3. Semi-stable deformation rings

$$
\begin{tikzcd}
\mathrm{Rep}_E\, G_{\mathbb{Q}_p} \arrow[d, dash] & & \\
\mathrm{Rep}_E^{HT}\, G_{\mathbb{Q}_p} \arrow[d, dash] & & \\
\mathrm{Rep}_E^{dR}\, G_{\mathbb{Q}_p} \arrow[d, dash] \arrow[dr, equal, "\text{p-adic monodromy thm.}"] & & \\
\mathrm{Rep}_E^{st}\, G_{\mathbb{Q}_p} \arrow[d, dash] & \mathrm{Rep}_E^{p.st}\, G_{\mathbb{Q}_p} \arrow[l] \arrow[d, dash] & \\
\mathrm{Rep}_E^{cris}\, G_{\mathbb{Q}_p} & \mathrm{Rep}_E^{pcris}\, G_{\mathbb{Q}_p} \arrow[l] &
\end{tikzcd}
$$

· <u>Def</u> A potentially semi-stable (resp. potentially crystalline) rep-n V of $G_{\mathbb{Q}_p}$ is called <u>L-semi-stable</u> (resp. <u>L-crystalline</u>) if $\exists\, L/\mathbb{Q}_p$ finite + Galois s.t. $V|_{G_L}$ is semi-stable (resp. crystalline)

· <u>Thm</u> (Savitt). $\mathrm{Rep}_E^{L\text{-}st}\, G_{\mathbb{Q}_p} \xrightarrow[\sim]{D_{st,L}} \{$adm. filtered (φ, N)-modules $/L\otimes_{\mathbb{Q}_p}E$ with descent data$\}$.

· Let V be a L-semi-stable rep-n of $G_{\mathbb{Q}_p}$.

$\Rightarrow$ · $D_{st,L}(V) := (\mathbb{B}_{st}\otimes_{\mathbb{Q}_p} V)^{G_L} \in$ free $L_0 \otimes_{\mathbb{Q}_p} E$-module

$\mathrm{Gal}(L/\mathbb{Q}_p) \circlearrowright$

$\uparrow$

descent data

· $\mathrm{Fil}^i D_{st,L}(V) := (\mathrm{Fil}^i\, \mathbb{B}_{st,L}\otimes_{\mathbb{Q}_p} V)^{G_L} \in L\otimes_{\mathbb{Q}_p} E$-module

· $\phi := \phi \otimes 1$

· $N := N \otimes 1$

* Galois type

$$\begin{tikzcd}
1 \arrow[r] & I_{\mathbb{Q}_p} \arrow[r] & G_{\mathbb{Q}_p} \arrow[r] & G_{\mathbb{F}_p} \arrow[r] & 0 \\
1 \arrow[r] & I_{\mathbb{Q}_p} \arrow[r] \arrow[u, equal] & W_{\mathbb{Q}_p} \arrow[r] \arrow[u, hook] & \mathrm{Frob}_p^{\mathbb{Z}} \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

$\mathrm{Frob}_p : x \longmapsto x^{\frac{1}{p}}$

Weil group

Topology of $W_{\mathbb{Q}_p} := \langle \{\text{subspace top}\} \cup \{I_{\mathbb{Q}_p}\} \rangle$

Def A Weil-Deligne rep-n of $W_{\mathbb{Q}_p}$ on a finite dim. E-v.s. V is a pair (r, N), where

- $r : W_K \longrightarrow GL(V)$ is continuous.
- $N \in End(V)$ st $\forall g \in W_{\mathbb{Q}_p}$

$$r(g) \cdot N \cdot r(g)^{-1} = p^{-\nu_{\mathbb{Q}_p}(g)} \cdot N,$$

where $\nu_{\mathbb{Q}_p} : W_{\mathbb{Q}_p} \twoheadrightarrow \mathbb{Z}$ is determined by $g|_{L_0} = \mathrm{Frob}_p^{\nu_{\mathbb{Q}_p}(g)}$

Remark !!

