* Breuil Modules

Assume $0 \le r < p-1$, and let $T_{st}^{r}(\ ) := T_{st}^{*}(\ )^{\vee} \otimes \epsilon_p^{r}$

$$\begin{array}{ccc} \mathrm{Rep}^{st,[0,r]}_{\mathcal{O}_E} G_{\mathbb{Q}_p} & \xleftarrow[\sim]{T_{st}^{*r}} & \{\text{Str. div. mod}/S_{\mathcal{O}_E} \text{ of wt } r\} \\ \downarrow & & \downarrow \\ \mathrm{Rep}_{\mathbb{F}} G_{\mathbb{Q}_p} & \xleftarrow{T_{st}^{*r}} & \{\text{Breuil modules}/\bar{S}_{\mathbb{F}} \text{ of wt } r\} \end{array}$$

- $\bar{S}_{\mathbb{F}} := S_{\mathcal{O}_E}/(\pi_E, \mathrm{Fil}^p S_{\mathcal{O}_E}) \cong \mathbb{F}[u]/u^p \cong \mathbb{F}_p[u]/u^p \otimes_{\mathbb{F}_p} \mathbb{F}$

- Def) A Breuil module$/\bar{S}_{\mathbb{F}}$ of weight r is

  a free $\bar{S}_{\mathbb{F}}$-module $\mathcal{M}$ of finite rank together with

  $(\mathrm{Fil}^r\mathcal{M}, \phi_r, N)$, where

  - $\mathrm{Fil}^r\mathcal{M}$ is $\bar{S}_{\mathbb{F}}$-submodule of $\mathcal{M}$ containing $u^r\mathcal{M}$.
  - $\phi \otimes 1$-semilinear map $\phi_r : \mathrm{Fil}^r\mathcal{M} \to \mathcal{M}$ with image generating $\mathcal{M}$ as $\bar{S}_{\mathbb{F}}$-module,

    where $\phi : \mathbb{F}_p[u]/u^p \to \mathbb{F}_p[u]/u^p$ is the p-th power map.
  - $N : \mathcal{M} \to \mathcal{M}$ ~~is~~ $\mathbb{F}$-linear map s.t.
    - $N(ux) = u \cdot N(x) - u \cdot x \quad \forall x \in \mathcal{M}$
    - $u^e N(\mathrm{Fil}^r\mathcal{M}) \subset \mathrm{Fil}^r\mathcal{M}$
    - $\phi_r(u^e N(x)) = c \cdot N(\phi_r(x)) \quad \forall x \in \mathrm{Fil}^r\mathcal{M}$,

      where
      $$\begin{array}{ccc} S & \longrightarrow & \mathbb{F}_p[u]/u^p \\ \psi & & \psi \\ \frac{1}{p}\phi(E(u)) & \longmapsto & c \end{array}$$

<u>Thm</u> $\exists$ an exact fully faithful functor

$$\mathrm{Rep}_{\mathbb{F}} G_{\mathbb{Q}_p} \xleftarrow{T_{st}^*} \{\text{Breuil modules}/\overline{S}_{\mathbb{F}} \text{ of weight } r\}$$

s.t. the diagram commutes.

· The map T_{st}^* above is only faithful, in general, for G_K.

If $er < p-1$ where $e = [K : K_0]$, then it is fully faithful.

e.g.) ⓪ Let $a \in \mathbb{Z}_{\geq 0}$ with $0 \leq a \leq r$, and $\alpha \in \mathbb{F}^\times$

· $\widetilde{\mathcal{M}}(a;\alpha) := \overline{S}_{\mathbb{F}}(\tilde{e})$

· $\mathrm{Fil}^r \widetilde{\mathcal{M}} := \langle u^a \tilde{e} \rangle$

· $\phi_r : \mathrm{Fil}^r \widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{M}}$
$\quad u^a \tilde{e} \longmapsto \alpha \cdot \tilde{e}$

· $N(\tilde{e}) = 0.$

$\Rightarrow$ · $\widetilde{\mathcal{M}}(a;\alpha)$ is a Breuil module of weight r.

· $T_{st}^*(\widetilde{\mathcal{M}}(a;\alpha))|_{I_{\mathbb{Q}_p}} \cong \omega^{r-a}$ (Caruso)

$\Rightarrow T_{st}^r(\widetilde{\mathcal{M}}(a;\alpha))|_{I_{\mathbb{Q}_p}} \cong \omega^a$

① Assume $r = 2a$. $\mathcal{M}(a;A)$.

