ϕ, N)-module$/S_E$

$\Rightarrow$ Let $s_0 : S_{\mathbb{Q}_p} \longrightarrow \mathbb{Q}_p$, $u \longmapsto 0$ $\qquad$ $s_p : S_{\mathbb{Q}_p} \longrightarrow \mathbb{Q}_p$, $u \longmapsto p$.

· $D := \mathcal{D} \otimes_{S_{\mathbb{Q}_p}, s_0} \mathbb{Q}_p = \mathcal{D} \otimes_{S_{\mathbb{Q}_p}, s_p} \mathbb{Q}_p$

$D \circlearrowleft \phi, N$

$D \supset \mathrm{Fil}^i D := \mathrm{Fil}^i \mathcal{D} \otimes_{S_{\mathbb{Q}_p}, s_p} \mathbb{Q}_p$.

$\Rightarrow$ D is an admissible filtered (ϕ, N)-module$/E$ with jumps $\in [0, r]$.

$\therefore$ We have an alternative definition of strongly div. modules from the data of adm. filtered (ϕ, N)-modules.

Def Let D be an adm. filtered (ϕ, N)-mod $/E$ with $\mathrm{Fil}^0 D = D$ and $\mathrm{Fil}^{r+1} D = 0$.

A <u>strongly div. module</u> $/S_{\mathcal{O}_E}$ of weight r in $\mathcal{D} := S \otimes_{\mathbb{Z}_p} D$ is a free $S_{\mathcal{O}_E}$-submodule of $\mathcal{D}$ of finite rank $\underline{\mathcal{M}}$ with $\underline{\mathcal{M}}[\frac{1}{p}] \cong \mathcal{D}$.

s.t. - $\underline{\mathcal{M}}$ is stable under ϕ and N.

- $\phi(\mathrm{Fil}^r \underline{\mathcal{M}}) \subseteq p^r \underline{\mathcal{M}}$, where

$$\mathrm{Fil}^r \underline{\mathcal{M}} := \underline{\mathcal{M}} \cap \mathrm{Fil}^r \mathcal{D}.$$

e.g.) Consider example (2) + assume $r=2$.

$E(u) := (u-p) \in S.$

$\gamma := \frac{(u-p)^p}{p} \in S$

$$\Rightarrow \cdot\ \phi(\gamma) = \frac{(u^p-p)^p}{p} \in p^{p-1} \cdot S.$$

$$\cdot\ \gamma - 1 \equiv \frac{u^p - p}{p} \quad (\mathrm{mod}\ pS)$$

$$\Rightarrow \phi(E(u)) = u^p - p \equiv p(\gamma-1) \quad (\mathrm{mod}\ p^2 S)$$

$$\cdot\ N(\gamma) = -u(u-p)^{p-1}$$

$$= -p\left[\gamma + (u-p)^{p-1}\right] \in p \cdot S.$$

· Let D be an adm. filtered (φ, N)-module$/E$ in example (2) with $r=2$. $(\Rightarrow v_p(\lambda)=\frac{1}{2})$

<u>Prop</u> $\mathcal{M} := S_{\mathcal{O}_E}(E_1, E_2)$ is a str. div. mod. in $\mathcal{D} := S \otimes_{\mathbb{Z}_p} D$

where ① if $v_p(\alpha-1) \geq \frac{1}{2}$

$$E_1 = pe_1 + \alpha e_2 + (\gamma-1) e_2$$
$$E_2 = \lambda e_2$$

② if $v_p(\alpha-1) < \frac{1}{2}$

$$E_1 = pe_1 + \alpha e_2 + (\gamma-1) e_2 - \frac{p\delta}{\alpha-1}(\gamma-1)^2 e_2$$
$$E_2 = \lambda(\alpha-1) e_2.$$

Here, δ is defined as follows: assume $v_p(\alpha-1) < \frac{1}{2}$.

· Define a sequence

$$G_0 := 1, \quad G_{n+1} := \frac{(\alpha-1)^2}{(\alpha-1)^2 - pG_n} \in E.$$

$\Rightarrow \{G_n\}$ converges to an element in $1+\mathfrak{m}_E$, ~~detected~~ denoted by δ.