- $r(I_{\mathbb{Q}_p}) = $ finite $\because$ $I_{\mathbb{Q}_p} = $ compact. + $V = $ discrete top.
- N is nilpotent.
- V has the discrete topology.

- Assume ~~L_0~~ $L_0 \subset E \Rightarrow \mathrm{Hom}(L_0, E) = \mathrm{Hom}(L_0, \overline{\mathbb{Q}}_p)$.

$$\Rightarrow \begin{array}{ccc} L_0 \otimes_{\mathbb{Q}_p} E & \xrightarrow{\sim} & \bigoplus\limits_{\tau: L_0 \hookrightarrow E} E \\ \Psi & & \Psi \\ x \otimes y & \longmapsto & (\tau(x)\cdot y)_\tau \\ e_\tau & \longleftrightarrow & (0, 0, \cdots, 1, 0, \cdots, 0) \end{array} \quad (\text{by C.R.T}).$$

$\uparrow$ τ-position.

Let V be L-semi-stable

$\Rightarrow D := D_{st,L}(V) \in$ a free $L_0 \otimes_{\mathbb{Q}_p} E$-mod of rank $= \dim_E V$.

$$\Rightarrow \begin{array}{ccc} D & \xrightarrow{\sim} & \bigoplus\limits_{\tau: L_0 \hookrightarrow E} De_\tau \\ \Psi & & \\ d & \longmapsto & (d\cdot e_\tau)_\tau \end{array}, \quad De_\tau = E\text{-v.s. with } \dim_E De_\tau = \dim_E V$$

Moreover,

the map respects the action of $\mathrm{Gal}(L/\mathbb{Q}_p)$:

$$\forall g \in \mathrm{Gal}(L/\mathbb{Q}_p),\ g.(d_\tau)_\tau := (g.d_\tau)_{\tau\circ g^{-1}} = (g\, d_{\tau\circ g})_\tau.$$

Let $g \in W_{\mathbb{Q}_p}$ and $\bar{g} \in \mathrm{Gal}(L/\mathbb{Q}_p)$ be the image of g under $W_{\mathbb{Q}_p} \longrightarrow \mathrm{Gal}(L/\mathbb{Q}_p)$, and

let $\alpha: W_{\mathbb{Q}_p} \longrightarrow \mathbb{Z}$ be the map $\bar{g}|_{L_0} = \mathrm{Frob}_p^{\alpha(g)}$

- Set $r_\tau(g): De_\tau \xrightarrow{\bar{g}} De_{\tau\circ g^{-1}} \xrightarrow{\phi^{\alpha(g)}} De_\tau$

$N_\tau := N|_{De_\tau}$ (Note N is $L_0 \otimes_{\mathbb{Q}_p} E$-linear)

$\Rightarrow$ $WD_I(V) := (r_I, N_I)$ forms a Weil-Deligne rep-n.

(does not depend on the choice of I)

<u>Def</u> Let V be a L-semi-stable rep-n of $G_{\mathbb{Q}_p}$

$\Rightarrow$ · $WD(V) :=$ the isom class of $WD_I(V)$.

· <u>Galois type</u> of $V := r|_{I_{\mathbb{Q}_p}}$, if $WD(V) = (r, N)$.

* <u>Galois deformation theory</u>

Fix $\bar{\rho}_0 : G_K \longrightarrow GL_n(\mathbb{F})$, and let $\mathcal{C}_{\mathcal{O}_E}$ be the category of complete Noeth. local $\mathcal{O}_E$-algebras with residue field $\mathbb{F}$.