- $\mathcal{M} := \overline{S}_{\mathbb{F}}(e_1, e_2)$ $\quad \underline{e} := (e_1, e_2)$
- $Mat_{\underline{e},\underline{f}}(Fil^r \mathcal{M}) = \begin{pmatrix} u^a & 0 \\ 0 & u^a \end{pmatrix} \Leftrightarrow \underline{f} := (f_1, f_2) = \underline{e} \cdot \begin{pmatrix} u^a & 0 \\ 0 & u^a \end{pmatrix}$
- $Mat_{\underline{e},\underline{f}}(\phi_r) = A \in Gl_2(\mathbb{F}) \Leftrightarrow \phi_r(\underline{f}) = \underline{e} \cdot A.$
- $Mat_{\underline{e}}(N) = O_{2\times 2}$

<u>Lemma</u> Let $\overline{\rho} := T_{st}^{*}(\mathcal{M}(a;A))$. Then

$$\overline{\rho}|_{I_{\mathbb{Q}_p}} \sim \begin{cases} \omega^a \oplus \omega^a & \text{if } A \sim \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix} \\ \omega^a - \omega^a & \text{if } A \sim \begin{pmatrix} \alpha & 1 \\ & \alpha \end{pmatrix} \end{cases}$$

<u>proof</u>) Assume $C \cdot A C^{-1} = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$ for $C \in Gl_2(\mathbb{F})$.

Let $\underline{e}' := \underline{e} \cdot C^{-1} = (e_1', e_2')$

$$\Rightarrow \phi_r\left(\underline{e}' \cdot \begin{pmatrix} u^a & \\ & u^a \end{pmatrix}\right) = \phi_r\left(\underline{e}' C \begin{pmatrix} u^a & \\ & u^a \end{pmatrix} C^{-1}\right)$$
$$= \phi_r\left(\underline{e} \begin{pmatrix} u^a & \\ & u^a \end{pmatrix} C^{-1}\right)$$
$$= \underline{e} \cdot A \cdot C^{-1}$$
$$= \underline{e}' \cdot C A C^{-1}$$
$$= \underline{e}' \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}.$$

$\Rightarrow \mathcal{M}(a, A) = \overline{S}_{\mathbb{F}} e_1' \oplus \overline{S}_{\mathbb{F}} e_2'$.

The case $A \sim \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$ is similar. ☒

② Assume $0 \leq b < a \leq r = a+b$

$\mathcal{M}(a, b; \alpha, \beta, \gamma)$

- $\mathcal{M} := S_{\mathbb{F}}(e_1, e_2)$ $\quad \underline{e} := (e_1, e_2)$
- $\mathrm{Mat}_{\underline{e}, \underline{f}}(\mathrm{Fil}^r \mathcal{M}) = \begin{pmatrix} u^a & \\ & u^b \end{pmatrix}$
- $\mathrm{Mat}_{\underline{e}, \underline{f}}(\phi_r) = \begin{pmatrix} \gamma & \beta \\ \alpha & 0 \end{pmatrix} \in GL_2(\mathbb{F})$, with $\gamma \neq 0$.
- $\mathrm{Mat}_{\underline{e}}(N) = O_{2\times 2}$

~~proof)~~

Lemma Let $\bar{\rho} := T_{st}^{*}(\mathcal{M}(a, b; \alpha, \beta, \gamma))$

$\Rightarrow \bar{\rho}|_{I_{\mathbb{Q}_p}} \sim \omega^a - \omega^b$.

proof) We have

$$0 \longrightarrow \tilde{\mathcal{M}}(a; \gamma) \longrightarrow \mathcal{M}(a, b; \alpha, \beta, \gamma) \longrightarrow \tilde{\mathcal{M}}(b; -\tfrac{\alpha\beta}{\gamma}) \longrightarrow 0.$$

$$\hat{e} \longmapsto \gamma e_1 + \alpha e_2$$

$$e_1 \longmapsto \alpha \tilde{e}$$

$$e_2 \longmapsto -\gamma \tilde{e}$$

Check that it is non-split.

∎

③ Assume $0 \le b < a \le r = a+b$

$M(a,b;\alpha,\beta)$

- $M := \overline{S}_{\mathbb{F}}(e_1, e_2)$, $\underline{e} := (e_1, e_2)$
- $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r M) = \begin{pmatrix} u^a & 0 \\ 0 & u^b \end{pmatrix}$
- $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_r) = \begin{pmatrix} & \beta \\ \alpha & \end{pmatrix} \in GL_2(\mathbb{F})$
- $\mathrm{Mat}_{\underline{e}}(N) = O_{2\times 2}$

Lemma Let $\overline{\rho} := T^*_{st}(M(a,b;\alpha,\beta))$.

$\Rightarrow \overline{\rho}|_{I_{\mathbb{Q}_p}} \sim \omega_2^{a+pb} \oplus \omega_2^{b+pa}$, i.e. $\overline{\rho}$ is abs. irred.