$\because$ · $G_{n+1} = 1 + \dfrac{pG_n}{(\alpha-1)^2 - pG_n} \in 1+\mathfrak{m}_E.$

· $G_{n+2} - G_{n+1} = \dfrac{p(\alpha-1)^2}{[(\alpha-1)^2 - pG_{n+1}][(\alpha-1)^2 - pG_n]}(G_{n+1} - G_n)$

$\Rightarrow \{G_n\}$ is Cauchy.

$\Rightarrow \{G_n\}$ converges in $1+\mathfrak{m}_E$ ∎

· δ satisfies $\boxed{p\delta^2 - (\alpha-1)^2\delta + (\alpha-1)^2 = 0.}$

We prove the case ② by a series of lemmas.

Lemma

- $\phi(E_1) \equiv E_2 \quad (\mathrm{mod}\ m_E \cdot \mathcal{M})$
- $\phi(E_2) \equiv 0 \quad (\;\; // \;\;)$
- $N(E_1) \equiv 0 \quad (\;\; // \;\;)$
- $N(E_2) = 0.$

proof)

· $\phi(E_1) = p\lambda e_1 + \lambda\alpha e_2 + \lambda(\phi(\gamma)-1)e_2 - \frac{\lambda p\delta}{\alpha-1}(\phi(\gamma)-1)^2 e_2$

$$= p\lambda\left[E_1 - \frac{\alpha}{\lambda(\alpha-1)}E_2 - \frac{\gamma-1}{\lambda(\alpha-1)}E_2 + \frac{p\delta}{\lambda(\alpha-1)^2}(\gamma-1)^2 E_2\right]$$

$$+ \frac{\alpha}{(\alpha-1)}E_2 + \frac{\phi(\gamma)-1}{\alpha-1}E_2 - \frac{p\delta}{(\alpha-1)^2}(\phi(\gamma)-1)^2 E_2$$

$$= p\lambda E_1 + \frac{\phi(\gamma)+(\alpha-1)-p(\gamma+\alpha-1)}{\alpha-1}E_2$$

$$- \frac{p\delta(\phi(\gamma)-1)^2 - p^2\delta(\gamma-1)^2}{(\alpha-1)^2}E_2 \equiv E_2 \quad (m_E \cdot \mathcal{M})$$

· $N(E_1) = pe_2 - u(up)^{M}e_2 - \frac{p\delta}{\alpha-1}2\cdot(\gamma-1)\left(-u(up)^{M}\right)e_2$

$$= p\left[1 - [\gamma+(up)^{M}]e_2 + \frac{2p\delta}{\alpha-1}(\gamma-1)[\gamma+(up)^{M}]e_2\right]$$

$$= \frac{p}{\lambda(\alpha-1)}\left[1 - [\gamma+(up)^{M}]\left(1 - \frac{2p\delta}{\alpha-1}(\gamma-1)\right)\right]E_2$$

$$\equiv 0 \quad (m_E \cdot \mathcal{M})$$

· Check $\phi(E_2)$, $N(E_2)$ ☐

Lemma $\quad \mathrm{Fil}^2 \underline{\mathcal{M}} = \langle F_1, F_2 \rangle + \mathrm{Fil}^2 S_{\mathcal{O}_E} \cdot \underline{\mathcal{M}}$, where

$$F_1 := \lambda E_1 - E_2 + \frac{p\delta}{(\alpha-1)^2} E_2 + \frac{p\alpha}{\alpha-1} E_2 + (u-p)\left(\frac{\lambda\delta}{\alpha-1} E_1 + \frac{p\delta\alpha}{(\alpha-1)^2} E_2\right)$$

$$F_2 := (u-p)\left(\lambda\delta E_1 - E_2 + \frac{p\delta\alpha}{\alpha-1} E_2\right).$$

proof). Recall: $\mathrm{Fil}^2 \mathcal{D} = S_E\left(e_1 + \alpha e_2 + \frac{u-p}{p} e_2\right) + \mathrm{Fil}^2 S_E \cdot \mathcal{D}$.