Define functors $D, D^\square : \mathcal{C}_{\mathcal{O}_E} \longrightarrow \underline{Sets}$ by

framed deformation $\rightarrow$ $\quad A \longmapsto D^\square(A) := \{\rho : G_K \to GL_n(A) \mid \bar{\rho} = \bar{\rho}_0\}$

$$\begin{tikzcd} G_K \arrow[r, "\rho"] \arrow[dr, "\bar{\rho}"'] & GL_n(A) \arrow[d] \\ & GL_n(\mathbb{F}) \end{tikzcd}$$

deformation $\rightarrow$ $\quad D(A) := D^\square(A)/\sim$, where

$\rho \sim \rho' \iff \exists\, c\rho c^{-1} = \rho'$ for $c \in \mathrm{Ker}(GL_n(A) \to GL_n(\mathbb{F}))$

<u>Thm</u> · $D^\square$ is representable by $R^\square_{\bar{\rho}_0} \in \mathcal{C}_{\mathcal{O}_E}$ (~~Mazur~~ Kisin)

· If $\mathrm{End}_{\mathbb{F}[G_K]}(\bar{\rho}_0) = \mathbb{F}$, D is representable by $R^{univ}_{\bar{\rho}_0} \in \mathcal{C}_{\mathcal{O}_E}$.

(Mazur/Kisin)

($R^\square_{\bar{\rho}_0} :=$ universal framed deformation ring,

$R^{univ}_{\bar{\rho}_0} :=$ universal deformation ring)

- $$\mathrm{Hom}(\mathcal{R}^{\square}_{\bar\rho_0}, \mathcal{R}^{\square}_{\bar\rho_0}) \xrightarrow{\sim} D^{\square}(\mathcal{R}^{\square}_{\bar\rho_0})$$
$$\mathrm{id}_{\mathcal{R}^{\square}_{\bar\rho_0}} \longleftrightarrow \rho^{\square} : G_K \to GL_n(\mathcal{R}^{\square}_{\bar\rho_0})$$

$$\Rightarrow \quad \mathrm{Hom}(\mathcal{R}^{\square}_{\bar\rho_0}, A) \xrightarrow{\sim} D^{\square}(A)$$
$$f \longmapsto f\circ\rho^{\square} : G_K \to GL_n(\mathcal{R}^{\square}_{\bar\rho_0}) \xrightarrow{f} GL_n(A)$$

$$\because \quad \begin{array}{ccc} \mathrm{Hom}(\mathcal{R}^{\square}_{\bar\rho_0}, \mathcal{R}^{\square}_{\bar\rho_0}) \ni \mathrm{id}_{\mathcal{R}^{\square}} & \xrightarrow{\sim} & D^{\square}(\mathcal{R}^{\square}_{\bar\rho_0}) \ni \rho^{\square} \\ \downarrow f & & \downarrow f \\ \mathrm{Hom}(\mathcal{R}^{\square}_{\bar\rho_0}, A) \ni f & \xrightarrow{\sim} & D^{\square}(A) \ni f\circ\rho^{\square} \end{array} \qquad \boxed{}$$

$\Rightarrow$ $\rho^{\square}$ is called the <u>universal framed deformation</u>

Similarly, $\exists\, \rho^{univ} : G_K \to GL_n(\mathcal{R}^{univ}_{\bar\rho_0})$, called the <u>universal deformation</u> for D.

- **Lemma** If $\mathrm{End}_{\mathbb{F}[G_K]}(\bar\rho_0) = \mathbb{F}$,

$$\mathcal{R}^{\square}_{\bar\rho_0} \cong \mathcal{R}_{\bar\rho_0}[\![X_1, \cdots, X_{n^2-1}]\!]$$

proof)

$$\rho^{\square} : G_K \longrightarrow GL_n\left(\mathcal{R}^{univ}_{\bar\rho_0}[\![X_{ij} \mid i,j\in\{1,\cdots,n\}]\!]/X_{11}\right)$$
$$g \longmapsto (I_n + (X_{ij}))\,\rho^{univ}(g)\,(I_n + (X_{ij}))^{-1}$$

is a univ. framed def.