Idea) $\overline{\rho}|_{I_{\mathbb{Q}_p}} \longleftrightarrow \overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} M(a,b;\alpha,\beta)$

$\cong$ by change of basis

$$\begin{array}{ccc} M_1 & \oplus & M_2 \\ \updownarrow T^*_{st} & & \updownarrow T^*_{st} \\ \omega_2^{a+pb} & & \omega_2^{b+pa} \end{array} \quad \Leftarrow \text{Caruso.}$$

□

Lemma $M(a,b;\alpha,\beta) \sim M(a',b';\alpha',\beta')$

$\iff a = a'$, $b = b'$, $\alpha\cdot\beta = \alpha'\beta'$.

- Let $\underline{\mathfrak{M}}$ be a str. div. module$/S_{\mathcal{O}_E}$ of weight r.

$\Rightarrow$ $\mathcal{M} := \underline{\mathfrak{M}} / (\pi_E, \mathrm{Fil}^p S_{\mathcal{O}_E}) \underline{\mathfrak{M}}$.

- $\mathrm{Fil}^r \mathcal{M}$ is the image of $\mathrm{Fil}^r \underline{\mathfrak{M}}$ in $\mathcal{M}$.
- ϕ_r is induced by $\frac{1}{p^r} \phi|_{\mathrm{Fil}^r \underline{\mathfrak{M}}}$
- N is —— by the one on $\underline{\mathfrak{M}}$.

$\Rightarrow$ $\mathcal{M}$ is a Breuil module$/\overline{S_{\mathbb{F}}}$ of weight r.

e.g.) Recall the str. div. mod. in the example (2) + $r=2$

$$\underline{\mathfrak{M}} := S_{\mathcal{O}_E}(E_1, E_2), \quad \mathrm{Fil}^2 \underline{\mathfrak{M}} := \langle F_1, F_2 \rangle + \mathrm{Fil}^2 S_{\mathcal{O}_E} \cdot \underline{\mathfrak{M}}$$

Assume $v_p(\lambda-1) < \frac{1}{2}$

- $\phi(E_1) \equiv E_2 \quad (\mathrm{mod}\ \mathfrak{m}_E \cdot \underline{\mathfrak{M}})$
- $\phi(E_2) \equiv 0 \quad ($ —— // —— $)$
- $N(E_1) \equiv 0 \quad ($ —— // —— $)$
- $N(E_2) = 0$
- $F_1 \equiv -E_2 \quad ($ —— // —— $)$
- $F_2 \equiv -u E_2 \quad ($ —— // —— $)$
- $\phi(F_1) \equiv p\lambda^2 E_1 \quad (\mathrm{mod}\ p^2 \mathfrak{m}_E \underline{\mathfrak{M}})$
- $\phi(F_2) \equiv 0 \quad ($ —— // —— $)$

$\Rightarrow$ Let $\underline{e} := (e_1, e_2) = (E_1, E_2) \mod (\pi_E, \mathrm{Fil}^p S_{\mathcal{O}_E})$.

- $\mathrm{Fil}^2 \mathcal{M} = \langle e_2, u^2 e_1 \rangle$ $\quad \because \ u^2 e_1 \equiv (u-p)^2 E_1 \ (\pi_E, \mathrm{Fil}^p S_{\mathcal{O}_E})$

- $\phi_2 : \mathrm{Fil}^2 \mathcal{M} \longrightarrow \mathcal{M}$

  $e_2 \longmapsto -\frac{p\lambda^2}{p^2} e_1 = \frac{-\lambda^2}{p} e_1$ $\quad \because \ \phi(F_1) \equiv p\lambda^2 E_1 \ (p^2 m_E \mathcal{M})$

  $u^2 e_1 \longmapsto e_2$ $\quad \because \ \phi((u-p)^2 E_1) \equiv p^2 \left(\frac{u^p - p}{p}\right)^2 E_2 \ (p^2 m_E \mathcal{M})$

- $N : \mathcal{M} \longrightarrow \mathcal{M}$

  $e_1 \longmapsto 0$

  $e_2 \longmapsto 0$

$$\Rightarrow \mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^2 \mathcal{M}) = \begin{pmatrix} u^2 & \\ & 1 \end{pmatrix}$$

$$\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_2) = \begin{pmatrix} 0 & -\overline{\left(\frac{\lambda^2}{p}\right)} \\ 1 & 0 \end{pmatrix}$$

$$\mathrm{Mat}_{\underline{e}}(N) = 0.$$

$$\therefore \ \mathcal{M} \sim \mathcal{M}\left(2, 0; 1, -\overline{\left(\frac{\lambda^2}{p}\right)}\right)$$

$$\Rightarrow T_{st}^{*}(\mathcal{M})\big|_{I_{\mathbb{Q}_p}} \sim \omega_2^{2} \oplus \omega_2^{2p}.$$