$\Rightarrow$ Modulo $\mathrm{Fil}^2 S_E$, every element in $\mathrm{Fil}^2 \mathcal{D}$ is written as

$$x = c_0\left(e_1 + \alpha e_2 + \frac{u-p}{p} e_2\right) + c_1 (u-p)\left(e_1 + \alpha e_2\right) \quad \text{for } c_i \in E.$$

Recall: $\mathrm{Fil}^2 \underline{\mathcal{M}} := \underline{\mathcal{M}} \cap \mathrm{Fil}^2 \mathcal{D}$.

$$\Rightarrow x \equiv c_0\left[\frac{1}{p}\left(E_1 - \frac{\alpha}{\lambda(\alpha-1)} E_2 + \frac{1}{\lambda(\alpha-1)} E_2 + \frac{p\delta}{\lambda(\alpha-1)^2} E_2\right) + \frac{\alpha}{\lambda(\alpha-1)} E_2 + (u-p)\frac{1}{p\lambda(\alpha-1)} E_2\right]$$

(mod $\mathrm{Fil}^2 S_E \cdot \mathcal{D}$)

$$+ c_1(u-p)\left[\frac{1}{p}\left(E_1 - \frac{\alpha}{\lambda(\alpha-1)} E_2 + \frac{1}{\lambda(\alpha-1)} E_2 + \frac{p\delta}{\lambda(\alpha-1)^2} E_2\right) + \frac{\alpha}{\lambda(\alpha-1)} E_2\right]$$

$$= c_0\left(\frac{1}{p} E_1 - \frac{1}{p\lambda} E_2 + \frac{\delta}{\lambda(\alpha-1)^2} E_2 + \frac{\alpha}{\lambda(\alpha-1)} E_2\right)$$

$$+ (u-p)\left[c_1\left(\frac{1}{p} E_1 - \frac{1}{p\lambda} E_2 + \frac{\delta}{\lambda(\alpha-1)^2} E_2 + \frac{\alpha}{\lambda(\alpha-1)} E_2\right) + \frac{c_0}{p\lambda(\alpha-1)} E_2\right]$$

$$= c_0\left(\frac{1}{p} E_1 - \frac{1}{p\lambda} E_2 + \frac{\delta}{\lambda(\alpha-1)^2} E_2 + \frac{\alpha}{\lambda(\alpha-1)} E_2\right)$$

$$+ (u-p)\left(\frac{c_1}{p} E_1 + \frac{(\alpha-1)c_0 + p\alpha(\alpha-1)c_1 - [(\alpha-1)^2 - p\delta]c_1}{p\lambda(\alpha-1)^2} E_2\right)$$

Recall: $p\delta^2 - (\alpha-1)^2\delta + (\alpha-1)^2 = 0$

$$\Rightarrow \frac{(\alpha-1)c_0 - [(\alpha-1)^2 - p\delta]c_1}{p\lambda(\alpha-1)^2} = \frac{\delta c_0 - (\alpha-1)c_1}{p\lambda\delta(\alpha-1)}$$

$$x \equiv C_0\left(\frac{1}{p}E_1 - \frac{(\alpha-1)^2 - p\delta - p\alpha(\alpha-1)}{p\lambda(\alpha-1)^2}E_2\right)$$

$$+ (wp)\left(\frac{c_1}{p}E_1 + \frac{p\alpha\, c_1}{p\lambda(\alpha-1)}E_2 + \frac{\delta C_0 - (\alpha-1)C_1}{p\lambda\delta(\alpha-1)}E_2\right)$$

$\Rightarrow$ · $v_p(C_0) \geq v_p(p\lambda) = \frac{3}{2}$

· $v_p(C_1) \geq 1$

· $v_p\left(\delta C_0 - (\alpha-1)C_1 + p\delta\alpha C_1\right) \geq v_p\left(p\lambda(\alpha-1)\right)$

$\Leftrightarrow v_p\left(\delta C_0 - (\alpha-1)C_1\right) \geq v_p\left(p\lambda(\alpha-1)\right)$