$\boxed{}$

· Consider $\det\bar{\rho}_0 : G_K \longrightarrow \mathbb{F}^\times$, and let $\chi : G_K \to \mathcal{O}_E^\times$ s.t.

$$\chi \pmod{\pi_E} = \det\bar{\rho}_0 .$$

Define functors $D^\chi, D^{\square,\chi} : \mathcal{C}_{\mathcal{O}_E} \longrightarrow \underline{\text{Sets}}$ by

$$\cdot\ D^\chi(A) := \{\rho \in D(A) \mid \det\rho = \chi\}$$

$$\cdot\ D^{\square,\chi}(A) := \{\rho \in D^\square(A) \mid \det\rho = \chi\}.$$

$\Rightarrow$ $D^{\square,\chi}$ (D^χ) are representable in $\mathcal{C}_{\mathcal{O}_E}$, i.e.

$$D^\chi(\cdot) \cong \mathrm{Hom}(\mathcal{R}_{\bar{\rho}_0}^{univ,\chi}, \ \ ) \quad \text{for } \mathcal{R}_{\bar{\rho}_0}^{univ,\chi} \in \mathcal{C}_{\mathcal{O}_E}$$

$$D^{\square,\chi}(\cdot) \cong \mathrm{Hom}(\mathcal{R}_{\bar{\rho}_0}^{\square,\chi}, \ \ ) \quad \text{for } \mathcal{R}_{\bar{\rho}_0}^{\square,\chi} \in \mathcal{C}_{\mathcal{O}_E}.$$

and $\mathcal{R}_{\bar{\rho}_0}^{univ} \twoheadrightarrow \mathcal{R}_{\bar{\rho}_0}^{univ,\chi}$ and $\mathcal{R}_{\bar{\rho}_0}^{\square} \twoheadrightarrow \mathcal{R}_{\bar{\rho}_0}^{\square,\chi}$.

<u>Thm</u> (Kisin). Let $\underline{v} \in \mathbb{Z}_+^n$, $\tau : I_K \to GL_n(E)$ be a Galois type, ~~and~~

$\cancel{\chi : G_K \to \mathcal{O}_E^\times \text{ be a lift of } \det\bar{\rho}_0}$.

$\Rightarrow$ $\exists!$ quotient $\mathcal{R}_{\bar{\rho}_0}^{univ} \overset{\pi}{\twoheadrightarrow} \mathcal{R}_{\bar{\rho}_0}^{v,\tau}$ s.t.

- $\mathcal{R}_{\bar{\rho}_0}^{v,\tau}$ is p-torsion free and has a reduced generic fiber
- ~~each~~ a point $x : \mathcal{R}_{\bar{\rho}_0}^{univ} \longrightarrow \overline{\mathbb{Q}}_p$ induces a potentially semi-stable lift of $\bar{\rho}_0$ with $HT = v$ and $GT = \tau$

$\Leftrightarrow$ x factors through π.

Moreover, the relative dimension of $\mathcal{R}_{\bar{\rho}_0}^{v,\tau}$ over $\mathcal{O}_E$

is $[K:\mathbb{Q}_p]\,\frac{n(n-1)}{2} + 1$.

Remark!!

① $R_{\bar{\rho}_0}^{v,\tau}[\frac{1}{p}]$ is equi-dimensional.

② $R_{\bar{\rho}_0}^{v,\tau}/(\pi_E)$ is "

③ A potentially semi-stable rep-n is semi-stable iff its Galois type is trivial.

④ $\exists !$ $R_{\bar{\rho}_0}^{v,\tau} \overset{\pi}{\twoheadrightarrow} R_{\bar{\rho}_0}^{v,\tau,\chi}$ s.t.

- $R_{\bar{\rho}_0}^{v,\tau,\chi}$ is p-torsion free and has a reduced generic fiber
- a point $x : R_{\bar{\rho}_0}^{univ} \longrightarrow \overline{\mathbb{Q}}_p$ induces a potentially semi-stable lift of $\bar{\rho}_0$ with $HT = v$, $GT = \tau$, and $\det = \chi$ $\Leftrightarrow$ x factors through π.

Lemma (Emerton - Gee). If $p > 2$,

$$R_{\bar{\rho}_0}^{v,\tau} \cong R_{\bar{\rho}_0}^{v,\tau,\chi}[\![X]\!]$$

* <u>Strongly divisible module</u> / $\mathcal{R}$.

- let $\mathcal{R} \in \mathcal{C}_{\mathcal{O}_E}$.

$\Rightarrow$ $S_{\mathcal{R}} := \mathfrak{m}_{\mathcal{R}}$-adic completion of $S \otimes_{\mathbb{Z}_p} \mathcal{R}$.

Extend the definitions of $\mathrm{Fil}^{\cdot}$, ϕ, N to $S_{\mathcal{R}}$ $\mathcal{R}$-linearly.

- Fix $0 < r < p-1$.

<u>Def</u>) A strongly div. module$/_{S_{\mathcal{R}}}$ of weight r is a free $S_{\mathcal{R}}$-module $\underline{\mathcal{M}}$ of finite rank together with $(\phi_r, \mathrm{Fil}^r \underline{\mathcal{M}}, N)$, where

- $\mathrm{Fil}^r \underline{\mathcal{M}}$ is a $S_{\mathcal{R}}$-submodule of $\underline{\mathcal{M}}$
- $\phi_r : \mathrm{Fil}^r \underline{\mathcal{M}} \to \underline{\mathcal{M}}$ is an additive map
- $N : \underline{\mathcal{M}} \to \underline{\mathcal{M}}$ —//—

s.t.

1. $\mathrm{Fil}^r \underline{\mathcal{M}}$ contains $\mathrm{Fil}^r S_{\mathcal{R}} \cdot \underline{\mathcal{M}}$;
2. $\mathrm{Fil}^r \underline{\mathcal{M}} \cap I \cdot \underline{\mathcal{M}} = I \cdot \mathrm{Fil}^r \underline{\mathcal{M}}$ $\quad \forall I \underset{\text{ideal}}{\subseteq} \mathcal{R}$
3. $\phi_r(sx) = \phi(s)\phi_r(x)$ $\quad \forall s \in S_{\mathcal{R}}$, $x \in \mathrm{Fil}^r \underline{\mathcal{M}}$
4. $\phi_r(\mathrm{Fil}^r \underline{\mathcal{M}}) \subset \underline{\mathcal{M}}$ and generates $\underline{\mathcal{M}}$ over $S_{\mathcal{R}}$.
5. $N(sx) = sN(x) + N(s) \cdot x$ $\quad \forall s \in S_{\mathcal{R}}$ $\forall x \in \underline{\mathcal{M}}$
6. $E(u) N(\mathrm{Fil}^r \underline{\mathcal{M}}) \subset \mathrm{Fil}^r \underline{\mathcal{M}}$
7. $c \cdot N\phi_r(x) = \phi_r(E(u) N(x))$ $\quad \forall x \in \mathrm{Fil}^r \underline{\mathcal{M}}$, where $c := \frac{1}{p}\phi(E(u))$.

Remark!!

① If $\mathcal{R}$ is p-torsion free, then

$\exists \phi : \underline{\mathcal{M}} \to \underline{\mathcal{M}}$ s.t. $-\ \phi(sx) = \phi(s)\cdot\phi(x)$ $\forall s \in S_{\mathcal{R}}$ $\forall x \in \underline{\mathcal{M}}$

$-\ \phi_r = \frac{1}{p^r}\phi$

$-\ N\phi = p\phi N$ $\quad (\Leftarrow ⑦)$.

② $\underline{\mathcal{M}} / (\mathfrak{m}_{\mathcal{R}}, \mathrm{Fil}^p S_{\mathcal{R}})$ is a Breuil module$/\overline{S}_{\mathbb{F}}$.

<u>Thm</u> (Breuil–Mézard). $\exists$ exact faithful covariant functor T_{st}^r

from the category of str. div. modules$/S_{\mathcal{R}}$ of wt r

to ~~$\mathcal{R}$-rep-ns of~~ the category of $\mathcal{R}$-rep-ns of $G_{\mathbb{Q}_p}$.

satisfying the following properties:

Let I be an ideal of $\mathcal{R}$ containing $\mathfrak{m}_{\mathcal{R}}^n$ for some $n>0$.

① If $\mathcal{R}'$ is a C.N.L. $\mathcal{O}_E$-algebra whose residue field

is a finite ext-n of $\mathbb{F}$ with a morphism $\mathcal{R}/I \to \mathcal{R}'$

then $T_{st}^r(\underline{\mathcal{M}} \otimes_{\mathcal{R}} \mathcal{R}') \cong T_{st}^r(\underline{\mathcal{M}}) \otimes_{\mathcal{R}} \mathcal{R}'$.