Since $$C_1 = \frac{p\lambda\delta}{\alpha-1}\cdot\frac{C_0}{p\lambda} - \frac{p\lambda\delta(\alpha-1)}{\alpha-1}\cdot\frac{\delta C_0 - (\alpha-1)C_1}{p\lambda\delta(\alpha-1)},$$

$$x \equiv \frac{C_0}{p\lambda}\left(\lambda E_1 - \frac{(\alpha-1)^2 - p\delta - p\alpha(\alpha-1)}{(\alpha-1)^2}E_2\right)$$

$$+ (wp)\left(\frac{p\lambda\delta}{\alpha-1}\frac{C_0}{p\lambda} - \frac{p\lambda\delta(\alpha-1)}{\alpha-1}\cdot\frac{\delta C_0 - (\alpha-1)C_1}{p\lambda\delta(\alpha-1)}\right)\left(\frac{1}{p}E_1 + \frac{p\alpha}{p\lambda(\alpha-1)}E_2\right)$$

$$+ (wp)\frac{\delta C_0 - (\alpha-1)C_1}{p\lambda\delta(\alpha-1)}E_2$$

$$\equiv \frac{C_0}{p\lambda}\underbrace{\left[\lambda E_1 - \frac{(\alpha-1)^2 - p\delta - p\alpha(\alpha-1)}{(\alpha-1)^2}E_2 + (wp)\left(\frac{\lambda\delta}{\alpha-1}E_1 + \frac{p\delta\alpha}{(\alpha-1)^2}E_2\right)\right]}_{=F_1}$$

$$- \frac{\delta C_0 - (\alpha-1)C_1}{p\lambda\delta(\alpha-1)}(wp)\underbrace{\left(\lambda\delta E_1 - E_2 + \frac{p\delta\alpha}{\alpha-1}E_2\right)}_{=F_2}$$

<u>Cor</u> $F_1 \equiv -E_1 \pmod{m_E \mathcal{M}}$

$F_2 \equiv$ ~~$\alpha \ell(\varphi) E_2$~~ $-uE_2$ (—— // ——)

<u>proof</u> obvious

<u>Lemma</u> $\phi(F_1) \equiv p\lambda^2 E_1 \pmod{p^2 m_E \cdot \mathcal{M}}$

$\phi(F_2) \equiv 0$ (—— // ——).

<u>proof</u>) Using our computation of $\phi(E_1)$, $\phi(E_2)$

$$\phi(F_1) = p\lambda^2 E_1 + \frac{\lambda[\phi(\gamma)-p(\gamma-1)]}{\alpha-1}E_2 - \frac{p\lambda\delta[\phi(\gamma)(\phi(\gamma)-2)-p(\gamma-1)^2]}{(\alpha-1)^2}E_2$$

$$+(u^p-p)\left(\frac{p\lambda^2\delta}{\alpha-1}E_1 + \frac{\lambda\delta(\phi(\gamma)+\alpha-1-p(\gamma-1))}{(\alpha-1)^2}E_2 - \frac{p\lambda\delta^2((\phi(\gamma)-1)^2-p(\gamma-1)^2)}{(\alpha-1)^3}E_2\right)$$

$$\equiv p\lambda^2 E_1 - \frac{p\lambda(\gamma-1)}{\alpha-1}E_2 + \frac{p^2\lambda\delta(\gamma-1)^2}{(\alpha-1)^2}E_2$$

$$+\frac{(u^p-p)}{p}\left(\frac{p\lambda\delta(\alpha-1-p(\gamma-1))}{(\alpha-1)^2}E_2 - \frac{p^2\lambda\delta^2}{(\alpha-1)^3}E_2\right)$$

$p\delta^2-(\alpha-1)\delta+(\alpha-1)^2$ $\overset{\text{!!}}{0}$

$$\overset{\vee}{=} p\lambda^2 E_1 - \frac{p\lambda(\gamma-1)}{\alpha-1}E_2 + \frac{p^2\lambda\delta(\gamma-1)^2}{(\alpha-1)^2}E_2$$

$$+\frac{u^p-p}{p}\left(\frac{p\lambda}{\alpha-1}E_2 - \frac{p^2\lambda\delta(\gamma-1)}{(\alpha-1)^2}E_2\right)$$

$$= p\lambda^2 E_1 - \frac{p\lambda}{\alpha-1}\left[(\gamma-1)-\frac{u^p-p}{p}\right]E_2 + \frac{p^2\lambda\delta(\gamma-1)}{(\alpha-1)^2}\left[(\gamma-1)-\frac{u^p-p}{p}\right]E_2$$

$$\equiv p\lambda^2 E_1 \pmod{p^2 \cdot m_E \cdot \mathcal{M}}.$$

$\phi(F_2)$ = exercise ▨