② The induced map $T_{st}^r(\underline{\mathcal{M}}) \to T_{st}^r(\underline{\mathcal{M}} \otimes_{\mathcal{R}} \mathcal{R}/I)$ is surjective.

③ If $\mathcal{M} := \underline{\mathcal{M}} / (\mathfrak{m}_{\mathcal{R}}, \mathrm{Fil}^p S_{\mathcal{R}})$, then

$$T_{st}^r(\mathcal{M}) \cong T_{st}^r(\underline{\mathcal{M}}) \otimes_{\mathcal{R}} \mathbb{F}.$$

eg.) Consider the example (2) + assume $\cdot\, r=2$ $\left(v_p(\lambda)=\frac{1}{2}\right)$

$\cdot\, v_p(\alpha-1)<\frac{1}{2}$

$\cdot$ Fix a residual rep-n $\bar{\rho}_0 : G_{\mathbb{Q}_p} \to GL_2(\mathbb{F})$ s.t. $\bar{\rho}_0|_{I_{\mathbb{Q}_p}} \sim \omega_2^{2} \oplus \omega_2^{2p}$

$\Leftarrow$ fix $M(2,0;\alpha,\beta)$, assuming.

$$\bar{\rho}_0 \sim T_{st}^{*}\left(M(2,0;\alpha,\beta)\right).$$

$\Rightarrow$ Fix $\mu \in \mathbb{F}$ with $-\mu = \alpha\cdot\beta$

$\Rightarrow$ $\mu = \left(\frac{\lambda^2}{p}\right) \pmod{\mathfrak{m}_E}$

$\cdot$ Fix a lift $\chi : G_{\mathbb{Q}_p} \to \mathcal{O}_E^{\times}$ of $\det \bar{\rho}_0$.

$\Leftrightarrow$ fix λ^2. (let $\eta := \lambda^2$)

$\because$ Any semi-stable lift ρ of $\bar{\rho}_0$ with $HT=(0,2)$

has $\det \rho = \varepsilon_p^{2}\cdot$ (Unramified character)

~~...~~ determined by λ^2.

$\Rightarrow$ $\lambda=\sqrt{\eta}$, $\lambda=-\sqrt{\eta}$

<u>lemma</u> let $X := \frac{p}{\lambda(\alpha-1)}$

① If $\lambda=\sqrt{\eta}$, the strongly div. mod$/S_{\mathcal{O}_E}$ discussed in the talk 2.

determines a str. div. mod$/S_{\mathcal{O}_E[\![X]\!]}$

② If $\lambda=-\sqrt{\eta}$, ——— // ———

① $\Rightarrow$ $\mathcal{M}^{+}(X)\,/\,S_{\mathcal{O}_E[\![X]\!]}$

② $\Rightarrow$ $\mathcal{M}^{-}(X)\,/\,S_{\mathcal{O}_E[\![X]\!]}$

proof) It is routine: for instance, in the case $\sqrt{\eta}=\lambda$

$$\cdot\ \mathcal{F}_2 := (\varpi p)\left(\lambda\delta E_1 - E_2 + \frac{p\delta\lambda}{\lambda-1}E_2\right)$$

$$= (\varpi p)\left(\sqrt{\eta}\cdot\delta_X E_1 - E_2 + p\delta_X E_2 + \delta_X\sqrt{\eta}\cdot X E_2\right),$$

where δ_X is the limit of the sequence

$$\mathcal{O}_E[\![X]\!] \ni \quad G_0=1,\quad G_{n+1} = \frac{(\lambda-1)^2}{(\lambda-1)^2 - p\,G_n} = \frac{1}{1-\frac{\eta}{p}X^2 G_n}$$

Similarly replace λ with $\sqrt{\eta}$, and $\frac{p}{\lambda(\lambda-1)}$ with X

in the coefficients of $\mathcal{F}_1$, $E(u)^2E_1$, $\phi_2(\mathcal{F}_1)$, $\phi_2(E(u)^2E_1)$

$\mathcal{N}(\mathcal{F}_1)$, $\mathcal{N}(E_2)$,

we get a str. div. mod$/S_{\mathcal{O}_E}$. ☐

Cor $\quad \mathcal{R}^{\mathrm{univ}}_{\bar{\rho}} \xrightarrow{\pi^{\pm}} \mathcal{O}_E[\![X]\!]$, $\quad(\pi^+$ for $\underline{\mathcal{M}}^+(X)$, π^- for $\underline{\mathcal{M}}^-(X))$.

$\because$ Consider $\rho := T^{2}_{st}(\underline{\mathcal{M}}^{\pm}(X))$

Lemma $\quad \pi^{\pm}: \mathcal{R}^{\mathrm{univ}}_{\bar{\rho}} \twoheadrightarrow \mathcal{O}_E[\![X]\!]$ is surjective

proof) Let $\mathcal{R}_0 := \mathcal{O}_E[\![X]\!]/(\pi_E, X^2) \cong \mathbb{F}[X]/(X^2)$.

By Nakayama Lemma, it is enough to check

the induced map $\mathcal{R}^{\mathrm{univ}}_{\bar{\rho}} \twoheadrightarrow \mathcal{R}_0$ is surjective.

By direct computation, when $\lambda = \sqrt{\eta}$,

$$\underline{\mathcal{M}}_0 := S_{\mathcal{R}_0}(E_1, E_2)$$

- $\mathrm{Fil}^2 \underline{\mathcal{M}}_0 = \langle u^2 E_1, E_2 \rangle + \mathrm{Fil}^p S_{\mathcal{R}_0} \cdot \underline{\mathcal{M}}$
- $\phi_2(u^2 E_1) = E_2$ and $\phi_2(E_2) = -\mu E_1$
- $N(E_1) = X\left(1 - [u^p + (wp)^M]\right) E_2$, ~~and~~ $N(E_2) = 0$

One can check that whatever change of basis you do,

X survives.

~~Thus~~ $\therefore$ The image of $\mathcal{R}^{univ}_{\bar{\rho}_0} \to \mathcal{R}_0$ contains X.

$\Rightarrow$ it is surjective $\boxed{}$

<u>Lemma</u> $\pi^+, \pi^- : \mathcal{R}^{univ}_{\bar{\rho}_0} \twoheadrightarrow \mathcal{O}[[X]]$ factor through

the quotient $q : \mathcal{R}^{univ}_{\bar{\rho}_0} \twoheadrightarrow \mathcal{R}^{(0,2),x}_{\bar{\rho}_0}$

<u>proof</u>)

$$\begin{tikzcd}
 & \mathcal{R}^{univ.}_{\bar{\rho}_0} \arrow[dl, two heads, "q"'] \arrow[dr, two heads, "\pi^+"] & \\
\mathcal{R}^{(0,2),x}_{\bar{\rho}_0} \arrow[rr, dashed, "\exists q^+"] \arrow[dr, dashed] & & \mathcal{O}[[X]] \arrow[dl, "x"] \\
 & \overline{\mathbb{Q}}_p &
\end{tikzcd}$$

- reduced generic fiber
- p-torsion free

It is fun! Check the existence of such a q^+ $\boxed{}$

Prop If $\bar{\rho}_0|_{I_{\mathbb{Q}_p}} \sim \omega_2^{1} \oplus \omega_2^{2p}$, then

$\mathcal{R}_{\bar{\rho}_0}^{(0,2)}$ has three irred. components:

- $\mathcal{O}_E[[P_1, P_2]]$ $\leftarrow$ crystalline from example (1), ©
- $\mathcal{O}_E[[P, X]]$ $\leftarrow$ $\lambda = \sqrt{\eta}$, (2) $v_p(\alpha - 1) < \frac{1}{2}$
- $\mathcal{O}_E[[P, X]]$ $\leftarrow$ $\lambda = -\sqrt{\eta}$ ——— // .